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ON NONSTATIONARY MOTION OF A FIXED MASS OF A VISCOUS COMPRESSIBLE BAROTROPIC FLUID BOUNDED BY A FREE BOUNDARY

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1. Introduction. In this paper we consider the global motion of a drop of a viscous barotropic fluid in the general case, i.e. without assuming any conditions on the form of the pressure $p = p(\varrho)$. Here $\varrho = \varrho(x,t)$ (where $x \in \Omega_t$, $t \in [0,T]$, $\Omega_t \subset \mathbb{R}^3$ is a bounded domain of the drop at time t) is the density of the drop.

Next, let $v = v(x,t)$ ($v = (v_i)_{i=1,2,3}$) denote the velocity of the fluid, $f = f(x,t)$ the external force field per unit mass, μ and ν the constant viscosity coefficients, and p_0 the external (constant) pressure. Then the motion of the drop is described by the following system of equations (see [2, Chs. 1, 2]):

$$
\begin{aligned}
&\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p) = \varrho f && \text{in } \widetilde{\Omega}^T,\\
&\varrho_t + \operatorname{div}(\varrho v) = 0 && \text{in } \widetilde{\Omega}^T,\\
&\mathbb{T}\overline{n} = -p_0\overline{n} && \text{on } \widetilde{S}^T,\\
&v\cdot\overline{n} = -\frac{\phi_t}{|\nabla\phi|} && \text{on } \widetilde{S}^T,\\
&\varrho|_{t=0} = \varrho_0, \quad v|_{t=0} = v_0 && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\widetilde{\Omega}^T = \bigcup_{t\in(0,T)} \Omega_t \times \{t\}$, $\widetilde{S}^T = \bigcup_{t\in(0,T)} S_t\times\{t\}$, $S_t = \partial\Omega_t$, $\phi(x,t) = 0$ describes S_t (at least locally), $\overline{n}$ is the unit outward vector normal to the boundary, i.e. $\overline{n} = \nabla\phi/|\nabla\phi|$, and $\Omega = \Omega_t|_{t=0} = \Omega_0$. In (1.1), $\mathbb{T} = \mathbb{T}(v,p) = \{T_{ij}\}_{i,j=1,2,3} = \{-p\delta_{ij} + \mu(v_{ix_j} + v_{jx_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} v\}_{i,j=1,2,3}$ is the stress tensor. Moreover, we assume $\nu > \frac{1}{3}\mu > 0$.

Let the domain Ω be given. Then by $(1.1)_4$, $\Omega_t = \{x\in\mathbb{R}^3 : x = x(\xi,t),\ \xi\in\Omega\}$, where $x = x(\xi,t)$ is the solution of the Cauchy problem

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(1.2) $$\frac{\partial x}{\partial t} = v(x,t), \quad x|_{t=0} = \xi \in \Omega, \quad \xi = (\xi_1, \xi_2, \xi_3).$$

Hence, we obtain the following relation between the Eulerian x and the Lagrangian ξ coordinates of the same fluid particle:

(1.3) $$x = \xi + \int_0^t u(\xi, t')\, dt' \equiv X_u(\xi, t),$$

where $u(\xi,t) = v(X_u(\xi,t),t)$. Moreover, by $(1.1)_4$, $S_t = \{x : x = x(\xi,t),\ \xi \in S = \partial\Omega\}$.

By the continuity equation $(1.1)_2$ and the kinematic condition $(1.1)_4$ the total mass is conserved, i.e.

(1.4) $$\int_{\Omega_t} \varrho(x,t)\, dx = \int_{\Omega} \varrho_0(\xi)\, d\xi = M,$$

where M is a given constant.

In [15] the local existence of a unique solution is proved for a problem analogous to (1.1), but describing the motion of a drop of a viscous heat–conducting fluid.

Let $u = u(\xi,t)$, $\eta = \eta(\xi,t)$ denote v and ϱ written in Lagrangian coordinates. In the same way as in [15] (see Theorem 4.2 of [15]) one can prove the local existence of a unique solution (v,ϱ) of problem (1.1) such that $u \in \mathcal{A}_{T,\Omega}$, $\eta \in \mathcal{B}_{T,\Omega}$, where $\mathcal{A}_{T,\Omega} \equiv \mathcal{A}_{T,\Omega_{0T}}$, $\mathcal{B}_{T,\Omega} \equiv \mathcal{B}_{T,\Omega_{0T}}$ and

(1.5) $$\begin{aligned}\mathcal{B}_{T,\Omega_{iT}} = \{f \in C(iT,(i+1)T; H^2(\Omega_{iT})) :\ & f_t \in C(iT,(i+1)T; H^1(\Omega_{iT})) \cap L_2(iT,(i+1)T; H^2(\Omega_{iT})),\\ & f_{tt} \in C(iT,(i+1)T; L_2(\Omega)) \cap L_2(iT,(i+1)T; H^1(\Omega_{iT}))\},\end{aligned}$$

(1.6) $$\mathcal{A}_{T,\Omega_{iT}} = \mathcal{B}_{T,\Omega_{iT}} \cap L_2(iT,(i+1)T; H^3(\Omega_{iT})),$$

$i \in \mathbb{N} \cup \{0\}$, $T \le T_*$, where $T_* > 0$ is a certain constant.

The aim of this paper is to prove the existence of a global-in-time solution of problem (1.1) near a constant state. Consider the equation

(1.7) $$p(\varrho) = p_0,$$

where $\varrho \in \mathbb{R}_+$, $p \in C^3(\mathbb{R}_+)$, and $p' > 0$.

We introduce the following definition of a constant state.

Definition 1.1. Let $f = 0$. Then by a *constant* (*equilibrium*) *state* we mean a solution (v,ϱ) of problem (1.1) such that $v = 0$, $\varrho = \varrho_e$, and $\Omega_t = \Omega_e$ for $t \ge 0$, where ϱ_e is a solution of (1.7) and $|\Omega_e| = M/\varrho_e$ ($|\Omega_e| = \operatorname{vol}\Omega_e$).

First, in Section 2 we derive a differential inequality (2.58) which enables extending the local solution of (1.1) step by step from the interval $[0,T]$ to $[0,\infty)$. To prove the global existence we also use Lemma 2.1, which gives an energy estimate (2.8), and Lemmas 3.3–3.4. The above lemmas yield in particular global estimates for $\|v\|^2_{L_2(\Omega_t)}$ and $\|p_\sigma\|^2_{L_2(\Omega_t)}$ (where $p_\sigma = p - p_0$), which are used in the proofs of Lemma 3.4 and Theorem 3.9, the main result of the paper.

The global motion of a fluid described by (1.1) has been considered earlier in papers [7] and [17].

In [17] the global existence for problem (1.1) is proved for a special form of $p = p(\varrho)$:

$$p = a_0 \varrho^{\alpha}, \tag{1.8}$$

where $a_0 > 0$ and $\alpha > 0$ are constants. The global solution obtained in [17] is more regular than the one obtained in this paper.

A result analogous to that of [17] is proved (under assumption (1.8)) in [18] for the fluid bounded by a free boundary the shape of which is governed by surface tension.

Paper [7] of V. A. Solonnikov and A. Tani is concerned with problem (1.1) with the boundary condition $\mathbb{T}\overline{n} - \sigma H \overline{n} = 0$ (where H is the double mean curvature of S_t, and $\sigma > 0$ is the constant coefficient of surface tension). In [7] the existence of a solution is proved in some anisotropic Sobolev–Slobodetskiĭ spaces; it is a little less regular than ours. To prove the local existence the authors of [7] apply potential techniques.

Both in [17] and in [7] the energy conservation law is used in order to derive a global estimate for $\|v\|^2_{L_2(\Omega_t)}$.

Papers [8]–[10] are concerned with the free boundary problem for a viscous barotropic self-gravitating fluid with p of the form (1.8).

Next, papers [11]–[14] are devoted to the free boundary problem for a viscous heat-conducting fluid under the assumption that the internal energy ε has a special form:

$$\varepsilon = a_0 \varrho^{\alpha} + h(\varrho, \theta),$$

where $a_0 > 0$, $\alpha > 0$, $h(\varrho,\theta) \ge h_* > 0$; a_0, α and h_* are constants.

The free boundary problem for a viscous incompressible fluid was examined by V. A. Solonnikov in [3]–[6].

Finally, we present the notation used in the paper. We denote by $\|\cdot\|_{l,Q}$ (where $l \ge 0$, $Q \subset \mathbb{R}^3$) the norms in the Sobolev spaces $H^l(Q)$, and by $\Gamma^l_k(Q)$ ($l > 0$, $k \ge 0$, $Q \subset \mathbb{R}^3$) the space of functions $u = u(x,t)$ ($x \in Q$, $t \in (0,T)$, $T > 0$) with the norm

$$\|u\|_{\Gamma^k_l(Q)} = \sum_{i \le l-k} \|\partial^i_t u\|_{l-i,Q} \equiv |u|_{l,k,Q}.$$

2. Differential inequality. Assume that the existence of a sufficiently smooth local solution of problem (1.1) has been proved and let

$$f = 0. \tag{2.1}$$

In this section we obtain a special differential inequality which enables us to prove the global existence. To get the inequality we consider the motion near the constant state. Let

$$p_\sigma = p - p_0, \qquad \varrho_\sigma = \varrho - \varrho_e, \tag{2.2}$$

where ϱ_e is introduced in Definition 1.1. Then problem (1.1) takes the form

$$\begin{aligned} &\varrho[v_t + (v\cdot\nabla)v] - \operatorname{div}\mathbb{T}(v,p_\sigma) = 0 && \text{in } \Omega_t,\ t\in(0,T),\\ &\varrho_{\sigma t} + \operatorname{div}(\varrho v) = 0 && \text{in } \Omega_t,\ t\in(0,T),\\ &\mathbb{T}(v,p_\sigma)\overline{n} = 0 && \text{on } S_t,\ t\in(0,T),\\ &\varrho_\sigma|_{t=0} = \varrho_{\sigma 0} = \varrho_0 - \varrho_e,\ v|_{t=0} = v_0, && \text{in } \Omega. \end{aligned} \tag{2.3}$$

In the sequel we use the following Taylor formula for p_σ:

$$p_\sigma = (\varrho - \varrho_e)\int_0^1 p'(\varrho_e + s(\varrho-\varrho_e))\,ds = p_1\varrho_\sigma, \tag{2.4}$$

where the function p_1 is positive.

Now, let ϱ_* and ϱ^* be positive constants such that

$$\varrho_* < \varrho < \varrho^* \quad \text{for } x\in\overline{\Omega}_t,\ t\in[0,T]. \tag{2.5}$$

In the lemmas below we denote by ε small constants, by $c_0<1$ a positive constant depending on μ, ν, and by c a positive constants depending on T (the time of local existence), $\varrho_*, \varrho^*, \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt'$, $\|S\|_{5/2}$, on the parameters which guarantee the existence of the inverse transformation to $x = x(\xi,t)$ and on the constants of imbedding theorems and Korn inqualities. We do not distinguish different ε's or c's.

We underline that all the estimates below are obtained under the assumption that there exists a local-in-time solution of problem (1.1), so all the quantities $\varrho_*, \varrho^*, T, \int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt', \|S\|_{5/2}$ are estimated by the data functions. Moreover, the existence of the inverse transformation to $x = x(\xi,t)$ is guaranteed by the estimates for the local solution (see [15]).

Now, assume the relations

$$\int_{\Omega_t} \varrho v\,dx = 0, \tag{2.6}$$

$$\int_{\Omega_t} \varrho v\cdot\eta\,dx = 0, \tag{2.7}$$

where $\eta = a + b\times x$ and a and b are arbitrary vectors.

Lemma 2.1. *Let (v,ϱ_σ) be a sufficiently smooth solution of* (2.3). *Then*

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}\left(\varrho v^2 + \frac{p_1}{\varrho}\varrho_\sigma^2\right)dx + c_0\|v\|^2_{1,\Omega_t} \le cX_1^2(1+X_1), \tag{2.8}$$

where $X_1 = \|v\|^2_{2,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t}$.

P r o o f. Multiplying $(2.3)_1$ by v, integrating over Ω_t and using the continuity equation $(2.3)_2$, boundary condition $(2.3)_4$ and (2.4) we obtain

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\varrho v^2\,dx+\frac{\mu}{2}E_{\Omega_t}(v)+(\nu-\mu)\|\operatorname{div}v\|^2_{0,\Omega_t}-\int\limits_{\Omega_t}p_1\varrho_\sigma\operatorname{div}v\,dx=0,\tag{2.9}$$

where $E_{\Omega_t}(v)=\int_{\Omega_t}\sum_{i,j=1}^3(v_{ix_j}+v_{jx_i})^2\,dx$.

In [13] it is proved that

$$\frac{\mu}{2}E_{\Omega_t}(v)+(\nu-\mu)\|\operatorname{div}v\|^2_{0,\Omega_t}\ge cE_{\Omega_t}(v),$$

where $c>0$ is a constant.

Next, by the continuity equation $(2.3)_2$ we have

$$-\int\limits_{\Omega_t}p_1\varrho_\sigma\operatorname{div}v\,dx=\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{p_1\varrho_\sigma^2}{\varrho}\,dx+J,\tag{2.10}$$

where

$$|J|\le\varepsilon(\|\varrho_{\sigma t}\|^2_{0,\Omega_t}+\|v\|^2_{1,\Omega_t})+cX_1^2(1+X_1).\tag{2.11}$$

Moreover, in view of assumptions (2.6) and (2.7), Lemma 5.2 of [17] yields

$$\|v\|^2_{1,\Omega_t}\le c(E_{\Omega_t}(v)+\|\varrho_\sigma\|^2_{0,\Omega_t}\|v\|^2_{0,\Omega_t})\tag{2.12}$$

and by the continuity equation $(2.3)_2$,

$$\|\varrho_{\sigma t}\|^2_{0,\Omega_t}\le c\|v\|^2_{1,\Omega_t}+c\|v\|^2_{1,\Omega_t}\|\varrho_\sigma\|^2_{2,\Omega_t}.\tag{2.13}$$

Taking into account (2.9)–(2.13) we get estimate (2.8). ∎

Lemma 2.2. *Let* (v,ϱ_σ) *be a sufficiently smooth solution of* (2.3). *Then*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_t^2+\frac{p_{\varrho\sigma}}{\varrho}\varrho_{\sigma t}^2\right)dx+c_0(\|v_t\|^2_{1,\Omega_t}+\|\varrho_{\sigma t}\|^2_{0,\Omega_t})\le c\|v\|^2_{1,\Omega_t}+cY_1^2(1+X_2),\tag{2.14}$$

where

$$X_2=|v|^2_{2,0,\Omega_t}+|\varrho_\sigma|^2_{2,0,\Omega_t}+\int\limits_0^t\|v\|^2_{3,\Omega_{t'}}\,dt',\tag{2.15}$$

$$Y_1=X_2-\int\limits_0^t\|v\|^2_{3,\Omega_{t'}}\,dt'.\tag{2.16}$$

P r o o f. Differentiating $(2.3)_1$ with respect to t, multiplying by v_t and integrating over Ω_t yields

$$
(2.17)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\varrho v_t^2\,dx+\frac{\mu}{2}E_{\Omega_t}(v_t)+(\nu-\mu)\|\operatorname{div}v_t\|_{0,\Omega_t}^2 - \int\limits_{\Omega_t} p_{\sigma\varrho}\varrho_{\sigma t}\operatorname{div}v_t\,dx\le cY_1^2(1+X_2),
$$

where we have used the boundary condition $(2.3)_4$.

By Lemma 5.3 of [17] we have the following Korn type inequality:

$$
(2.18)\qquad \|v_t\|_{1,\Omega_t}^2\le c[E_{\Omega_t}(v_t)+Y_1^2(1+Y_1)].
$$

Finally, using the continuity equation $(2.3)_3$ we get

$$
(2.19)\qquad -\int\limits_{\Omega_t} p_{\sigma\varrho}\varrho_{\sigma t}\operatorname{div}v_t\,dx=\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2\,dx+J,
$$

where

$$
(2.20)\qquad |J|\le\varepsilon(\|v_t\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2)+cY_1^2(1+Y_1).
$$

In view of inequalities (2.17)–(2.20) and (2.13) we obtain (2.14). ■

Lemmas 2.1 and 2.2 yield

LEMMA 2.3. *Let* (v,ϱ_σ) *be a sufficiently smooth solution of* (2.3). *Then*

$$
(2.21)\qquad \frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left[\varrho(v^2+v_t^2)+\frac{p_1}{\varrho}\varrho_\sigma^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma t}^2\right]dx + c_0(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2)\le cY_1^2(1+X_2),
$$

where X_2 *and* Y_1 *are given by* (2.15) *and* (2.16), *respectively.*

Next, we obtain

LEMMA 2.4. *Let* v,ϱ_σ *be a sufficiently smooth solution of* (2.3). *Then*

$$
\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_{tt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma tt}^2\right)dx+c_0(\|v_{tt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma tt}\|_{0,\Omega_t}^2) \le c(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2)+cX_2Y_2(1+X_2^2),
$$

where X_2 *is given by* (2.15) *and*

$$
(2.22)\qquad Y_2=|v|_{3,1,\Omega_t}^2+\|\varrho_\sigma\|_{2,\Omega_t}^2+\|\varrho_{\sigma t}\|_{2,\Omega_t}^2+\|\varrho_{\sigma tt}\|_{1,\Omega_t}^2.
$$

The above lemma can be proved in the same way as Lemmas 2.1 and 2.2. To estimate $E_{\Omega_t}(v_{tt})$ we use here Lemma 5.4 of [17].

In order to obtain estimates for derivatives with respect to x we rewrite problem (2.3) in Lagrangian coordinates. We have

$$
(2.23)\qquad
\begin{aligned}
&\eta u_{it} - \nabla_{u_j} T_{uij}(u, p_\sigma) = 0 \ (i = 1,2,3) && \text{in } \Omega^T \equiv \Omega \times (0,T),\\
&\eta_{\sigma t} + \eta \nabla_u \cdot u = 0 && \text{in } \Omega^T,\\
&\mathbb{T}_u(u, p_\sigma)\overline{n}_u = 0 && \text{on } S^T \equiv S \times (0,T),\\
&u|_{t=0} = v_0, \quad \eta_\sigma|_{t=0} = \varrho_{\sigma 0}, && \text{in } \Omega,
\end{aligned}
$$

where $\eta(\xi,t) = \varrho(X_u(\xi,t),t)$, $u(\xi,t) = v(X_u(\xi,t),t)$ (X_u is given by (1.3)), $\eta_\sigma = \eta - \varrho_e$, $\varrho_{\sigma 0} = \varrho_0 - \varrho_e$, $\mathbb{T}_u(u,p_\sigma) = \{T_{uij}(u,p_\sigma)\}_{i,j=1,2,3} = \{-p_\sigma\delta_{ij} + \mu(\partial_{x_i}\xi_k\partial_{\xi_k}u_j + \partial_{x_j}\xi_k\partial_{\xi_k}u_i) + (\nu-\mu)\delta_{ij}\,\mathrm{div}_u\, u\}_{i,j=1,2,3}$, $\mathrm{div}_u\, u = \nabla_u \cdot u = \partial_{x_i}\xi_k\partial_{\xi_k}u_i$, $\nabla_u = (\xi_{kx_i}\partial_{\xi_k})_{i=1,2,3}$, $\nabla_{u_j} = \xi_{kx_j}\partial_{\xi_k}$, $\partial_{x_i}\xi_k$ are the elements of the matrix ξ_x which is inverse to $x_\xi = I + \int_0^t u_\xi(\xi,t')\,dt'$, $I = \{\delta_{ij}\}_{i,j=1,2,3}$ is the unit matrix, $\overline{n}_u = \overline{n}(X_u(\xi,t),t) = \nabla_x\phi(x,t)/|\nabla_x\phi(x,t)|_{x=X_u(\xi,t)}$ (S_t is determined at least locally by the equation $\phi(x,t) = 0$) and summation over repeated indices is assumed.

By (2.4) we have $p_\sigma = p_1\eta_\sigma$, where $p_1 = p_1(\eta)$.

Now, introduce a partition of unity $(\{\widetilde{\Omega}_i\},\{\zeta_i\})$, $\Omega = \bigcup_i \widetilde{\Omega}_i$. Let $\widetilde{\Omega}$ be one of the $\widetilde{\Omega}_i$'s and $\zeta(\xi) = \zeta_i(\xi)$ be the corresponding function. If $\widetilde{\Omega}$ is an interior subdomain then let $\widetilde{\omega}$ be a set such that $\widetilde{\omega} \subset \widetilde{\Omega}$ and $\zeta(\xi) = 1$ for $\xi \in \widetilde{\omega}$. Otherwise, we assume that $\overline{\widetilde{\Omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \cap S \neq \emptyset$, $\overline{\widetilde{\omega}} \subset \overline{\widetilde{\Omega}}$. Take any $\beta \in \overline{\widetilde{\omega}} \cap S = \overline{\widetilde{S}}$ and introduce local coordinates $\{y\}$ associated with $\{\xi\}$ by

$$
(2.24)\qquad y_k = \alpha_{kl}(\xi_l - \beta_l), \quad \alpha_{3k} = n_k(\beta), \quad k = 1,2,3,
$$

where $\{\alpha_{kl}\}$ is a constant orthogonal matrix such that $\widetilde{S}$ is determined by the equation $y_3 = F(y_1,y_2)$, $F \in H^{5/2}$ and

$$
\widetilde{\Omega} = \{y : |y_i| < d,\ i = 1,2,\ F(y') < y_3 < F(y') + d,\ y' = (y_1,y_2)\}.
$$

Next, we introduce u', η', η'_σ by

$$
\begin{aligned}
&u'_i(y) = \alpha_{ij}u_j(\xi)|_{\xi=\xi(y)} \quad (i = 1,2,3), \quad \eta'(y) = \eta(\xi)|_{\xi=\xi(y)},\\
&\eta'_\sigma(y) = \eta'(y) - \varrho_e,
\end{aligned}
$$

where $\xi = \xi(y)$ is the inverse transformation to (2.24).

Next, we introduce new variables by

$$
z_i = y_i \ (i = 1,2), \quad z_3 = y_3 - \widetilde{F}(y), \quad y \in \widetilde{\Omega},
$$

which will be denoted by $z = \Phi(y)$ (where $\widetilde{F} \in H^3$ is an extension of F). Let

$$
(2.25)\qquad \widehat{\Omega} = \Phi(\widetilde{\Omega}) = \{z : |z_i| < d,\ i = 1,2,\ 0 < z_3 < d\} \quad \text{and} \quad \widehat{S} = \Phi(\widetilde{S}).
$$

Define

$$
\widehat{u}(z) = u'(y)|_{y=\Phi^{-1}(z)}, \quad \widehat{\eta}(z) = \eta'(y)|_{y=\Phi^{-1}(z)}, \quad \widehat{\eta}_\sigma(z) = \widehat{\eta}(z) - \varrho_e.
$$

Set $\widehat{\nabla}_k = \xi_{lx_k} z_{i\xi_l} \nabla_{z_i}|_{\xi=\chi^{-1}(z)}$, where $\chi(\xi) = \Phi(\psi(\xi))$ and $y = \psi(\xi)$ is described by (2.24). We also introduce the following notation:

$$\widetilde{u}(\xi) = u(\xi)\zeta(\xi), \quad \widetilde{\eta}(\xi) = \eta(\xi)\zeta(\xi), \quad \widetilde{\eta}_\sigma(\xi) = \eta_\sigma(\xi)\zeta(\xi)$$

for $\xi \in \widetilde{\Omega}$, $\widetilde{\Omega} \cap S = \emptyset$ and

$$\widetilde{u}(z) = \widehat{u}(z)\widehat{\zeta}(z), \quad \widetilde{\eta}(z) = \widehat{\eta}(z)\widehat{\zeta}(z), \quad \widetilde{\eta}_\sigma(z) = \widehat{\eta}_\sigma(z)\widehat{\zeta}(z)$$

for $z \in \widehat{\Omega} = \Phi(\widetilde{\Omega})$, $\overline{\widetilde{\Omega}} \cap S \neq \emptyset$, where $\widehat{\zeta}(z) = \zeta(\xi)|_{\xi=\chi^{-1}(z)}$.

Using the above notation we rewrite problem (2.23) in the following form in an interior subdomain:

$$\begin{aligned}
&\eta\widetilde{u}_{it} - \nabla_{u_j} T_{uij}(\widetilde{u}, \widetilde{p}_\sigma) = -\nabla_{u_j} B_{uij}(u,\zeta) - T_{uij}(u, p_\sigma)\nabla_{u_j}\zeta \equiv k_1, \quad i = 1,2,3,\\
&\widetilde{\eta}_{\sigma t} + \eta\nabla_u \cdot \widetilde{u} = \eta u \cdot \nabla_u \zeta \equiv k_2,
\end{aligned}$$

where $\widetilde{p}_\sigma = p_\sigma\zeta$ and $\mathbb{B}_u(u,\zeta) = \{B_{uij}(u,\zeta)\}_{i,j=1,2,3} = \{\mu(u_i\nabla_{u_j}\zeta + u_j\nabla_{u_i}\zeta) + (\nu-\mu)\delta_{ij} u \cdot \nabla_u\zeta\}_{i,j=1,2,3}$.

In boundary subdomains we have

$$\begin{aligned}
&\widehat{\eta}\,\widetilde{u}_{it} - \widehat{\nabla}_j \widehat{T}_{ij} = -\widehat{\nabla}_j \widehat{B}_{ij}(\widehat{u}, \widehat{\zeta}) - \widehat{T}_{ij}(\widehat{u}, p_\sigma)\widehat{\nabla}_j\widehat{\zeta} \equiv k_{3i}, \qquad i = 1,2,3,\\
&\widetilde{\eta}_{\sigma t} + \widehat{\eta}\widehat{\nabla} \cdot \widetilde{u} = \widehat{\eta}\,\widehat{u} \cdot \widehat{\nabla}\widehat{\zeta} \equiv k_4,\\
&\widehat{\mathbb{T}}(\widetilde{u}, \widetilde{p}_\sigma)\widehat{n} = k_5,
\end{aligned} \tag{2.26}$$

where $k_{5i} = \widehat{B}_{ij}(\widehat{u}, \widehat{\zeta})\widehat{n}_j$, $\widehat{\nabla} = (\widehat{\nabla}_j)_{j=1,2,3}$ and $\widehat{\mathbb{T}}$ and $\widehat{\mathbb{B}}$ indicate that the operator ∇_u is replaced by $\widehat{\nabla}$.

In Lemmas 2.5–2.7 below we denote z_1, z_2, by τ, i.e. $\tau = (z_1, z_2)$, and z_3 by n.

Lemma 2.5. *Let* (v, ϱ_σ) *be a sufficiently smooth solution of* (2.3). *Then*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_x^2 + \frac{p_{\sigma\varrho}}{\varrho}\varrho_{\sigma x}^2\right)dx + c_0(\|v\|_{2,\Omega_t}^2 + \|\varrho_{\sigma x}\|_{0,\Omega_t}^2)\\
&\qquad \le c(\|v\|_{1,\Omega_t}^2 + \|v_t\|_{1,\Omega_t}^2 + \|\varrho_{\sigma t}\|_{0,\Omega_t}^2 + \|p_\sigma\|_{0,\Omega_t}) + cX_2^2(1 + X_2),
\end{aligned} \tag{2.27}$$

where X_2 *is given by* (2.15), $v_x^2 = \sum_{i,j=1}^3 v_{ix_j}^2$, *and* $\varrho_{\sigma x}^2 = \sum_{i=1}^3 \varrho_{\sigma x_i}^2$.

Proof. First, we consider the following elliptic problem:

$$\begin{aligned}
&\mu\nabla_u^2 u + \nu\nabla_u\nabla_u \cdot u - p_{\sigma\eta}\nabla_u\eta = \eta u_t && \text{in } \Omega,\\
&\operatorname{div}_u u = \operatorname{div}_u u && \text{in } \Omega,\\
&\mathbb{T}_u(u, p_\sigma)\overline{n}_u = 0 && \text{on } S.
\end{aligned} \tag{2.28}$$

Since the complementarity condition for (2.28) is satisfied we can apply to problem (2.28) the Agmon–Douglis–Nirenberg theory (see [1]). Thus, we get

$$\begin{aligned}
\|u\|_{2,\Omega}^2 + \|\eta_\sigma\|_{1,\Omega}^2 &\le c(\|\eta u_t\|_{0,\Omega}^2 + \|\operatorname{div}_u u\|_{1,\Omega}^2)\\
&\le c(\|u_t\|_{0,\Omega}^2 + \|\operatorname{div} u\|_{1,\Omega}^2 + cX_2^2(\Omega)(1 + X_2(\Omega))),
\end{aligned} \tag{2.29}$$

where we have used the fact that $\|\operatorname{div}_u u - \operatorname{div} u\|^2_{1,\Omega} \le \varepsilon\|u\|^2_{2,\Omega}$ ($\varepsilon > 0$ is sufficiently small), and

$$X_2(\Omega) = |u|^2_{2,0,\Omega} + |\eta_\sigma|^2_{2,0,\Omega} + \int_0^t \|u\|^2_{3,\Omega}\,dt'. \tag{2.30}$$

In view of (2.29) we see that in order to obtain inequality (2.27) it remains to get appropriate estimates for $\|\operatorname{div} u\|^2_{1,\Omega}$ and for $\frac{1}{2}\frac{d}{dt}\int_{\Omega_t}(\varrho v_x^2 + (p_{\sigma\varrho}/\varrho)\varrho^2_{\sigma x})\,dx$. To do this, consider first boundary subdomains. Differentiate $(2.26)_1$ with respect to τ, multiply the result by $\widetilde{u}_\tau J$ (J is the Jacobian of the transformation $x = x(z)$) and integrate over $\widehat{\Omega}$. Hence using the Korn inequality and equation $(2.26)_2$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}} \widehat{\eta}\widetilde{u}^2_\tau J\,dz + c_0\|\widetilde{u}_\tau\|^2_{1,\widehat{\Omega}} \\ &\quad - \int_{\widehat{S}} (\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma)\widehat{n})_{,\tau}\widetilde{u}_\tau J\,dz - \int_{\widehat{\Omega}} \widetilde{p}_{\sigma\tau}\nabla\cdot\widetilde{u}_\tau J\,dz \\ &\le \varepsilon(\|\widehat{\eta}_\sigma\|^2_{0,\widehat{\Omega}} + \|\widetilde{u}_\tau\|^2_{1,\widehat{\Omega}}) + c(\|\widehat{u}\|^2_{1,\widehat{\Omega}} + \|p_\sigma\|^2_{0,\widehat{\Omega}}) + cX_2^2(\widehat{\Omega})(1 + X_2(\widehat{\Omega})), \end{aligned} \tag{2.31}$$

where

$$X_2(\widehat{\Omega}) = |\widehat{u}|^2_{2,0,\widehat{\Omega}} + |\widehat{\eta}_\sigma|^2_{2,0,\widehat{\Omega}} + \int_0^t \|\widehat{u}\|^2_{3,\widehat{\Omega}}\,dt', \qquad \widetilde{u}^2_\tau = \sum_{i=1}^3\sum_{j=1}^2 \widetilde{u}_{iz_j}. \tag{2.32}$$

Using the boundary condition $(2.26)_3$ we have

$$\begin{aligned} &-\int_{\widehat{S}} (\widehat{\mathbb{T}}(\widetilde{u},\widetilde{p}_\sigma)\widehat{n})_{,\tau}\widetilde{u}_\tau J\,d\tau = -\int_{\widehat{S}} (\widehat{B}_{ij}(\widehat{u},\widehat{\zeta})\widehat{n}_j)_{,\tau}\widetilde{u}_{i\tau} J\,d\tau \\ &= \int_{\widehat{S}} \partial_\tau^{1/2}(\widehat{B}_{ij}(\widehat{u},\widehat{\zeta})\widehat{n}_j)\partial_\tau^{1/2}(\widetilde{u}_{i\tau}J)\,d\tau \le \varepsilon\|\widetilde{u}_\tau\|^2_{1,\widehat{\Omega}} + \|\widehat{u}\|^2_{1,\widehat{\Omega}} + cX_2^2(\widehat{\Omega}), \end{aligned} \tag{2.33}$$

where to use the derivative $\partial_\tau^{1/2}$ we have to apply the Fourier transformation.

Next,

$$-\int_{\widehat{\Omega}} \widetilde{p}_{\sigma\tau}\nabla_u\cdot\widetilde{u}_\tau J\,dz = -\int_{\widehat{\Omega}} p_{\sigma\widehat{\eta}}\widetilde{\eta}_{\sigma\tau}\widehat{\nabla}\cdot\widetilde{u}_\tau J\,dz + J_1, \tag{2.34}$$

where $|J_1| \le \varepsilon\|\widetilde{u}_\tau\|^2_{1,\widehat{\Omega}} + c\|p_\sigma\|^2_{0,\widehat{\Omega}}$ and

$$-\int_{\widehat{\Omega}} p_{\sigma\widehat{\eta}}\widetilde{\eta}_{\sigma\tau}\widehat{\nabla}\cdot\widetilde{u}_\tau J\,dz = \frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}} \frac{p_{\sigma\widehat{\eta}}}{\widehat{\eta}}\widetilde{\eta}^2_{\sigma\tau} J\,dz + J_2, \tag{2.35}$$

where

$$|J_2| \le \varepsilon\|\widetilde{\eta}_{\sigma\tau}\|^2_{0,\widehat{\Omega}} + c\|\widehat{u}\|^2_{1,\widehat{\Omega}} + cX_2^2(\widehat{\Omega}). \tag{2.36}$$

Taking into account (2.31), (2.33)–(2.36) and assuming that ε is sufficiently small we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\left(\widehat{\eta}\widetilde{u}_\tau^2+\frac{p_{\sigma\widehat{\eta}}}{\widehat{\eta}}\widetilde{\eta}_{\sigma\tau}^2\right)J\,dz+c_0\|\widetilde{u}_\tau\|_{1,\widehat{\Omega}}^2 \le \varepsilon\|\widehat{\eta}_{\sigma\tau}\|_{0,\widehat{\Omega}}^2+c(\|\widehat{u}\|_{1,\widehat{\Omega}}^2+\|p_\sigma\|_{0,\widehat{\Omega}}^2)+cX_2^2(\widehat{\Omega})(1+X_2(\widehat{\Omega})). \tag{2.37}$$

Now, applying the operator $(\mu+\nu)\nabla_{z_i}$ to $(2.26)_2$, dividing the result by $\widehat{\eta}$, adding to $(2.26)_1$ and multiplying both sides of the result by $p_{\sigma\widehat{\eta}}$ gives

$$\begin{aligned}\frac{\mu+\nu}{\widehat{\eta}}p_{\sigma\widehat{\eta}}\nabla_{z_i}\widetilde{\eta}_{\sigma t}+p_{\sigma\widehat{\eta}}^2\nabla_{z_i}\widetilde{\eta}_\sigma &= p_{\sigma\widehat{\eta}}^2\widehat{\eta}_\sigma\nabla_{z_i}\widehat{\zeta}-p_1p_{\sigma\widehat{\eta}}\widehat{\eta}_\sigma\nabla_{z_i}\widehat{\zeta}+p_{\sigma\widehat{\eta}}k_{3i}+\mu p_{\sigma\widehat{\eta}}(\widehat{\nabla}^2\widetilde{u}_i-\widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u})\\ &\quad+(\mu+\nu)p_{\sigma\widehat{\eta}}(\widehat{\nabla}_i-\nabla_{z_i})\widehat{\nabla}\cdot\widetilde{u}+\frac{\mu+\nu}{\widehat{\eta}}p_{\sigma\widehat{\eta}}\nabla_{z_i}(\widehat{\eta}\widehat{u}\cdot\widehat{\nabla}\widehat{\zeta})\\ &\quad-p_{\sigma\widehat{\eta}}\widehat{\eta}\widetilde{u}_{it}-\frac{\mu+\nu}{\widehat{\eta}}p_{\sigma\widehat{\eta}}\nabla_{z_i}\widehat{\eta}\widehat{\nabla}\cdot\widetilde{u},\qquad i=1,2,3.\end{aligned} \tag{2.38}$$

Multiplying the normal component of (2.38) by $\eta_{\sigma n}J$ and integrating over $\widehat{\Omega}$ we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\frac{p_{\sigma\widehat{\eta}}}{\widehat{\eta}}\widetilde{\eta}_{\sigma n}^2J\,dz+c_0\|\widetilde{\eta}_{\sigma n}\|_{0,\widehat{\Omega}}^2\\ &\le(\varepsilon+cd)\|\widetilde{u}_{nn}\|_{0,\widehat{\Omega}}^2+\varepsilon\|\widetilde{\eta}_{\sigma n}\|_{0,\widehat{\Omega}}^2\\ &\quad+c(\|\widetilde{u}_{z\tau}\|_{0,\widehat{\Omega}}^2+\|\widehat{u}\|_{1,\widehat{\Omega}}^2+\|\widetilde{u}_t\|_{0,\widehat{\Omega}}^2+\|p_\sigma\|_{0,\widehat{\Omega}}^2)+cX_2^2(\widehat{\Omega})(1+X_2(\widehat{\Omega})),\end{aligned} \tag{2.39}$$

where d is from formula (2.25).

Now, we write $(2.26)_1$ in the form

$$\widehat{\eta}\widetilde{u}_{it}-\mu\Delta\widetilde{u}_i-\nu\nabla_{z_i}\nabla\cdot\widetilde{u}=\widehat{\nabla}_i\widetilde{p}_\sigma+k_{3i}-k_{6i}, \tag{2.40}$$

where $k_{6i}=(\mu\Delta\widetilde{u}_i+\nu\nabla_{z_i}\nabla\cdot\widetilde{u})-(\mu\widehat{\nabla}^2\widetilde{u}_i+\nu\widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u})$.

Multiplying the third component of (2.40) by $\widetilde{u}_{3nn}J$ and integrating over $\widehat{\Omega}$ yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\widehat{\Omega}}\widehat{\eta}\widetilde{u}_{3n}^2J\,dz+c_0\|\widetilde{u}_{3nn}\|_{0,\widehat{\Omega}}^2\\ &\le(\varepsilon+cd)\|\widetilde{u}_{nn}\|_{0,\widehat{\Omega}}^2+c(\|\widetilde{u}_{z\tau}\|_{0,\widehat{\Omega}}^2+\|\widehat{u}\|_{1,\widehat{\Omega}}^2\\ &\quad+\|\widetilde{u}_t\|_{1,\widehat{\Omega}}^2+\|\widetilde{\eta}_{\sigma n}\|_{0,\widehat{\Omega}}^2+\|p_\sigma\|_{0,\widehat{\Omega}}^2)+cX_2^2(\widehat{\Omega})(1+X_2(\widehat{\Omega})).\end{aligned} \tag{2.41}$$

For an interior subdomain the following estimate is obtained in the same way as (2.37):

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\widetilde{\Omega}}\left(\eta\widetilde{u}_\xi^2+\frac{p_{\sigma\eta}}{\eta}\widetilde{\eta}_{\sigma\xi}^2\right)A\,d\xi+c_0\|\widetilde{u}\|_{2,\widetilde{\Omega}}^2 \le \varepsilon(\|\widetilde{\eta}_{\sigma\xi}\|_{0,\widetilde{\Omega}}^2+\|\widetilde{u}_{\xi\xi}\|_{0,\widetilde{\Omega}}^2) + c(\|u\|_{1,\widetilde{\Omega}}^2+\|p_\sigma\|_{0,\Omega_t}^2)+cX_2^2(\widetilde{\Omega})(1+X_2(\widetilde{\Omega})), \tag{2.42}$$

where

$$X_2(\widetilde{\Omega})=|u|_{2,0,\widetilde{\Omega}}^2+|\eta_\sigma|_{2,0,\widetilde{\Omega}}^2+\int\limits_0^t\|u\|_{3,\widetilde{\Omega}}^2\,dt' \tag{2.43}$$

and A is the Jacobian of the transformation $x=x(\xi)$.

Finally, we have

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega}\eta u_\xi^2A\,d\xi\le c(\|u\|_{1,\widetilde{\Omega}}^2+\|u_t\|_{1,\widetilde{\Omega}}^2), \tag{2.44}$$

where we have used $(2.23)_1$.

Going back to the old variables ξ in estimates (2.37), (2.39), (2.41) and summing them and (2.42) over all neighbourhoods of the partition of unity, using (2.29) and (2.44), assuming that ε and d are sufficiently small and passing to the variables x we obtain (2.27). ■

LEMMA 2.6. *Let (v,ϱ_σ) be a sufficiently smooth solution of* (2.3). *Then*

$$\frac{1}{2}\frac{d}{dt}\int\limits_{\Omega_t}\left(\varrho v_{xt}^2+\frac{p_{\sigma\varrho}}{\varrho}\varrho_{xt}^2\right)dx+c_0(\|v_t\|_{2,\Omega_t}^2+\|\varrho_{\sigma t}\|_{1,\Omega_t}^2) \le c(\|v\|_{1,\Omega_t}^2+\|v_t\|_{1,\Omega_t}^2+\|v_{tt}\|_{1,\Omega_t}^2+\|\varrho_{\sigma t}\|_{0,\Omega_t}^2+\|p_\sigma\|_{0,\Omega_t}^2) + cX_2Y_2(1+X_2^2),$$

where X_2 is given by (2.15) *and Y_2 is given by* (2.22).

Proof. Differentiating problem (2.28) with respect to t we get the following elliptic problem:

$$\begin{aligned}
&\mu\nabla_u^2u_t+\nu\nabla_u\nabla_u\cdot u_t-p_{\sigma\eta}\nabla_u\eta_{\sigma t}=\eta_{\sigma t}u_t+\eta u_{tt}-\nu(\nabla_u\nabla_u)_{,t}\cdot u\\
&\quad-\mu(\nabla_u^2)_{,t}u+p_{\sigma\eta\eta}\eta_{\sigma t}\nabla_u\eta_\sigma+p_{\sigma\eta}(\nabla_u)_{,t}\eta_\sigma\equiv K_1 && \text{in } \Omega,\\
&\operatorname{div}_u u_t=\operatorname{div}_u u_t && \text{in } \Omega,\\
&\mathbb{T}_u(u_t,p_{\sigma t})\overline{n}_u=-(\mathbb{T}_u)_{,t}(u,p_\sigma)\overline{n}_u-\mathbb{T}_u(u,p_\sigma)(\overline{n}_u)_{,t}\equiv K_2 && \text{on } S.
\end{aligned}$$

By the Agmon–Douglis–Nirenberg theory (see [1]) we have the estimate

$$\|u_t\|_{2,\Omega}^2+\|\eta_{\sigma t}\|_{1,\Omega}^2\le c(\|K_1\|_{0,\Omega}^2+\|K_2\|_{1/2,S}^2+\|\operatorname{div}_u u_t\|_{1,\Omega}^2),$$

where

$$\|K_1\|_{0,\Omega}^2+\|K_2\|_{1/2,S}^2\le c(\|\eta_{\sigma\varsigma}\|_{0,\Omega}^2+\|u_{tt}\|_{0,\Omega}^2+\|p_\sigma\|_{0,\Omega}^2) + X_2(\Omega)Y_2(\Omega)(1+X_2^2(\Omega)),$$

with $X_2(\Omega)$ given by (2.30) and

$$Y_2(\Omega)=|u|^2_{3,1,\Omega}+\|\eta_\sigma\|^2_{2,\Omega}+\|\eta_{\sigma t}\|^2_{2,\Omega}+\|\eta_{\sigma tt}\|^2_{1,\Omega}. \tag{2.45}$$

The remaining part of the proof is analogous to that in Lemma 2.5. ∎

LEMMA 2.7. *Let (v,ϱ_σ) be a sufficiently smooth solution of* (2.3). *Then*

$$\begin{aligned}\frac12\frac{d}{dt}\int_{\Omega_t}\bigg(\varrho v^2_{xx}+\frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xx}\bigg)\,dx+c_0(\|v\|^2_{3,\Omega_t}+\|\varrho_{\sigma x}\|^2_{1,\Omega_t})\\ \le c(\|v\|^2_{2,\Omega_t}+\|v_t\|^2_{1,\Omega_t}+\|\varrho_{\sigma x}\|^2_{0,\Omega_t}+\|p_\sigma\|^2_{0,\Omega_t})\\ +\varepsilon\|v_t\|^2_{2,\Omega_t}+cX_2Y_2(1+X_2^2),\end{aligned} \tag{2.46}$$

where X_2 and Y_2 are given by (2.15) *and* (2.22), *respectively*, *and*

$$v^2_{xx}=\sum_{i,j,k=1}^3 v^2_{ix_jx_k},\qquad \varrho^2_{\sigma xx}=\sum_{j,k=1}^3\varrho^2_{\sigma x_jx_k}.$$

Proof. First, we consider problem (2.28). By the Agmon–Douglis–Nirenberg theory (see [1]) we have

$$\begin{aligned}\|u\|^2_{3,\Omega}+\|\eta_\sigma\|^2_{2,\Omega}\le c(\|u_t\|^2_{1,\Omega}+\|\operatorname{div}u\|^2_{2,\Omega})\\ +cX_2(\Omega)Y_2(\Omega)(1+X_2^2(\Omega)),\end{aligned} \tag{2.47}$$

where $X_2(\Omega)$ and $Y_2(\Omega)$ are given by (2.30) and (2.45), respectively. Thus, to obtain (2.46) we have to estimate $\|\operatorname{div}u\|^2_{2,\Omega}$ and $\frac12\frac{d}{dt}\int_{\Omega_t}(\varrho v^2_{xx}+\frac{p_{\sigma\varrho}}{\varrho}\varrho^2_{\sigma xx})\,dx$. To do this, consider first boundary subdomains. Differentiate $(2.26)_1$ twice with respect to τ, multiply the result by $\widetilde u_{\tau\tau}J$ and integrate over $\widehat\Omega$. Using the Korn inequality, the continuity equation $(2.26)_2$, and the boundary condition $(2.26)_3$ we get

$$\begin{aligned}\frac12\frac{d}{dt}\int_{\widehat\Omega}\bigg(\widehat\eta\widetilde u^2_{\tau\tau}+\frac{p_{\sigma\widehat\eta}}{\widehat\eta}\widetilde\eta^2_{\sigma\tau\tau}\bigg)J\,dz+c_0\|\widetilde u_{\tau\tau}\|^2_{1,\widehat\Omega}\\ \le\varepsilon(\|\widehat\eta_{\sigma\tau\tau}\|^2_{0,\widehat\Omega}+\|\widetilde u_{\tau\tau}\|^2_{1,\widehat\Omega})+c(\|\widehat u\|^2_{2,\widehat\Omega}+\|\widehat\eta_{\sigma z}\|^2_{0,\widehat\Omega})\\ +cX_2(\widehat\Omega)Y_2(\widehat\Omega)(1+X_2^2(\widehat\Omega)),\end{aligned} \tag{2.48}$$

where $X_2(\widehat\Omega)$ is given by (2.32) and

$$Y_2(\widehat\Omega)=|\widehat u|^2_{3,1,\widehat\Omega}+\|\widehat\eta_\sigma\|^2_{2,\widehat\Omega}+\|\widehat\eta_{\sigma t}\|^2_{2,\widehat\Omega}+\|\widehat\eta_{\sigma tt}\|^2_{1,\widehat\Omega}.$$

In the same way we obtain the following estimate in an interior subdomain:

$$\begin{aligned}\frac12\frac{d}{dt}\int_{\widetilde\Omega}\bigg(\eta\widetilde u^2_{\xi\xi}+\frac{p_{\sigma\eta}}{\eta}\widetilde\eta^2_{\sigma\xi\xi}\bigg)A\,d\xi+c_0\|\widetilde u\|^2_{3,\widetilde\Omega}\\ \le\varepsilon(\|\widetilde\eta_{\sigma\xi\xi}\|^2_{0,\widetilde\Omega}+\|\widetilde u_{\xi\xi\xi}\|^2_{0,\widetilde\Omega})\\ +c(\|u\|^2_{2,\widetilde\Omega}+\|\eta_{\sigma\xi}\|^2_{0,\widehat\Omega}+\|p_\sigma\|^2_{0,\widetilde\Omega})+cX_2(\widetilde\Omega)Y_2(\widetilde\Omega)(1+X_2^2(\widetilde\Omega)),\end{aligned} \tag{2.49}$$

where $X_2(\widetilde{\Omega})$ is given by (2.43) and

$$Y_2(\widetilde{\Omega}) = |u|^2_{3,1,\widetilde{\Omega}} + \|\eta_\sigma\|^2_{2,\widetilde{\Omega}} + \|\eta_{\sigma t}\|^2_{2,\widetilde{\Omega}} + \|\eta_{\sigma tt}\|^2_{1,\widetilde{\Omega}}.$$

Now, differentiate the third component of (2.38) in τ, multiply the result by $\widetilde{\eta}_{\sigma n\tau} J$ and integrate over $\widehat{\Omega}$ to get

$$\begin{aligned}(2.50)\qquad & \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \frac{p_{\sigma\widehat{\eta}}}{\widehat{\eta}} \widetilde{\eta}^2_{\sigma n\tau} J\,dz + \int\limits_{\widehat{\Omega}} p^2_{\sigma\widehat{\eta}} \widetilde{\eta}^2_{\sigma n\tau} J\,dz \\ &\le \varepsilon \|\widetilde{\eta}_{\sigma n\tau}\|^2_{0,\widehat{\Omega}} + c(\|\widehat{u}\|^2_{2,\widehat{\Omega}} + \|\widehat{u}_t\|^2_{1,\widehat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\widehat{\Omega}} + \|p_\sigma\|^2_{0,\widehat{\Omega}}) \\ &\quad + cd\|\widetilde{u}\|^2_{3,\widehat{\Omega}} + c\|\widetilde{u}_{z\tau\tau}\|^2_{0,\widehat{\Omega}} + cX_2(\widehat{\Omega})Y_2(\widehat{\Omega})(1 + X_2^2(\widehat{\Omega})),\end{aligned}$$

where d is from formula (2.25).

In the same way we obtain

$$\begin{aligned}(2.51)\qquad & \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \frac{p_{\sigma\widehat{\eta}}}{\widehat{\eta}} \widetilde{\eta}^2_{\sigma nn} J\,dz + \int\limits_{\widehat{\Omega}} p^2_{\sigma\widehat{\eta}} \widetilde{\eta}^2_{\sigma nn} J\,dz \\ &\le \varepsilon \|\widetilde{\eta}_{\sigma nn}\|^2_{0,\widehat{\Omega}} + c(\|\widehat{u}\|^2_{2,\widehat{\Omega}} + \|\widehat{u}_t\|^2_{1,\widehat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\widehat{\Omega}} + \|p_\sigma\|^2_{0,\widehat{\Omega}}) \\ &\quad + cd\|\widetilde{u}\|^2_{3,\widehat{\Omega}} + c\|\widetilde{u}_{zn\tau}\|^2_{0,\widehat{\Omega}} + cX_2(\widehat{\Omega})Y_2(\widehat{\Omega})(1 + X_2^2(\widehat{\Omega})).\end{aligned}$$

Next, differentiating the third component of (2.40) in τ, multiplying by $\widetilde{u}_{3nn\tau} J$ and integrating over $\widehat{\Omega}$ we have

$$\begin{aligned}(2.52)\qquad & \frac{1}{2}\frac{d}{dt}\int\limits_{\widehat{\Omega}} \widehat{\eta}\widetilde{u}^2_{3n\tau} J\,dz + c_0\|\widetilde{u}_{3nn\tau}\|^2_{0,\widehat{\Omega}} \\ &\le \varepsilon\|\widetilde{u}_{3nn\tau}\|^2_{0,\widehat{\Omega}} + \varepsilon\|\widetilde{u}_t\|^2_{2,\widehat{\Omega}} + c(\|\widetilde{u}\|^2_{2,\widehat{\Omega}} + \|\widetilde{u}_t\|^2_{1,\widetilde{\Omega}} + \|\widetilde{u}_{z\tau\tau}\|^2_{0,\widehat{\Omega}} \\ &\quad + \|\widehat{\eta}_{\sigma n\tau}\|^2_{0,\widehat{\Omega}} + \|\widehat{\eta}_{\sigma z}\|^2_{0,\widehat{\Omega}} + \|p_\sigma\|^2_{0,\widehat{\Omega}}) + cd\|\widehat{u}\|^2_{3,\widehat{\Omega}} \\ &\quad + cX_2(\widehat{\Omega})Y_2(\widehat{\Omega})(1 + X_2^2(\widehat{\Omega})).\end{aligned}$$

In order to estimate $\|(\operatorname{div}\widetilde{u})_{,nn}\|^2_{0,\widehat{\Omega}}$ rewrite equation $(2.26)_1$ in the form

$$\begin{aligned}(2.53)\qquad (\nu+\mu)\nabla_{z_i}\operatorname{div}\widetilde{u} &= -\mu(\Delta\widetilde{u}_i - \nabla_{z_i}\operatorname{div}\widetilde{u}) + \widehat{\eta}\widetilde{u}_{it} - k_{3i} \\ &\quad + (\mu\Delta\widetilde{u}_i + \nu\nabla_{z_i}\operatorname{div}\widetilde{u} - \mu\widehat{\nabla}^2\widetilde{u}_i - \nu\widehat{\nabla}_i\widehat{\nabla}\cdot\widetilde{u}) \\ &\quad + p_1\widehat{\eta}_\sigma\widehat{\nabla}_i\widehat{\zeta} + \widehat{\zeta}p_{\sigma\widehat{\eta}}\widehat{\nabla}_i\widehat{\eta}_\sigma, \qquad i = 1,2,3.\end{aligned}$$

Differentiating the third component of (2.53) with respect to n gives

$$\begin{aligned}(2.54)\ \|(\operatorname{div}\widetilde{u})_{,nn}\|^2_{0,\widehat{\Omega}} &\le cd\|\widetilde{u}_{nnn}\|^2_{0,\widehat{\Omega}} + c(\|\widetilde{u}_\tau\|^2_{2,\widehat{\Omega}} + \|\widehat{u}\|^2_{2,\widehat{\Omega}} + \|\widetilde{u}_t\|^2_{1,\widehat{\Omega}} \\ &\quad + \|\widehat{\eta}_{\sigma n}\|^2_{1,\widehat{\Omega}} + \|p_\sigma\|^2_{0,\widehat{\Omega}}) + cX_2(\widehat{\Omega})Y_2(\widehat{\Omega}).\end{aligned}$$

To obtain an estimate for $\|\widetilde{u}_\tau\|^2_{2,\widehat{\Omega}}$ consider the following elliptic problem:

$$
(2.55)\quad \begin{aligned}
&\mu\widehat{\nabla}^2\widetilde{u}+\nu\widehat{\nabla}\widehat{\nabla}\cdot\widetilde{u}-p_{\sigma\widehat{\eta}}\widetilde{\eta}_\sigma=\widehat{\eta}\widetilde{u}_t+(p_1-p_{\sigma\widehat{\eta}})\widehat{\eta}_\sigma\widehat{\nabla}\widehat{\zeta}\\
&\qquad\qquad+\widehat{\nabla}\cdot\widehat{\mathbb{B}}(\widehat{u},\widehat{\zeta})+\widehat{\mathbb{T}}(\widehat{u},p_\sigma)\cdot\widehat{\nabla}\widehat{\zeta},\\
&\widehat{\nabla}\cdot\widetilde{u}=\widehat{\nabla}\cdot\widetilde{u},\\
&\widehat{\mathbb{T}}(\widetilde{u},p_\sigma)\widehat{n}=k_5,
\end{aligned}
$$

where $\widehat{\nabla}\cdot\widehat{\mathbb{B}}(\widehat{u},\widehat{\zeta})=\{\widehat{\nabla}_j\widehat{\mathbb{B}}_{ij}(\widehat{u},\widehat{\zeta})\}_{i=1,2,3}$, $\widehat{\mathbb{T}}(\widehat{u},p_\sigma)\cdot\widehat{\nabla}\widehat{\zeta}=\{\widehat{T}_{ij}(\widehat{u},p_\sigma)\widehat{\nabla}_j\widehat{\zeta}\}_{i=1,2,3}$.

Differentiating (2.55) with respect to τ and next using the Agmon–Douglis–Nirenberg theory we get

$$
(2.56)\quad \begin{aligned}
&\|\widetilde{u}_\tau\|^2_{2,\widehat{\Omega}}+\|\widetilde{\eta}_{\sigma\tau}\|^2_{1,\widehat{\Omega}}\\
&\le c(\|\widetilde{u}_{\tau\tau}\|^2_{1,\widehat{\Omega}}+\|\widetilde{u}_{3nn\tau}\|^2_{0,\widehat{\Omega}}+\|\widehat{u}\|^2_{2,\widehat{\Omega}}+\|\widetilde{u}_t\|^2_{1,\widehat{\Omega}}\\
&\quad+\|\widehat{\eta}_{\sigma z}\|^2_{0,\widehat{\Omega}}+\|p_\sigma\|^2_{0,\widehat{\Omega}})+cX_2(\widehat{\Omega})Y_2(\widehat{\Omega})(1+X_2(\widehat{\Omega})).
\end{aligned}
$$

Finally, we have

$$
(2.57)\qquad \frac12\frac{d}{dt}\int_\Omega \eta u^2_{\xi\xi}A\,d\xi\le c\|u\|^2_{2,\Omega}+\varepsilon\|u_t\|^2_{2,\Omega}.
$$

Going back to the old variables ξ in estimates (2.48), (2.50)–(2.52), (2.54), (2.56) and summing them and (2.49) over all neighbourhoods of the partition of unity, using (2.47) and (2.57), assuming that ε and d are sufficiently small and passing to the variables x we obtain (2.46). ∎

Lemmas 2.1–2.7 and the estimates

$$
\|\varrho_{\sigma tt}\|^2_{1,\Omega_t}\le c\|v_t\|^2_{2,\Omega_t}+c(\|\varrho_{\sigma t}\|^2_{2,\Omega_t}\|v\|^2_{2,\Omega_t}+\|\varrho_\sigma\|^2_{2,\Omega_t}\|v_t\|^2_{2,\Omega_t})
$$

and

$$
\|\varrho_{\sigma t}\|^2_{2,\Omega_t}\le c\|v\|^2_{3,\Omega_t}+cX_2Y_2(1+X_2)
$$

(which follow from equations $(2.3)_2$ and $(2.23)_2$, respectively) imply the following theorem.

THEOREM 2.8. *Let $\nu>\frac13\mu>0$ and let relations* (2.6) *and* (2.7) *be satisfied. Then for a sufficiently smooth solution (v,ϱ_σ) of problem* (2.3) *we have*

$$
(2.58)\quad \begin{aligned}
&\frac{d\overline{\phi}}{dt}+c_0\Phi\le c_1\Big(\phi+\int_0^t\|v\|^2_{3,\Omega_{t'}}\,dt'\Big)\\
&\qquad\cdot\Big[1+\Big(\phi+\int_0^t\|v\|^2_{3,\Omega_{t'}}\,dt'\Big)^2\Big]\Phi+c_2\Psi\quad \textit{for } t\le T,
\end{aligned}
$$

where

$$(2.59)\quad \begin{aligned}\overline{\phi}(t) &= \int\limits_{\Omega_t} \varrho \sum_{0\le|\alpha|+i\le 2} |D_x^\alpha \partial_t^i v|^2\,dx + \int\limits_{\Omega_t} \frac{p_1}{\varrho}\varrho_\sigma^2\,dx \\ &\quad + \int\limits_{\Omega_t} \frac{p_{\sigma\varrho}}{\varrho} \sum_{1\le|\alpha|+i\le 2} |D_x^\alpha \partial_t^i \varrho_\sigma|^2\,dx, \\ \phi(t) &= |v|^2_{2,0,\Omega_t} + |\varrho_\sigma|^2_{2,0,\Omega_t}, \\ \Phi(t) &= |v|^2_{3,1,\Omega_t} + \|\varrho_\sigma\|^2_{2,\Omega_t} + \|\varrho_{\sigma t}\|^2_{2,\Omega_t} + \|\varrho_{\sigma tt}\|^2_{1,\Omega_t}, \\ \Psi(t) &= \|p_\sigma\|^2_{0,\Omega_t},\end{aligned}$$

c_i $(i = 1, 2)$ *are positive constants depending on* ϱ_*, ϱ^*, μ, ν, $\int_0^t \|v\|^2_{3,\Omega_{t'}}\,dt'$, $\|S\|_{5/2}, T$ *and on the constants of imbedding theorems and Korn inequalities*; $c_0 < 1$ *is a positive constant depending on* μ *and* ν; *and* ϱ_σ *and* p_σ *are given by* (2.2).

3. Global existence. Assume (2.1) and rewrite problem (1.1) in Lagrangian coordinates as follows (see problem (2.23)):

$$(3.1)\quad \begin{aligned} &\eta u_t - \mu\nabla_u^2 u - \nu\nabla_u\nabla_u\cdot u + \nabla p = 0 && \text{in } \Omega^T,\\ &\eta_t + \eta\nabla_u\cdot u = 0 && \text{in } \Omega^T,\\ &\mathbb{T}_u(u,p)\overline{n}_u = -p_0\overline{n}_u && \text{on } S^T,\\ &u|_{t=0} = v_0, \quad \eta|_{t=0} = \varrho_0, && \text{in } \Omega.\end{aligned}$$

The local existence of a solution of problem (3.1) can be proved by the method of successive approximations (see [15]), taking as a zero step function the solution $u^0 \in \mathcal{A}_{T,\Omega}$ ($\mathcal{A}_{T,\Omega}$ is given by (1.6)) of the following parabolic problem:

$$(3.2)\quad \begin{aligned} &u^0_t - \operatorname{div}\mathbb{D}(u^0) = 0 && \text{in } \Omega^T,\\ &\mathbb{D}(u^0)\overline{n}_0 = (p(\varrho_0) - p_0)\overline{n}_0 && \text{on } S^T,\\ &u^0|_{t=0} = v_0 && \text{in } \Omega,\end{aligned}$$

where $\mathbb{D}(u^0) = \{\mu(u^0_{i\xi_j} + u^0_{j\xi_i}) + (\nu-\mu)\delta_{ij}\operatorname{div} u^0\}_{i,j=1,2,3}$ and $\overline{n}_0$ is the unit outward vector normal to S.

Assume that

$$(3.3)\quad l > 0 \text{ is a constant such that } \varrho_e - l > 0 \text{ and } \varrho_1 < \varrho_0 < \varrho_2,$$

where $\varrho_1 = \varrho_e - l$, $\varrho_2 = \varrho_e + l$, and ϱ_e is given in Definition 1.1.

The function u^0 satisfies the estimate (see [15], estimate (4.3))

$$(3.4)\quad \begin{aligned} &\|u^0\|^2_{\mathcal{A}_{T,\Omega}}\\ &\le C_1(T)(\|(p(\varrho_0) - p_0)\overline{n}_0\|^2_{3/2,S} + \|v_0\|^2_{2,\Omega} + \|u^0_t(0)\|^2_{1,\Omega} + \|u^0_{tt}(0)\|^2_{0,\Omega})\\ &< C_1(T)(\widetilde{c}\overline{\phi}(0) + \|v_0\|^2_{2,\Omega} + \|u^0_t(0)\|^2_{1,\Omega} + \|u^0_{tt}(0)\|^2_{0,\Omega}) \equiv A_0,\end{aligned}$$

where $C_1(T)$ is a positive constant; $\widetilde{c} > 0$ is a constant depending on ϱ_1, ϱ_2 and on the volume and shape of Ω; $\overline{\phi}$ is defined in (2.59); $u_t^0(0)$, $u_{tt}^0(0)$ are calculated from (3.2); and to obtain A_0 in (3.4) we have used (2.4).

Next, define

$$
\begin{aligned}
H_0 &= \frac{1}{\varrho_1} + \|\varrho_0\|^2_{2,\Omega} + \|v_0\|^2_{2,\Omega} + \|u_t(0)\|^2_{1,\Omega} + \|u_{tt}(0)\|^2_{0,\Omega} \\
&\le \frac{1}{\varrho_1} + \overline{c}\overline{\phi}(0) + |\Omega|\varrho_e^2 < \widetilde{H}_0,
\end{aligned} \tag{3.5}
$$

where $u_t(0)$, $u_{tt}(0)$ are calculated from $(3.1)_1$; $\overline{c} > 0$ is a constant depending on ϱ_1, ϱ_2; and $\widetilde{H}_0 > 0$ is a constant. Then the following theorem holds.

THEOREM 3.1. (see [15, Theorem 4.2]). *Assume that* $\varrho_0, v_0 \in H^2(\Omega)$, $\varrho_0 > 0$, $u_t(0), u_t^0(0) \in H^1(\Omega)$, $u_{tt}(0), u_{tt}^0(0) \in L_2(\Omega)$ (*where* $u_t(0)$, $u_{tt}(0)$ *are calculated from* (3.1)), $S \in H^{5/2}$, *and* $p \in C^3(\mathbb{R}^2_+)$. *Let assumption* (3.3) *and the following compatibility conditions be satisfied*:

$$
\mathbb{D}(v_0)\overline{n}_0 = (p(\varrho_0) - p_0)\overline{n}_0 \quad \text{on } S. \tag{3.6}
$$

Assume that $A_0 < A$, *where* $A > 0$ *is a constant depending also on* $\widetilde{H}_0$ (*i.e. there exists a positive continuous increasing function* $F = F(\widetilde{H}_0)$ *satisfying* $F(\widetilde{H}_0) < A$). *Then there exists* $T_* > 0$ (*depending on* A) *such that for* $T \le T_*$ *there exists a unique solution of* (1.1) *such that* $u \in \mathcal{A}_{T,\Omega}$, $\eta \in \mathcal{B}_{T,\Omega}$ *and*

$$
\|u\|^2_{\mathcal{A}_{T,\Omega}} \le A, \tag{3.7}
$$

$$
\|\eta\|^2_{\mathcal{B}_{T,\Omega}} \le \psi_1(A), \tag{3.8}
$$

where ψ_1 *is a positive continuous increasing function of* A ($\mathcal{A}_{T,\Omega}$ *and* $\mathcal{B}_{T,\Omega}$ *are given by* (1.6) *and* (1.5), *respectively*).

Now, we shall derive an estimate for the local solution (u, η_σ) of problem (2.23). Using (3.7) and (3.8) and the interpolation inequality we have

$$
\begin{aligned}
&\|\nabla p_\sigma\|^2_{1,2,2,\Omega_T} + \|\nabla p_{\sigma t}\|^2_{0,\Omega_T} + \varepsilon_*\|\nabla p_{\sigma tt}\|^2_{0,\Omega_T} \\
&\qquad + \sup_t \|\nabla p_\sigma\|^2_{0,\Omega} + \|p_\sigma \overline{n}_u\|^2_{3/2,2,2,S^T} + \|(p_\sigma \overline{n}_u)_{,t}\|^2_{1/2,2,2,S^T} \\
&\qquad + \varepsilon_*\|(p_\sigma \overline{n}_u)_{,tt}\|^2_{0,S^T} + \sup_t \|p_\sigma \overline{n}_u\|^2_{0,S} \\
&\quad \le \psi'(A,T)(\|\varrho_{\sigma 0}\|^2_{2,\Omega} + \|v_0\|^2_{2,\Omega} + \|u_t(0)\|^2_{1,\Omega}) \\
&\qquad + (\varepsilon + T)\psi''(A,T)\|u\|^2_{\mathcal{A}_{T,\Omega}},
\end{aligned} \tag{3.9}
$$

where ψ' and ψ'' are positive continuous increasing functions of their arguments, and ε_*, $\varepsilon \in (0,1)$ are sufficiently small constants.

By estimate (3.9), Lemmas 3.5 and 2.3 of [15] and by Theorem 3.1 the local solution (u,η_σ) of problem (2.23) satisfies, for sufficiently small ε and T,

$$\|u\|^2_{\mathcal{A}_{T,\Omega}}+\|\eta_\sigma\|^2_{\mathcal{B}_{T,\Omega}} \le \psi_2(A,T)(\|\varrho_{\sigma 0}\|^2_{2,\Omega}+\|v_0\|^2_{2,\Omega}+\|u_t(0)\|^2_{1,\Omega}+\|u_{tt}(0)\|^2_{0,\Omega}), \tag{3.10}$$

where ψ_2 is a positive continuous function.

Now, let $\overline{\phi}(t)$, $\phi(t)$ and $\Phi(t)$ be defined by (2.59). Introduce the spaces

$$\mathfrak{N}(t)=\{(v,\varrho_\sigma):\phi(t)<\infty\},$$
$$\mathfrak{M}(t)=\Big\{(v,\varrho_\sigma):\phi(t)+\int_0^t\Phi(t')\,dt'<\infty\Big\}.$$

Notice that $(v,\varrho_\sigma)\in\mathfrak{N}(t)$ iff $\overline{\phi}(t)<\infty$, and $(v,\varrho_\sigma)\in\mathfrak{M}(t)$ iff $\overline{\phi}(t)+\int_0^t\Phi(t')\,dt'\le\infty$. Moreover,

$$c'\phi(t)\le\overline{\phi}(t)\le c''\phi(t), \tag{3.11}$$

where $c',c''>0$ are constants depending on ϱ_*, ϱ^* given by (2.5).

From inequality (3.10) and from the definitions of $\mathfrak{N}(t)$ and $\mathfrak{M}(t)$ it follows that the local solution satisfies the estimate

$$\phi(t)+\int_0^t\Phi(t')\,dt'\le c_3\overline{\phi}(0), \tag{3.12}$$

where $c_3>0$ is a constant depending on the same quantities as c_1 and c_2 from Theorem 2.8.

Hence we obtain the following lemma.

LEMMA 3.2. *Let* $(v,\varrho_\sigma)\in\mathfrak{N}(0)$, $S\in H^{5/2}$, $u^0_t(0)\in H^1(\Omega)$, $u^0_{tt}(0)\in L_2(\Omega)$ *(*u^0 *is the solution of problem* (3.2)*), and* $p\in C^3(\mathbb{R}^2_+)$. *Let assumption* (3.3) *and the compatibility condition* (3.6) *be satisfied. Moreover, assume*

$$\overline{\phi}(0)\le\alpha, \tag{3.13}$$

where $\alpha>0$ *is sufficiently small. Then the local solution* (v,ϱ) *of problem* (1.1) *is such that* $(v,\varrho_\sigma)\in\mathfrak{M}(t)$ *for* $t\le T$, *where* $T>0$ *is the time of local existence, and the following estimate holds*:

$$\phi(t)+\int_0^t\Phi(t')\,dt'\le c_3\alpha,$$

where $c_3>0$ *is a constant depending on the same quantities as* c_1 *and* c_2 *from Theorem* 2.8.

Next, we prove

LEMMA 3.3. *Let the assumptions of Lemma* 3.2 *be satisfied. Then there exist constants* $\mu_1>1$ *and* $\mu_2>0$ *(depending on the same quantities as* c_1

and c_2 *from* (2.58)) *such that*

$$\overline{\phi}(t) \le \mu_1 \overline{\phi}(0) e^{-\mu_2 t} \quad \textit{for } t \le T, \tag{3.14}$$

where $T > 0$ *is the time of local existence.*

Proof. Consider inequality (2.58) and assume that α from (3.13) is so small that

$$c_1\Big(\phi + \int_0^t \|v\|^2_{3,\Omega_{t'}}\, dt'\Big)\Big[1 + \Big(\phi + \int_0^t \|v\|^2_{3,\Omega_{t'}}\, dt'\Big)^2\Big] < \frac{c_0}{4}. \tag{3.15}$$

Then inequality (2.58) implies

$$\frac{d\overline{\phi}}{dt} + \frac{3}{4} c_0 \Phi < c_2 \|p_\sigma\|^2_{0,\Omega_t}. \tag{3.16}$$

Applying the same argument as in the proof of Lemma 6.2 of [17] yields

$$\|p_\sigma\|^2_{0,\Omega_t} \le \varepsilon(\|p_{\sigma x}\|^2_{0,\Omega_t} + \|v_{xx}\|^2_{0,\Omega_t}) + c(\varepsilon)(\|v\|^2_{0,\Omega_t} + \|v_t\|^2_{0,\Omega_t}). \tag{3.17}$$

Since $\|p_{\sigma x}\|^2_{0,\Omega_t} \le c_4 \|\varrho_{\sigma x}\|^2_{0,\Omega_t}$, inequalities (3.16) and (3.17) imply, for sufficiently small ε,

$$\frac{d\overline{\phi}}{dt} + \frac{3}{4} c_0 \Phi < c_5(\|v\|^2_{0,\Omega_t} + \|v_t\|^2_{0,\Omega_t}). \tag{3.18}$$

Now, multiplying (2.21) by a constant c_6 so large that $c_0c_6 - c_5 > 0$ and $c_6 > 1$, adding to (3.18) and using Lemma 3.2 we obtain

$$\begin{aligned}\frac{d}{dt}(\overline{\phi} + c_6 J) + \frac{3}{4} c_0 \Phi \\ + (c_0c_6 - c_5)(\|v\|^2_{1,\Omega_t} + \|v_t\|^2_{1,\Omega_t} + \|\varrho_{\sigma t}\|^2_{0,\Omega_t}) < c_7 \alpha \phi,\end{aligned} \tag{3.19}$$

where

$$J = \frac{1}{2} \int_{\Omega_t} \Big[\varrho(v^2 + v_t^2) + \frac{p_1}{\varrho} \varrho_\sigma^2 + \frac{p_{\sigma\varrho}}{\varrho} \varrho_{\sigma t}^2\Big]\, dx.$$

Since $\overline{\phi}/c'' \le \phi \le \Phi$ and $\overline{\phi} \ge J$ for sufficiently small α (so small that $c_7\alpha < \frac{1}{4} c_0$), inequality (3.19) implies

$$\frac{d}{dt}(\overline{\phi} + c_6 J) + c_8(\overline{\phi} + c_6 J) < 0, \tag{3.20}$$

where $c_8 = c_0/(4c''c_6)$ ($c'' > 0$ is the constant from (3.11)).

Inequality (3.20) yields (3.14) with $\mu_1 = c_6 + 1$ and $\mu_2 = c_8$. ∎

By using Lemma 3.3 we prove

LEMMA 3.4. *Let the assumptions of Lemma* 3.2 *be satisfied. Moreover*, *assume*

$$C_0 \equiv \|v_0\|^2_{0,\Omega} + \|\varrho_{\sigma 0}\|^2_{0,\Omega} \le \delta, \tag{3.21}$$

where $\varrho_{\sigma 0} = \varrho_0 - \varrho_e$. *Then*

$$\|v\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t} \le c_9\alpha^2 + c_{10}c_{11}\delta \quad \textit{for } t \le T, \tag{3.22}$$

where $c_9 = \frac{c_{11}\mu_1^2}{c'\mu_2} c_3 c(1 + c_3\alpha)$; c' *is the constant from inequality* (3.11); α *and* c_3 *are the constants from Lemma* 3.2; μ_1, μ_2 *are the constants from Lemma* 3.3; c *is the constant from Lemma* 2.1 *and* $c_{10}, c_{11} > 0$ *are constants depending on* ϱ_*, ϱ^* *such that*

$$\begin{aligned}\frac{1}{c_{11}}(\|v\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t}) &\le \frac{1}{2}\int\limits_{\Omega_t}\left(\varrho v^2 + \frac{p_1}{\varrho}\varrho_\sigma^2\right)dx \\ &\le c_{10}(\|v\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t}) \quad \textit{for } t \le T;\end{aligned}$$

and $T > 0$ *is the time of local existence. Moreover*,

$$\|p_\sigma\|^2_{0,\Omega_t} \le c_{12}(c_9\alpha^2 + c_{10}c_{11}\delta), \tag{3.23}$$

where $c_{12} > 0$ *is a constant depending on* p, ϱ_*, ϱ^*.

Proof. Integrating (2.8) with respect to t over $(0,t)$ $(t \le T)$ we get

$$\begin{aligned}&\|v\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t} \\ &\quad \le c_{11}c \sup_{0\le t'\le t}\phi(t')\int\limits_0^t \phi(t')\,dt'\,(1 + \sup_{0\le t'\le t}\phi(t')) + c_{10}c_{11}C_0.\end{aligned} \tag{3.24}$$

Using Lemmas 3.2–3.3 and assumption (3.21) we obtain

$$\begin{aligned}\|v\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t} &\le \frac{c_{11}c\mu_1}{c'}c_3\alpha^2(1 + c_3\alpha)\int\limits_0^t e^{-\mu_2 t'}\,dt' + c_{10}c_{11}C_0 \\ &\le c_9\alpha^2 + c_{10}c_{11}\delta.\end{aligned} \tag{3.25}$$

Estimate (3.23) follows from (3.22) and (2.4). ■

REMARK 3.5. Estimate (3.12) and assumption (3.13) yield

$$\begin{aligned}\Big|\int\limits_0^t u(\xi,t')\,dt'\Big| &< c_{13}T^{1/2}\Big(\int\limits_0^T \|u\|^2_{2,\Omega}\,dt'\Big)^{1/2} \\ &\le c_{13}\psi_3(A,T)T^{1/2}\alpha^{1/2} \equiv c_{14}T^{1/2}\alpha^{1/2},\end{aligned} \tag{3.26}$$

where ψ_3 is a positive continuous function; $c_{13} > 0$ is a constant from the imbedding theorem depending on Ω. Hence, relation (1.3) implies that both the shape and the volume of Ω_t do not change much for $t \le T$ and the constants c_i $(i = 1,\dots,12)$, μ_i $(i = 1,2)$ (from Lemma 3.3) and c (from Lemma 3.4) can be chosen independent of time for $t \le T$.

REMARK 3.6. Under assumption (2.1) one can prove the following momentum conservation law (see [18]):

$$\frac{d}{dt}\int_{\Omega_t} \varrho v\cdot\eta\,dx = 0, \tag{3.27}$$

where $\eta = a + b\times x$ and a, b are arbitrary constant vectors. Moreover,

$$\frac{d}{dt}\int_{\Omega_t} \varrho x\,dx = \int_{\Omega_t} \varrho v\,dx. \tag{3.28}$$

Assuming

$$\int_{\Omega} \varrho_0 v_0\cdot\eta\,d\xi = 0, \qquad \int_{\Omega} \varrho_0\xi\,d\xi = 0, \tag{3.29}$$

in view of (3.27) and (3.28) we get (2.6) and (2.7), respectively. Condition (2.6) guarantees that the barycentre of Ω_t coincides with the origin of coordinates.

Now, we can prove

LEMMA 3.7. *Let the assumptions of Lemma* 3.2 *and estimate* (3.22) *be satisfied. Then*

$$\overline{\phi}(t)\le\alpha \quad \textit{for } t\le T, \tag{3.30}$$

where α is sufficiently small (*so that* (3.15) *and* (3.32) *are satisfied*), *and $T>0$ is the time of local existence.*

Proof. For α so small that (3.15) is satisfied, the differential inequality (2.58) implies (3.16). Hence by estimate (3.23) of Lemma 3.4 we have

$$\frac{d\overline{\phi}}{dt} + \frac{3}{4}c_0\Phi < c_2c_{12}(c_9\alpha^2 + c_{10}c_{11}\delta).$$

Therefore, since $\overline{\phi}/c''\le\Phi$ (where c'' is the constant from inequality (3.11)) we obtain

$$\frac{d\overline{\phi}}{dt} + \frac{3}{4}\frac{c_0}{c''}\overline{\phi} < c_2c_{12}(c_9\alpha^2 + c_{10}c_{11}\delta). \tag{3.31}$$

Now, assume that $t_* = \inf\{t\in[0,T] : \overline{\phi}(t) > \alpha\}$ and consider (3.31) in the interval $(0,t_*]$. From the definition of t_* we have $\overline{\phi}(t_*)=\alpha$. Therefore (3.31) yields

$$\frac{d\overline{\phi}}{dt}(t_*) < -\frac{3}{4}\frac{c_0}{c''}\alpha + c_2c_{12}(c_9\alpha^2 + c_{10}c_{11}\delta).$$

Let α and δ be so small that

$$c_2c_{12}(c_9\alpha^2 + c_{10}c_{11}\delta) < \frac{3}{4}\frac{c_0}{c''}\alpha. \tag{3.32}$$

Then $(d\overline{\phi}/dt)(t_*)<0$, a contradiction. Therefore, (3.30) holds. ∎

Lemma 3.7 suggests that the solution can be continued to the interval $[T, 2T]$. However, to do this we also need the analogous lemma for the solution of (3.2), to have the sum on the right-hand side of (3.4) with initial condition at T estimated by A.

Set

$$\phi_1(t) = |u^0(t)|^2_{2,0,\Omega}, \quad \Phi_1(t) = |u^0(t)|^2_{3,1,\Omega} - \|u^0(t)\|^2_{3,\Omega},$$

where u^0 is the solution of (3.2).

LEMMA 3.8. *Let the assumptions of Lemma* 3.7 *and* (3.21) *be satisfied. Moreover, assume that* $\phi_1(0) \le \alpha_1$, *where* $\alpha_1 > 0$ *is a constant. Then if the constants* δ *from Lemma* 3.4 *and* α *are sufficiently small we have*

$$\phi_1(t) \le \alpha_1 \quad \text{for } t \le T. \tag{3.33}$$

Proof. First, we shall obtain a differential inequality similar to (2.58). Multiplying $(3.2)_1$ by u^0, integrating over Ω and using the boundary condition $(3.2)_2$ and (2.4) (where $p_1 = p_1(\varrho_0)$) we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0)^2\,d\xi + \frac{\mu}{2}E_\Omega(u^0) + \int_S p_1\varrho_{\sigma 0}\overline{n}_0 u^0\,d\xi_s = 0, \tag{3.34}$$

where $E_\Omega(u^0) = \int_\Omega \sum_{i,j=1}^3 (u^0_{ix_j} + u^0_{jx_i})^2\,d\xi$.

In view of assumptions (3.29), Lemma 5.2 of [14] and the interpolation inequality, equality (3.34) yields

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0)^2\,d\xi + c_0\|u^0\|^2_{1,\Omega} \\ &\quad \le c\|\varrho_{\sigma 0}\|^2_{0,\Omega}\|u^0\|^2_{0,\Omega} + \varepsilon\|\varrho_{\sigma 0}\|^2_{1,\Omega} + c(\varepsilon)\|\varrho_{\sigma 0}\|^2_{0,\Omega}, \quad \text{where } \varepsilon \in (0,1). \end{aligned} \tag{3.35}$$

Next, differentiating $(3.2)_1$ with respect to t, multiplying by u^0_t, integrating over Ω and using the Korn inequality we get

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0_t)^2\,d\xi + c_0\|u^0_t\|^2_{1,\Omega} \le c\|u^0_t\|^2_{0,\Omega} \tag{3.36}$$

and from $(3.2)_1$ we obtain

$$\|u^0_t\|^2_{0,\Omega} \le \varepsilon\|u^0_t\|^2_{1,\Omega} + \varepsilon\|\varrho_{\sigma 0}\|^2_{1,\Omega} + c(\varepsilon)\|\varrho_{\sigma 0}\|^2_{0,\Omega} + c\|u^0\|^2_{1,\Omega}. \tag{3.37}$$

By (3.36) and (3.37) we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0_t)^2\,d\xi + c_0\|u^0_t\|^2_{1,\Omega} \le \varepsilon\|\varrho_{\sigma 0}\|^2_{1,\Omega} + c(\varepsilon)\|\varrho_{\sigma 0}\|^2_{0,\Omega} + c\|u^0\|^2_{1,\Omega}. \tag{3.38}$$

In the same way we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0_{tt})^2\,d\xi + c_0\|u_{tt}\|^2_{1,\Omega} \le c\|u^0_t\|^2_{1,\Omega}. \tag{3.39}$$

Now, consider the elliptic problem

$$-\operatorname{div}\mathbb{D}(u^0) = -u^0_t,$$
$$\mathbb{D}(u^0)\overline{n}_0 = (p(\varrho_0)-p_0)\overline{n}_0.$$

By the Agmon–Douglis–Nirenberg theory (see [1])

$$\|u^0\|^2_{2,\Omega} \le c(\|u^0_t\|^2_{0,\Omega} + \|u^0\|^2_{0,\Omega}) + \varepsilon\|\varrho_{\sigma 0}\|^2_{2,\Omega} + c(\varepsilon)\|\varrho_{\sigma 0}\|^2_{0,\Omega}. \tag{3.40}$$

Moreover,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0_\xi)^2\,d\xi \le c(\|u^0\|^2_{1,\Omega} + \|u^0_t\|^2_{1,\Omega}). \tag{3.41}$$

Using the same argument we get the estimates

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0_{t\xi})^2\,d\xi + c_0\|u^0_t\|^2_{2,\Omega} \le c(\|u^0_t\|^2_{1,\Omega} + \|u^0_{tt}\|^2_{1,\Omega}), \tag{3.42}$$

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (u^0_{\xi\xi})^2\,d\xi \le c(\|u^0\|^2_{2,\Omega} + \|u^0_t\|^2_{2,\Omega}). \tag{3.43}$$

Now, estimates (3.35) and (3.38)–(3.43) yield the following differential inequality:

$$\frac{d}{dt}\phi_1(t) + c_0\Phi_1(t) \le c_{15}\|\varrho_{\sigma 0}\|^2_{0,\Omega}\Phi_1(t) + \varepsilon\|\varrho_{\sigma 0}\|^2_{2,\Omega} + c_{16}\|\varrho_{\sigma 0}\|^2_{0,\Omega}. \tag{3.44}$$

By using the same argument as in Lemma 3.7, inequality (3.44) and assumptions (3.13) and (3.21) yield (3.33) for sufficiently small ε, δ and α. ∎

Now, we prove the main result of the paper.

Theorem 3.9. *Let* $\nu > \frac{1}{3}\mu > 0$, $f = 0$, *and* $p \in C^3(\mathbb{R}_+)$ *with* $p' > 0$. *Let* $(v,\varrho_\sigma) \in \mathfrak{N}(0)$, $S \in H^{5/2}$, $u^0_t(0) \in H^1(\Omega)$, $u^0_{tt}(0) \in L_2(\Omega)$ (u^0 *is a solution of* (3.2)) *and let the following compatibility condition be satisfied*:

$$[\mathbb{D}(v_0) - (p(\varrho_0) - p_0)]\overline{n}_0 = 0 \quad \text{on } S.$$

Moreover, *let the following assumptions be satisfied*:

(3.45) $\bar{\phi}(0) \le \alpha$;

(3.46) $\|v_0\|^2_{0,\Omega} + \|\varrho_{\sigma 0}\|^2_{0,\Omega} \le \delta$, *where* $\varrho_{\sigma 0} = \varrho_0 - \varrho_e$;

(3.47) $l > 0$ *is a constant such that* $\varrho_e - l > 0$ *and* $\varrho_1 < \varrho_0 < \varrho_2$, *where* $\varrho_1 = \varrho_e - l$, $\varrho_2 = \varrho_e + l$;

(3.48) $\int_\Omega \varrho_0 v_0\cdot\eta\,d\xi = 0, \quad \int_\Omega \varrho_0\xi\,d\xi = 0,$ *where* $\eta = a + b\times x$ *and* a, b *are arbitrary constant vectors*;

(3.49) $\int_\Omega \varrho_0\,d\xi = M.$

Then for sufficiently small constants α *and* δ *there exists a global solution of* (1.1) *such that* $(v,\varrho_\sigma)\in\mathfrak{M}(t)$ *for* $t\in\mathbb{R}^1_+$, $S_t\in H^{5/2}$ *for* $t\in\mathbb{R}^1_+$ *and*

$$\overline{\phi}(t)\le\alpha\quad \textit{for}\ t\in\mathbb{R}^1_+. \tag{3.50}$$

P r o o f. The theorem is proved step by step using the local existence in a fixed interval. In order to extend the solution to the interval $[T,2T]$ we first prove that

$$\varrho_1<\varrho(x,t)<\varrho_2\quad \forall x\in\overline{\Omega}_t,\ t\in[0,T]. \tag{3.51}$$

By (3.10) and assumption (3.45) we have

$$\|u(t)\|^2_{2,\Omega}+\|\eta_\sigma(t)\|^2_{2,\Omega}\le\psi_2(A,T)\alpha. \tag{3.52}$$

Hence

$$|u|^2_{\infty,\Omega^T}+|\eta_\sigma|^2_{\infty,\Omega^T}\le\alpha c(\Omega)\psi_2(A,T), \tag{3.53}$$

where $c(\Omega)>0$ is a constant from the imbedding lemma.

Assume now that α is so small that

$$[\alpha c(\Omega)\psi_2(A,T)]^{1/2}<l, \tag{3.54}$$

where l is the constant from assumption (3.47). Then by (3.53) we obtain (3.51) and this means that $\varrho_*=\varrho_1$ and $\varrho^*=\varrho_2$. Thus, the assumptions of the theorem and Lemmas 3.4, 3.7 yield

$$\overline{\phi}(t)\le\alpha\quad \text{for } t\le T, \tag{3.55}$$

where α and δ are so small that (3.15) and (3.32) are satisfied (with constants c_1, c_2, c_8, c_9, c_{10}, c_{11}, c_{12} and c'' depending on Ω, ϱ_1, ϱ_2). Hence, in view of Theorem 3.1, Lemma 3.8 and estimates (3.4)–(3.5) (with initial conditions at T) for A so large that

$$C_1(T)(\widetilde{c}\,\overline{\phi}(0)+\alpha)<A \tag{3.56}$$

and for α sufficiently small (so that (3.56) and (3.5) hold with $\overline{\phi}(0)$ replaced by α) there exists a local solution of (1.1) in the interval $[T,2T]$ and

$$\begin{aligned}\|u\|^2_{\mathcal{A}_{T,\Omega_T}}+\|\eta_\sigma\|^2_{\mathcal{B}_{T,\Omega_T}}&\le\psi_2(A,T)(\|\varrho_\sigma(T)\|^2_{2,\Omega_T}+\|u(T)\|^2_{2,\Omega_T}\\&\quad+\|u_t(T)\|^2_{1,\Omega_T}+\|u_{tt}(T)\|^2_{0,\Omega_T})\\&\le\psi_2(A,T)\alpha\end{aligned} \tag{3.57}$$

(where $\mathcal{A}_{T,\Omega_T}$ and $\mathcal{B}_{T,\Omega_T}$ are given by (1.6) and (1.5), respectively), which yields $(v,\varrho_\sigma)\in\mathfrak{M}(t)$ for $t\le 2T$.

To extend the solution to $[2T,3T]$ we have to prove

$$\overline{\phi}(t)\le\alpha\quad \text{for } t\le 2T. \tag{3.58}$$

First, we show the estimate

$$\varrho_1<\varrho(x,t)<\varrho_2\quad \forall x\in\overline{\Omega}_t,\ t\in[0,2T]. \tag{3.59}$$

In view of (3.51) we prove

$$\varrho_1 < \eta(\xi,t) < \varrho_2 \quad \forall \xi \in \overline{\Omega}_T,\ t \in [T,2T],$$

where by η we denote ϱ written in the Lagrangian coordinates $\xi \in \Omega_T$ connected with the Eulerian coordinates x by the relation

$$x = \xi + \int_T^t v(x,t')\,dt' = \xi + \int_T^t u(\xi,t')dt'.$$

In view of (3.55) and (3.57) we get

$$\|u(t)\|^2_{2,\Omega_T} + \|\eta_\sigma(t)\|^2_{2,\Omega_T} \le \psi_2(A,T)\alpha.$$

Hence

$$|u|^2_{\infty,\Omega_T\times(T,2T)} + |\eta_\sigma|^2_{\infty,\Omega_T\times(T,2T)} \le \alpha c(\Omega_T)\psi_2(A,T), \tag{3.60}$$

where $c(\Omega_T)$ is a constant from the imbedding lemma and by Remark 3.5,

$$[\alpha c(\Omega_T)\psi_2(A,T)]^{1/2} < l,$$

where l is the constant from assumption (3.47). Therefore, (3.60) implies (3.59).

Now, we prove that the volume and shape of Ω_t change in $[0,2T]$ no more than they do in $[0,T]$. To do this we consider $\int_0^t v(x,t')\,dt'$ for $0\le t\le 2T$. We estimate $\int_0^T v(x,t')\,dt'$ by applying Lemma 3.3, and to estimate $\int_T^{2T} v(x,t')\,dt'$ we use inequality (3.57) for the local solution in $[T,2T]$. Thus we have

$$\begin{aligned}\Big|\int_0^t v(x,t')\,dt'\Big| &\le \int_0^T |u(\xi,t')|\,dt' + \int_T^{2T} |u(\xi,t')|\,dt' \\ &< c_{13}T^{1/2}\Big[\Big(\int_0^T \|u\|^2_{2,\Omega}\,dt'\Big)^{1/2} + \Big(\int_T^{2T} \|u\|^2_{2,\Omega_T}\,dt'\Big)^{1/2}\Big] \\ &\le T^{1/2}\Big[\Big(c_{17}\int_0^T \|v\|^2_{2,\Omega_{t'}}\,dt'\Big)^{1/2} + c_{14}\alpha^{1/2}\Big] \\ &\le T^{1/2}\Big[\frac{c_{17}}{(c')^{1/2}}\Big(\int_0^T \overline{\phi}(t')\,dt'\Big)^{1/2} + c_{14}\alpha^{1/2}\Big] \\ &\le T^{1/2}\alpha^{1/2}\Big[c_{17}\Big(\frac{\mu_1}{c'}\Big)^{1/2}\Big(\int_0^T e^{-\mu_2 t'}\,dt'\Big)^{1/2} + c_{14}\Big] \\ &\le T^{1/2}\alpha^{1/2}\Big(\frac{c_{17}\mu_1}{(c'\mu_2)^{1/2}} + c_{14}\Big),\end{aligned} \tag{3.61}$$

where c_{13} and c_{14} are the constants from Remark 3.5, c' is the constant from (3.11) and we have used the fact that $\mu_1 > 1$.

If α is sufficiently small then estimates (3.61) and (3.59) imply that the differential inequality (2.58) can be derived in $[T,2T]$ with the same constants c_1 and c_2 as in $[0,T]$. Similarly, the other constants c_i and c', c'', μ_1, μ_2 are the same in $[T,2T]$ as in $[0,T]$.

Next, we prove that assumption (3.21) implies (3.22) for $t\le 2T$. To do this integrate (2.8) with respect to t over $(0,t)$ $(t\le 2T)$. Using Lemmas 3.2–3.3 we get

$$
\begin{aligned}
(3.62)\qquad & \|v\|^2_{0,\Omega_t}+\|\varrho_\sigma\|^2_{0,\Omega_t}\\
&\le c_{11}c\sup_{0\le t'\le t}\phi(t')\int_0^t\phi(t')\,dt'\,(1+\sup_{0\le t'\le t}\phi(t'))+c_{10}c_{11}C_0\\
&\le \frac{c_{11}c}{c'}c_3\mu_1(1+c_3\alpha)\alpha\Big(\int_0^T\overline{\phi}(0)e^{-\mu_2t'}\,dt'+\int_T^{2T}\overline{\phi}(T)e^{-\mu_2(t'-T)}\,dt'\Big)+c_{10}c_{11}\delta\\
&\le \frac{c_{11}cc_3\mu_1}{c'}(1+c_3\alpha)\alpha\Big(\alpha\int_0^T e^{-\mu_2t'}\,dt'+\mu_1\int_T^{2T}\overline{\phi}(0)e^{-\mu_2T}e^{-\mu_2(t'-T)}\,dt'\Big)\\
&\quad+c_{10}c_{11}\delta\\
&\le \frac{c_{11}cc_3\mu_1}{c'\mu_2}(1+c_3\alpha)\alpha^2[1-e^{-\mu_2T}+\mu_1(e^{\mu_2T}-e^{-2\mu_2T})]+c_{10}c_{11}\delta\\
&\le \frac{c_{11}cc_3\mu_1^2}{c'\mu_2}(1+c_3\alpha)\alpha^2+c_{10}c_{11}\delta,
\end{aligned}
$$

where c_{10}, c_{11} are the constants from Lemma 3.4 and c_3 is the constant from Lemma 3.2. Therefore (3.22) is satisfied for $t\le 2T$, so by (3.55) and Lemma 3.7 we obtain (3.58) and the existence of a local solution (v,ϱ) such that $(v,\varrho)\in\mathfrak{M}(t)$ for $t\le 3T$.

Finally, assume that there exists a local solution in $[0,kT]$ (where $k\ge 3$) satisfying

$$(3.63)\qquad \|u\|^2_{\mathcal{A}_{T,\Omega_{iT}}}\le A\qquad \text{for } i=1,\dots,k-1,$$

$$(3.64)\qquad \|\eta\|^2_{\mathcal{B}_{T,\Omega_{iT}}}\le \psi_1(A)\qquad \text{for } i=1,\dots,k-1,$$

$$(3.65)\qquad \overline{\phi}(t)\le\alpha\qquad \text{for } t\le(k-1)T,$$

$$(3.66)\qquad \|u\|^2_{\mathcal{A}_{T,\Omega_{iT}}}+\|\eta_\sigma\|^2_{\mathcal{B}_{T,\Omega_{iT}}}\le\psi_2(A,T)\alpha\qquad \text{for } i=1,\dots,k-1.$$

Moreover, assume that the volume and shape of Ω_t change in $[0,(k-1)T]$ no more than they do in $[0,T]$ and estimate (3.51) holds for $t\le(k-1)T$ (so the constants c_i, $i=1,\dots,17$, c', c'', μ_1, μ_2 are the same in each $[(i-1)T,iT]$, $i=1,\dots,k-1$). Since the argument used to show estimate (3.51) for $t\le kT$ is the same as for $t\le T$ and for $t\le 2T$, to prove the existence of a local solution in $[0,(k+1)T]$ it remains to show that the volume and shape of Ω_t change in $[0,kT]$ no more than they do in $[0,T]$ and that assumption

(3.21) implies (3.22) for $t \le kT$. In fact, applying Lemma 3.3 and estimates (3.63)–(3.66) we have, for $t \in [0, kT]$,

$$
\begin{aligned}
(3.67)\quad & \Big|\int_0^t v(x,t')\,dt'\Big| \\
&\le \sum_{i=0}^{k-1} \int_{iT}^{(i+1)T} |u(\xi,t')|\,dt' < c_{13}T^{1/2}\sum_{i=0}^{k-1}\Big(\int_{iT}^{(i+1)T} \|u\|^2_{2,\Omega_{iT}}\,dt'\Big)^{1/2} \\
&\le T^{1/2}\Big[c_{17}\sum_{i=0}^{k-2}\Big(\int_{iT}^{(i+1)T}\|v\|^2_{2,\Omega_{t'}}\,dt'\Big)^{1/2} + c_{14}\alpha^{1/2}\Big] \\
&\le T^{1/2}\Big[\frac{c_{17}}{(c')^{1/2}}\sum_{i=0}^{k-2}\Big(\int_{iT}^{(i+1)T}\overline{\phi}(t')\,dt'\Big)^{1/2} + c_{14}\alpha^{1/2}\Big] \\
&\le T^{1/2}\Big[c_{17}\Big(\frac{\mu_1}{c'}\Big)^{1/2}\sum_{i=0}^{k-2}\Big(\overline{\phi}(iT)\int_{iT}^{(i+1)T} e^{-\mu_2(t'-iT)}\,dt'\Big)^{1/2} + c_{14}\alpha^{1/2}\Big] \\
&\le T^{1/2}\Big[c_{17}\Big(\frac{\mu_1}{c'\mu_2}\Big)^{1/2}(1-e^{-\mu_2 T})^{1/2}\sum_{i=0}^{k-2}(\overline{\phi}(iT))^{1/2} + c_{14}\alpha^{1/2}\Big] \\
&\le T^{1/2}\Big\{c_{17}\Big(\frac{\mu_1}{c'\mu_2}\Big)^{1/2}(1-e^{-\mu_2 T})^{1/2}[\overline{\phi}(0)(1+\mu_1 e^{-\mu_2 T} \\
&\qquad\qquad + \mu_1 e^{-2\mu_2 T} + \ldots)]^{1/2} + c_{14}\alpha^{1/2}\Big\} \\
&\le T^{1/2}\alpha^{1/2}\Big[\frac{c_{17}\mu_1}{(c'\mu_2)^{1/2}}(1-e^{-\mu_2 T})^{1/2}\frac{1}{(1-e^{-\mu_2 T})^{1/2}} + c_{14}\Big] \\
&= T^{1/2}\alpha^{1/2}\Big(\frac{c_{17}\mu_1}{(c'\mu_2)^{1/2}} + c_{14}\Big),
\end{aligned}
$$

where c_{13}, c_{14} are the constants from Remark 3.5, c_{17} is the same constant as in inequality (3.61), c' is the constant from (3.11) and we have used the fact that $\mu_1 > 1$.

Thus, the right-hand side of (3.67) is the same as the right-hand side of (3.61). Therefore, for α sufficiently small the shape of Ω_t changes in $[0, kT]$ no more than it does in $[0, T]$ and the constants c_i $(i = 1, \ldots, 17)$, c', c'', μ_1, μ_2 from Theorem 2.8, Lemmas 3.2–3.4, 3.7, 3.8, Remark 3.5 and inequality (3.11) are the same in each $[iT, (i+1)T]$ for $i = 0, \ldots, k-1$.

In the same way we prove

$$(3.68)\qquad \|v\|^2_{0,\Omega_t} + \|\varrho_\sigma\|^2_{0,\Omega_t} \le c_9\alpha^2 + c_{10}c_{11}\delta$$

for $t \le kT$, where c_i $(i = 9, 10, 11)$ are the constants from Lemma 3.4.

Estimates (3.67)–(3.68), (3.65) and Lemma 3.7 yield $\overline{\phi}(t) \le \alpha$ for $t \le kT$ and hence we obtain the existence of a local solution (v, ϱ) of (1.1) such that $(v, \varrho_\sigma) \in \mathfrak{M}(t)$ for $t \le (k+1)T$. ■

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