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A REMARK ON A MODIFIED SZÁSZ–MIRAKJAN OPERATOR

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Abstract. We prove that, for a sequence of positive numbers $\delta(n)$, if $n^{1/2}\delta(n) \not\to \infty$ as $n \to \infty$, to guarantee that the modified Szász–Mirakjan operators $S_{n,\delta}(f,x)$ converge to $f(x)$ at every point, f must be identically zero.

1. Introduction. Let C_α be the set of all continuous functions on $[0,\infty)$ satisfying $|f(t)| \le Mt^{\alpha t}$ for some real numbers $M > 0$ and $\alpha > 0$. For $f \in C_\alpha$ and $x \in [0,\infty)$, the well-known Szász–Mirakjan operator is defined by

$$S_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) e^{-nx} \frac{(nx)^k}{k!} =: \sum_{k=0}^{\infty} f\left(\frac{k}{n}\right) p_k(nx).$$

From Hermann [2], we know that for $f \in C_\alpha$, $S_n(f,x)$ converges to $f(x)$ uniformly on any closed subset of $[0,\infty)$, hence in particular at every point x in $[0,\infty)$. At the same time, Hermann also pointed out that C_α, for all $\alpha > 0$, are the largest sets, in the usual sense, which guarantee $S_n(f,x)$ to exist.

For computational reasons, Gróf [1] and Lehnhoff [3] suggested using a partial sum of $S_n(f,x)$ (which only has a finite number of terms depending upon n and x) to approximate $f(x)$. Let $\delta = \delta(n)$ be a sequence of positive numbers. Lehnhoff examined the operator

$$S_{n,\delta}(f,x) = \sum_{k=0}^{[n(x+\delta)]} f\left(\frac{k}{n}\right) p_k(nx),$$

and he proved that, for all f in C_α satisfying $|f(t)| \le M_1 + M_2 t^{2m}$ for some positive numbers M_1, M_2 and some natural number m, $S_{n,\delta}(f,x)$ converges to $f(x)$ at every point on $[0,\infty)$ if

$$\lim_{n\to\infty} n^{1/2}\delta(n) = \infty. \tag{1}$$

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Recently, Sun [4] showed that the condition (1) is a sharp necessary and sufficient condition for $S_{n,\delta}(f,x)$ to converge pointwise to $f(x)$ for f in C_α. More precisely, he showed that for $f\in C_\alpha$ the condition (1) is sufficient for $S_{n,\delta}(f,x)$ to converge to $f(x)$ uniformly on any closed subset of $[0,\infty)$, and he also proved that if (1) does not hold, then for the function $f_0(x)=x^{\alpha x}\in C_\alpha$, $S_{n,\delta}(f_0,x)$ does not converge to $f_0(x)$ at some point x.

A natural question is whether (1) can be weakened if we consider a subset of C_α (for example, the subset that Lehnhoff studied). Our result exhibits a surprising phenomenon that if (1) does not hold, then to guarantee that $S_{n,\delta}(f,x)$ converges to $f(x)$ at every point, f must be identically zero.

2. Result and proof. In what follows, we always use C to indicate a positive constant, whose value may be different in different situations.

THEOREM. *Let $\delta=\delta(n)$, $n=1,2,\ldots$, be a sequence of positive numbers such that $n^{1/2}\delta(n)\not\to\infty$ as $n\to\infty$, and assume that $f\in C_\alpha$. If $f(x_0)\neq 0$ for some $x_0\in[0,\infty)$, then $S_\delta(f,x_0)\not\to f(x_0)$ as $n\to\infty$.*

Proof. Suppose $n^{1/2}\delta(n)\not\to\infty$ as $n\to\infty$. Without loss of generality, there exists a constant $A>0$ and a sequence $\{n_j\}$ of positive integers such that $n_j^{1/2}\delta(n_j)\le A$, and $f(x_0)>0$ for $x_0\in(0,\infty)$, say. There are $M_0>0$ and $\varepsilon_0>0$ such that $f(x)>M_0$ for all $x\in(x_0-\varepsilon_0,x_0+\varepsilon_0)\subset(0,\infty)$. Write

$$f^+(x)=\tfrac12(f(x)+|f(x)|),\qquad f^-(x)=\tfrac12(f(x)-|f(x)|).$$

Then

$$R_n(f,x):=S_n(f,x)-S_{n,\delta}(f,x)=R_n(f^+,x)+R_n(f^-,x)$$

since R_n is a linear operator. At the same time, noting that R_n is also a positive operator, we calculate that

$$\begin{aligned}R_{n_j}(f^+,x_0)&=\sum_{k=[n_j(x_0+\delta)]+1}^{\infty} f(k/n_j)p_k(n_jx_0)\\ &\ge \sum_{n_jx_0+An_j^{1/2}+1\le k\le n_jx_0+2An_j^{1/2}+2} f(k/n_j)p_k(n_jx_0).\end{aligned}$$

For $n_jx_0+An_j^{1/2}+1\le k\le n_jx_0+2An_j^{1/2}+2$ and sufficiently large j, $k/n_j\in(x_0-\varepsilon_0,x_0+\varepsilon_0)$, so that

$$\begin{aligned}R_{n_j}(f^+,x_0)&\ge M_0\sum_{n_jx_0+An_j^{1/2}+1\le k\le n_jx_0+2An_j^{1/2}+2} p_k(n_jx_0)\\ &\ge AM_0n_j^{1/2}p_{[n_jx_0+2An_j^{1/2}]}(n_jx_0)\end{aligned}$$

by the monotonicity of $\{p_k(nx_0)\}$ for $k \geq nx_0$. It is easy to obtain

$$p_{[n_jx_0+2An_j^{1/2}]}(n_jx_0) \geq C(n_jx_0)^{-1/2}e^{-4A^2/x_0}$$

from Stirling's formula, hence

$$R_{n_j}(f^+, x_0) \geq CAM_0x_0^{-1/2}e^{-4A^2/x_0} > 0,$$

that is,

$$R_{n_j}(f^+, x_0) \not\to 0 \quad \text{as } j \to \infty. \tag{2}$$

For $R_n(f^-, x_0)$, we see that

$$R_n(f^-, x_0) = \sum_{k/n-x_0 \geq \varepsilon_0} f(k/n)p_k(nx_0)$$

in view of $f^-(x) = 0$ for $x \in (x_0 - \varepsilon_0, x_0 + \varepsilon_0)$. Thus for any given $\varepsilon > 0$, there is an $N > 0$ such that

$$\Big| \sum_{k=N+1}^{\infty} f(k/n)p_k(nx_0) \Big| < \varepsilon.$$

Similarly to the standard proof of the Korovkin theorem, we have

$$\begin{aligned}\Big| \sum_{k=[nx_0+n\varepsilon_0]+1}^{N} f(k/n)p_k(nx_0) \Big| &\leq \max_{0 \leq t \leq N/n} |f(t)| \sum_{k/n-x_0 \geq \varepsilon_0} p_k(nx_0) \\ &\leq MN^{\alpha N}\varepsilon_0^{-2}S_n((t-x_0)^2, x_0) \to 0\end{aligned}$$

as $n \to \infty$, or

$$R_n(f^-, x_0) \to 0 \quad \text{as } n \to \infty,$$

therefore, with (2),

$$R_{n_j}(f, x_0) \geq R_{n_j}(f^+, x_0) - |R_{n_j}(f^-, x_0)| \geq C > 0,$$

or $R_n(f, x_0) \not\to 0$ as $n \to \infty$. Consequently, $f(x_0) - S_{n,\delta}(f, x_0) = R_n(f, x_0) + f(x_0) - S_n(f, x_0) \not\to 0$ as $n \to \infty$. The theorem is proved. ■

Corollary. *Let $\delta = \delta(n)$, $n = 1, 2, \ldots,$ be a sequence of positive numbers such that $n^{1/2}\delta(n) \not\to \infty$ as $n \to \infty$, and assume that $f \in C_\alpha$. Then* $\lim_{n\to\infty} S_{n,\delta}(f, x) = f(x)$ *holds for every $x \in [0, \infty)$ if and only if $f \equiv 0$.*

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