A REMARK ON A MODIFIED SZÁSZ–MIRAKJAN OPERATOR

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Abstract. We prove that, for a sequence of positive numbers \(\delta(n)\), if \(n^{1/2}\delta(n) \not\to \infty\) as \(n \to \infty\), to guarantee that the modified Szász–Mirakjan operators \(S_{n,\delta}(f, x)\) converge to \(f(x)\) at every point, \(f\) must be identically zero.

1. Introduction. Let \(C_\alpha\) be the set of all continuous functions on \([0, \infty)\) satisfying \(|f(t)| \leq Mt^\alpha\) for some real numbers \(M > 0\) and \(\alpha > 0\). For \(f \in C_\alpha\) and \(x \in [0, \infty)\), the well-known Szász–Mirakjan operator is defined by

\[
S_n(f, x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_k(nx).
\]

From Hermann [2], we know that for \(f \in C_\alpha\), \(S_n(f, x)\) converges to \(f(x)\) uniformly on any closed subset of \([0, \infty)\), hence in particular at every point \(x\) in \([0, \infty)\). At the same time, Hermann also pointed out that \(C_\alpha\), for all \(\alpha > 0\), are the largest sets, in the usual sense, which guarantee \(S_n(f, x)\) to exist.

For computational reasons, Gróf [1] and Lehnhoff [3] suggested using a partial sum of \(S_n(f, x)\) (which only has a finite number of terms depending upon \(n\) and \(x\)) to approximate \(f(x)\). Let \(\delta = \delta(n)\) be a sequence of positive numbers. Lehnhoff examined the operator

\[
S_{n,\delta}(f, x) = \sum_{k=0}^{[n(x+\delta)]} f\left(\frac{k}{n}\right) p_k(nx),
\]

and he proved that, for all \(f \in C_\alpha\) satisfying \(|f(t)| \leq M_1 + M_2 t^{2m}\) for some positive numbers \(M_1, M_2\) and some natural number \(m\), \(S_{n,\delta}(f, x)\) converges to \(f(x)\) at every point on \([0, \infty)\) if

\[
\lim_{n \to \infty} n^{1/2}\delta(n) = \infty.
\]

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Recently, Sun [4] showed that the condition (1) is a sharp necessary and sufficient condition for $S_{n,\delta}(f, x)$ to converge pointwise to $f(x)$ for $f$ in $C_\alpha$. More precisely, he showed that for $f \in C_\alpha$ the condition (1) is sufficient for $S_{n,\delta}(f, x)$ to converge to $f(x)$ uniformly on any closed subset of $[0, \infty)$, and he also proved that if (1) does not hold, then for the function $f_0(x) = x^{\alpha x} \in C_\alpha$, $S_{n,\delta}(f_0, x)$ does not converge to $f_0(x)$ at some point $x$.

A natural question is whether (1) can be weakened if we consider a subset of $C_\alpha$ (for example, the subset that Lehnhoff studied). Our result exhibits a surprising phenomenon that if (1) does not hold, then to guarantee that $S_{n,\delta}(f, x)$ converges to $f(x)$ at every point, $f$ must be identically zero.

2. Result and proof. In what follows, we always use $C$ to indicate a positive constant, whose value may be different in different situations.

Theorem. Let $\delta = \delta(n)$, $n = 1, 2, \ldots$, be a sequence of positive numbers such that $n^{1/2}\delta(n) \nrightarrow \infty$ as $n \rightarrow \infty$, and assume that $f \in C_\alpha$. If $f(x_0) \neq 0$ for some $x_0 \in [0, \infty)$, then $S_\delta(f, x_0) \nrightarrow f(x_0)$ as $n \rightarrow \infty$.

Proof. Suppose $n^{1/2}\delta(n) \nrightarrow \infty$ as $n \rightarrow \infty$. Without loss of generality, there exists a constant $A > 0$ and a sequence $\{n_j\}$ of positive integers such that $n_j^{1/2}\delta(n_j) \leq A$, and $f(x_0) > 0$ for $x_0 \in (0, \infty)$, say. There are $M_0 > 0$ and $\varepsilon_0 > 0$ such that $f(x) > M_0$ for all $x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \subset (0, \infty)$. Write

$$f^+(x) = \frac{1}{2}(f(x) + |f(x)|), \quad f^-(x) = \frac{1}{2}(f(x) - |f(x)|).$$

Then

$$R_n(f, x) := S_n(f, x) - S_{n,\delta}(f, x) = R_n(f^+, x) + R_n(f^-, x)$$

since $R_n$ is a linear operator. At the same time, noting that $R_n$ is also a positive operator, we calculate that

$$R_{n_j}(f^+, x_0) = \sum_{k=\lceil n_j(x_0+\delta) \rceil + 1}^{\infty} f(k/n_j)p_k(n_jx_0) \geq \sum_{n_jx_0 + An_j^{1/2} + 1 \leq k \leq n_jx_0 + 2An_j^{1/2} + 2} f(k/n_j)p_k(n_jx_0).$$

For $n_jx_0 + An_j^{1/2} + 1 \leq k \leq n_jx_0 + 2An_j^{1/2} + 2$ and sufficiently large $j$, $k/n_j \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$, so that

$$R_{n_j}(f^+, x_0) \geq M_0 \sum_{n_jx_0 + An_j^{1/2} + 1 \leq k \leq n_jx_0 + 2An_j^{1/2} + 2} p_k(n_jx_0) \geq AM_0n_j^{1/2}p_{n_jx_0 + 2An_j^{1/2}}(n_jx_0).$$
by the monotonicity of \( \{p_k(nx_0)\} \) for \( k \geq nx_0 \). It is easy to obtain

\[
p_{n_jx_0+2An_j^{1/2}}(n_jx_0) \geq C(n_jx_0)^{-1/2}e^{-4A^2/x_0}
\]

from Stirling’s formula, hence

\[
R_{n_j}(f^+, x_0) \geq CAM_nx_0^{-1/2}e^{-4A^2/x_0} > 0,
\]

that is,

\[
(2) \quad R_{n_j}(f^+, x_0) \not\rightarrow 0 \quad \text{as } j \rightarrow \infty.
\]

For \( R_n(f^-, x_0) \), we see that

\[
R_n(f^-, x_0) = \sum_{k/n-x_0 \geq \varepsilon_0} f(k/n)p_k(nx_0)
\]

in view of \( f^-(x) = 0 \) for \( x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0) \). Thus for any given \( \varepsilon > 0 \), there is an \( N > 0 \) such that

\[
\left| \sum_{k=N+1}^{\infty} f(k/n)p_k(nx_0) \right| < \varepsilon.
\]

Similarly to the standard proof of the Korovkin theorem, we have

\[
\left| \sum_{k=\lfloor nx_0 + n\varepsilon_0 \rfloor + 1}^{N} f(k/n)p_k(nx_0) \right| \leq \max_{0 \leq t \leq N/n} |f(t)| \sum_{k/n-x_0 \geq \varepsilon_0} p_k(nx_0)
\]

\[
\leq MN^{\alpha}\varepsilon^2S_n((t-x_0)^2, x_0) \rightarrow 0
\]

as \( n \rightarrow \infty \), or

\[
R_n(f^-, x_0) \rightarrow 0 \quad \text{as } n \rightarrow \infty,
\]

therefore, with (2),

\[
R_{n_j}(f, x_0) \geq R_{n_j}(f^+, x_0) - |R_{n_j}(f^-, x_0)| \geq C > 0,
\]

or \( R_n(f, x_0) \not\rightarrow 0 \) as \( n \rightarrow \infty \). Consequently, \( f(x_0) - S_{n,\delta}(f, x_0) = R_n(f, x_0) + f(x_0) - S_n(f, x_0) \not\rightarrow 0 \) as \( n \rightarrow \infty \). The theorem is proved.

**Corollary.** Let \( \delta = \delta(n), n = 1, 2, \ldots, \) be a sequence of positive numbers such that \( n^{1/2}\delta(n) \not\rightarrow \infty \) as \( n \rightarrow \infty \), and assume that \( f \in C_\alpha \). Then

\[
\lim_{n \rightarrow \infty} S_{n,\delta}(f, x) = f(x) \quad \text{holds for every } x \in [0, \infty) \text{ if and only if } f \equiv 0.
\]

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