C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 79 1999 NO. 1

GEOMETRY OF MODULES OVER TAME QUASI-TILTED ALGEBRAS

BY

GRZEGORZ BOBIŃSKI AND ANDRZEJ SKOWROŃSKI (TORUŃ)

Introduction. The class of finite-dimensional algebras (associative, with an identity) over an algebraically closed field K may be divided into two disjoint classes (see [10], [11]). One class consists of tame algebras for which the indecomposable modules occur, in each dimension, in a finite number of discrete and a finite number of one-parameter families. The second class is formed by the wild algebras whose representation theory is as complicated as the study of finite-dimensional vector spaces together with two noncommuting endomorphisms, for which the classification up to isomorphism is a well-known unsolved problem. Hence, we can realistically hope to describe modules only for tame algebras. Given a finite-dimensional algebra over K, it is an interesting task to study its associated affine varieties of modules of fixed dimension-vectors and the actions of the corresponding products of general linear groups (see [5]–[9], [13], [17], [19]–[21], [23], [26], [27], [31], [33] for some results in this direction). For example, we may ask when these affine varieties are irreducible, smooth, complete intersections, Gorenstein, Cohen–Macaulay, normal, etc.

An important role in recent investigations of finite-dimensional algebras is played by quasi-tilted algebras. It is the class of algebras of the form $A = \mathrm{End}_{\mathcal{H}}(T)$, where T is a tilting object in a hereditary abelian K-category $\mathcal{H}$. It was shown in [15] that an algebra A is quasi-tilted if and only if A is of global dimension at most two and each indecomposable finite-dimensional A-module has projective dimension at most one or injective dimension at most one. Important classes of quasi-tilted algebras are provided by all tilted algebras, tubular algebras, canonical algebras [28] and their relatives. The structure of an arbitrary quasi-tilted algebra is not yet known. However, the class of tame quasi-tilted algebras has been described recently by the second named author in [30]. In particular, one knows that the dimension-vector of an indecomposable module over a tame quasi-tilted algebra is either a

1991 *Mathematics Subject Classification*: 14L30, 14M10, 16G20, 16G60, 16G70.

Supported by the Polish Scientific Grant KBN No. 2P03A 012 14.

root or a radical vector of the associated Euler (equivalently, Tits) integral quadratic form.

The main aim of this paper is to describe the geometry of modules over tame quasi-tilted algebras whose dimension-vectors are the dimension-vectors of the indecomposable modules. The geometry of module varieties in the dimension-vectors of directing modules over tame algebras has been described in our paper [3]. In the present paper we deal with modules whose dimension-vectors are the dimension-vectors of nondirecting modules over tame quasi-tilted algebras.

The paper is organized as follows. In Section 1 we present our main results and recall the related background. Section 2 contains some results on the module categories of tame quasi-tilted algebras. Section 3 is devoted to some geometric preliminary results on affine varieties of modules. In Sections 4 and 5 we study the geometry of module varieties of tame quasi-tilted algebras in the dimension-vectors of indecomposable modules lying in stable tubes and nonstable tubes of the associated Auslander–Reiten quiver. In Section 6 we sum up the considerations of the previous sections to get the proofs of the main results. The final Section 7 contains examples illustrating different cases appearing in our considerations.

1. The main results and related background. Throughout the paper K will denote a fixed algebraically closed field. By an *algebra* is meant an associative finite-dimensional K-algebra with an identity, which we shall assume (without loss of generality) to be basic and connected. For such an algebra A, there exists an isomorphism $A \simeq KQ/I$, where KQ is the path algebra of the Gabriel quiver $Q = Q_A$ of A and I is an admissible ideal of KQ, generated by a (finite) system of forms $\sum_{1\le j\le t} \lambda_j \alpha_{m_j,j} \ldots \alpha_{1,j}$ (called *K-linear relations*), where $\lambda_1, \ldots, \lambda_t$ are elements of K and $\alpha_{m_j,j} \ldots \alpha_{1,j}$, $1 \le j \le t$, are paths of length ≥ 2 in Q having a common source and a common end. Denote by Q_0 the set of vertices of Q, by Q_1 the set of arrows of Q, and by $s, e : Q_1 \to Q_0$ the maps which assign to each arrow $\alpha \in Q_1$ its source $s(\alpha)$ and its end $e(\alpha)$. The category mod_A of all finite-dimensional (over K) left A-modules is equivalent to the category $\mathrm{rep}_K(Q, I)$ of all finite-dimensional K-linear representations $V = (V_i, \varphi_\alpha)_{i\in Q_0,\alpha\in Q_1}$ of Q, where V_i, $i \in Q_0$, are finite-dimensional K-vector spaces and $\varphi_\alpha : V_{s(\alpha)} \to V_{e(\alpha)}$, $\alpha \in Q_1$, are K-linear maps satisfying the equations $\sum_{1\le j\le t} \lambda_j \varphi_{\alpha_{m_j,j}} \ldots \varphi_{\alpha_{1,j}} = 0$ for all K-linear relations $\sum_{1\le j\le t} \lambda_j \alpha_{m_j,j} \ldots \alpha_{1,j} \in I$ (see [14, Section 4] for details). We shall identify mod_A with $\mathrm{rep}_K(Q, I)$ and call the finite-dimensional left A-modules briefly *A-modules*. The Grothendieck group $K_0(A)$ of A is then identified with the group $\mathbb{Z}^{Q_0}$, and we may assign to each A-module $V = (V_i, \varphi_\alpha)$ its *dimension-vector* $\mathbf{dim}\, V = (\dim_K V_i)_{i\in Q_0}$.

Moreover, we denote by Γ_A the Auslander–Reiten quiver of A, and by τ_A and τ_A^- the Auslander–Reiten translations $\mathrm{D}\,\mathrm{Tr}$ and $\mathrm{Tr}\,\mathrm{D}$, respectively. We shall not distinguish between an indecomposable A-module and the vertex of Γ_A corresponding to it. A component in Γ_A of the form $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$, $r \ge 1$, is said to be a *stable tube* (of rank r). We refer to [2] for basic background on the Auslander–Reiten theory. Finally, following [11], an algebra A is said to be *tame* if, for any dimension d, there exist a finite number of A-$K[X]$-bimodules M_i, $1 \le i \le n_d$, which are finite rank free right modules over the polynomial algebra $K[X]$ in one variable and all but finitely many isomorphism classes of indecomposable A-modules of dimension d are of the form $M_i \otimes_{K[X]} K[X]/(X-\lambda)$ for some $\lambda \in K$ and some i.

Fix now a vector $\mathbf{d} = (d_i) \in K_0(A) = \mathbb{Z}^{Q_0}$ with nonnegative coordinates. Denote by $\mathrm{mod}_A(\mathbf{d})$ the set of all representations $V = (V_i, \varphi_i)$ in $\mathrm{rep}_K(Q, I)$ with $V_i = K^{d_i}$ for all $i \in Q_0$. A representation V in $\mathrm{mod}_A(\mathbf{d})$ is given by $d_{e(\alpha)} \times d_{s(\alpha)}$-matrices $V(\alpha)$ determining the maps $\varphi_\alpha : K^{s(\alpha)} \to K^{e(\alpha)}$, $\alpha \in Q_1$, in the canonical bases of K^{d_i}, $i \in Q_0$. Moreover, the matrices $V(\alpha)$, $\alpha \in Q_1$, satisfy the equations

$$\sum_{1 \le j \le t} \lambda_j V(\alpha_{m_j,j}) \dots V(\alpha_{1,j}) = 0$$

for all K-linear relations $\sum_{1 \le j \le t} \lambda_j \alpha_{m_j,j} \dots \alpha_{1,j} \in I$. Therefore, $\mathrm{mod}_A(\mathbf{d})$ is a closed subset of $\prod_{\alpha \in Q_1} K^{d_{e(\alpha)} \times d_{s(\alpha)}}$ in the Zariski topology, and so $\mathrm{mod}_A(\mathbf{d})$ is an affine variety. We note that $\mathrm{mod}_A(\mathbf{d})$ is not necessarily irreducible. Clearly, it is the case when $I = 0$. The affine (reductive) algebraic group $G(\mathbf{d}) = \prod_{i \in Q_0} \mathrm{Gl}_{d_i}(K)$ acts on the variety $\mathrm{mod}_A(\mathbf{d})$ by conjugation:

$$(gV)(\alpha) = g_{e(\alpha)} V(\alpha) g_{s(\alpha)}^{-1}$$

for $g = (g_i) \in G(\mathbf{d})$, $V \in \mathrm{mod}_A(\mathbf{d})$, $\alpha \in Q_1$. We shall identify an A-module V of dimension-vector $\mathbf{d}$ with the corresponding point of the variety $\mathrm{mod}_A(\mathbf{d})$. The $G(\mathbf{d})$-orbit $G(\mathbf{d})M$ of a module M in $\mathrm{mod}_A(\mathbf{d})$ will be denoted by $\mathcal{O}(M)$. Observe that two A-modules M and N are isomorphic if and only if $\mathcal{O}(M) = \mathcal{O}(N)$.

For $M, N \in \mathrm{mod}_A(\mathbf{d})$, we say that N is a *degeneration* of M if N belongs to the Zariski closure $\overline{\mathcal{O}(M)}$ of $\mathcal{O}(M)$ in $\mathrm{mod}_A(\mathbf{d})$. If $N \in \overline{\mathcal{O}(M)}$ implies $\mathcal{O}(N) = \mathcal{O}(M)$, then the orbit $\mathcal{O}(N)$ is said to be *maximal*. Clearly, an orbit in $\mathrm{mod}_A(\mathbf{d})$ of maximal dimension is maximal, but the converse is not true in general. It is known that the union of all $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$ of maximal dimension is an open subset of $\mathrm{mod}_A(\mathbf{d})$, called its *open sheet* (see [19], [20]).

Assume now that $A = KQ/I$ is of finite global dimension. Then there is a (nonsymmetric) bilinear form $\langle -, - \rangle_A$ on $K_0(A)$ such that

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N\rangle_A = \sum_{i=0}^{\infty} (-1)^i \dim_K \operatorname{Ext}_A^i(M,N)$$

for all A-modules M, N (see [28, 2.4]). Then the corresponding quadratic form $\chi_A(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}\rangle_A$, $\mathbf{x} \in K_0(A)$, is called the *Euler form* of A. If A is *triangular* (Q has no oriented cycles), we may also consider the *Tits form* q_A of A, defined for $\mathbf{x} = (x_i) \in \mathbb{Z}^{Q_0} = K_0(A)$ as follows:

$$q_A(\mathbf{x}) = \sum_{i\in Q_0} x_i^2 - \sum_{\alpha\in Q_1} x_{s(\alpha)} x_{e(\alpha)} + \sum_{i,j\in Q_0} r_{ij} x_i x_j$$

where r_{ij} is the number of K-linear relations with source i and end j in a minimal set $\mathcal{R}$ of K-linear relations generating the ideal I. If gl.dim $A \le 2$ then q_A coincides with χ_A (see [4]). Moreover, we set

$$a(\mathbf{x}) = \sum_{\alpha\in Q_1} x_{s(\alpha)} x_{e(\alpha)} - \sum_{i,j\in Q_0} r_{ij} x_i x_j$$

for $\mathbf{x} = (x_i) \in \mathbb{Z}^{Q_0}$. We note that all quasi-tilted algebras A are triangular of global dimension at most 2, and so the forms χ_A, q_A are defined, and in fact they coincide. A vector $\mathbf{d} \in \mathbb{Z}^{Q_0}$ is called *connected* if the full subquiver of Q given by its *support* $\operatorname{supp}(\mathbf{d}) = \{i \in Q_0 : d_i \ne 0\}$ is connected. Moreover, we say that $\mathbf{d}$ is *positive* if $\mathbf{d}$ is nonzero and $d_i \ge 0$ for all $i \in Q_0$.

We can now state the main results of the paper.

THEOREM 1. *Let A be a tame quasi-tilted algebra and $\mathbf{d}$ the dimension-vector of an indecomposable A-module. Then*

(i) $\operatorname{mod}_A(\mathbf{d})$ *is a complete intersection of dimension $a(\mathbf{d})$ and has at most two irreducible components.*

(ii) *The maximal $G(\mathbf{d})$-orbits in* $\operatorname{mod}_A(\mathbf{d})$ *consist of nonsingular modules.*

In fact, in the course of our proofs and in [3] we describe completely all irreducible components and maximal $G(\mathbf{d})$-orbits in the considered varieties $\operatorname{mod}_A(\mathbf{d})$. We also note that, if the support algebra of $\mathbf{d}$ is not a hereditary algebra, then the semisimple module of dimension-vector $\mathbf{d}$ is a singular point of $\operatorname{mod}_A(\mathbf{d})$, and consequently $\operatorname{mod}_A(\mathbf{d})$ is not smooth (see [5, Proposition 1]).

THEOREM 2. *Let A be a tame quasi-tilted algebra and $\mathbf{d}$ the dimension-vector of an indecomposable A-module. Then the following conditions are equivalent:*

(i) $\operatorname{mod}_A(\mathbf{d})$ *is irreducible.*

(ii) $\operatorname{mod}_A(\mathbf{d})$ *is normal.*

(iii) $\mathbf{d}$ *is not one of the following forms:*

(a) $\mathbf{d} = \mathbf{h}+\mathbf{z}$, *where* $\mathbf{h}$, $\mathbf{z}$ *are connected positive vectors with disjoint supports*, $\chi_A(\mathbf{h}) = 0$, $\chi_A(\mathbf{z}) = 1$ *and* $z_i \leq 1$ *for any* $i \in Q_0$.

(b) $\mathbf{d} = \mathbf{h} + \mathbf{h}'$, *where* $\mathbf{h}$, $\mathbf{h}'$ *are connected positive vectors with* $\chi_A(\mathbf{h}) = 0$, $\chi_A(\mathbf{h}') = 0$, $\langle \mathbf{h}, \mathbf{h}' \rangle_A = 1$, $\langle \mathbf{h}', \mathbf{h} \rangle_A = 0$.

We also get the following consequences of the above theorems and their proofs.

COROLLARY 3. *Let* A *be a tame quasi-tilted algebra and* $\mathbf{d}$ *the dimension-vector of an indecomposable* A*-module lying in a stable tube of* Γ_A. *Then* $\operatorname{mod}_A(\mathbf{d})$ *is irreducible, complete intersection and normal. In particular, this holds for any positive connected vector* $\mathbf{d} \in K_0(A)$ *with* $\chi_A(\mathbf{d}) = 0$.

COROLLARY 4. *Let* A *be a tame quasi-tilted algebra. The following conditions are equivalent*:

(i) A *is tilted*, *all tubes in* Γ_A *are stable and*, *for any nonisomorphic indecomposable* τ_A*-invariant* A*-modules* H_1 *and* H_2, *either* $\dim_K \operatorname{Hom}_A(H_1, H_2) \neq 1$ *or* $\dim_K \operatorname{Ext}^1_A(H_2, H_1) \neq \dim_K \operatorname{Ext}^2_A(H_2, H_1)$.

(ii) *For any indecomposable* A*-module* M, $\operatorname{mod}_A(\operatorname{\mathbf{dim}} M)$ *is irreducible.*

(iii) *For any indecomposable* A*-module* M, $\operatorname{mod}_A(\operatorname{\mathbf{dim}} M)$ *is normal.*

(iv) *For any indecomposable* A*-module* M, $\operatorname{mod}_A(\operatorname{\mathbf{dim}} M)$ *is irreducible, complete intersection and normal.*

Finally, we also get the following geometric characterization of tame quasi-tilted algebras.

COROLLARY 5. *A quasi-tilted algebra* A *is tame if and only if every indecomposable* A*-module* M *belongs to the open sheet of* $\operatorname{mod}_A(\operatorname{\mathbf{dim}} M)$.

2. Tame quasi-tilted algebras. In this section we recall some facts on tame quasi-tilted algebras and their module categories, needed in the proofs of our main results. For details we refer to [15], [18], [24], [25], [28], [30].

2.1. Following [15] an algebra A is said to be *quasi-tilted* if $\operatorname{gl.dim} A \leq 2$ and, for any indecomposable A-module X, either $\operatorname{pd}_A X \leq 1$ or $\operatorname{id}_A X \leq 1$.

One important class of quasi-tilted algebras is formed by the *tilted algebras*, that is, the algebras of the form $\operatorname{End}_H(T)$, where H is a hereditary algebra and T is a tilting H-module [28]. The structure of module categories over tame tilted algebras has been described in [18]. In particular, we know that the Auslander–Reiten quiver Γ_B of a tame tilted algebra B consists of a connecting component, a finite number of $\mathbb{P}_1(K)$-families of ray tubes, coray tubes, and a finite number of preprojective components and preinjective components.

We now describe the second important class of tame quasi-tilted algebras. Let C be a tame concealed algebra and $\mathcal{T} = (\mathcal{T}_\lambda)_{\lambda\in\mathbb{P}_1(K)}$ be the unique $\mathbb{P}_1(K)$-family of stable tubes in Γ_C [28, 4.3]. Following [30], by a *semiregular branch enlargement* of C we mean an algebra of the form

$$\Lambda = \begin{bmatrix} F & M & 0 \\ 0 & C & \mathrm{D}(N) \\ 0 & 0 & B \end{bmatrix}$$

where

$$\Lambda^+ = \begin{bmatrix} F & M \\ 0 & C \end{bmatrix} \qquad \left(\text{respectively, } \Lambda^- = \begin{bmatrix} C & \mathrm{D}(N) \\ 0 & B \end{bmatrix}\right)$$

is a tubular extension (respectively, tubular coextension) of C in the sense of [28, 4.7], and no tube in $\mathcal{T}$ admits both a direct summand of M and a direct summand of N. It is known that such an algebra Λ is quasi-tilted. Moreover, Λ is tame if and only if both Λ^+ and Λ^- are tame, or equivalently are tubular algebras or tilted algebras of Euclidean type (see [1], [28]).

2.2. The following characterization of tame quasi-tilted algebras has been established in [30, Theorem A].

Theorem. *Let A be a quasi-tilted algebra. The following conditions are equivalent*:

(i) *A is tame.*

(ii) *A is tame tilted or a tame semiregular branch enlargement of a tame concealed algebra.*

(iii) *χ_A is weakly nonnegative.*

(iv) $\dim_K \mathrm{Ext}^1_A(X,X) \le \dim_K \mathrm{End}_A(X)$ *for any indecomposable A-module X.*

Recall that χ_A is called *weakly nonnegative* if $\chi_A(\mathbf{x}) \ge 0$ for any positive vector $\mathbf{x} \in K_0(A)$. We also note that every representation-finite quasi-tilted algebra is tilted [15, II.3.6].

2.3. As a direct consequence of the above theorem, [18, Section 4, 6.2], [28, 2.4(8), 4.9, 5.2], and arguments applied in the proof of [24, 2.1] one gets the following proposition.

Proposition. *Let A be a quasi-tilted algebra. Then A is tame if and only if χ_A controls the category* mod_A, *that is*, *has the following properties*:

(i) *For any indecomposable A-module X, $\chi_A(\mathbf{dim}\, X) \in \{0,1\}$.*

(ii) *For any connected positive vector $\mathbf{d} \in K_0(A)$ with $\chi_A(\mathbf{d}) = 1$ there is a unique* (*up to isomorphism*) *indecomposable A-module X with $\mathbf{dim}\, X = \mathbf{d}$.*

(iii) *For any connected positive vector $\mathbf{d} \in K_0(A)$ with $\chi_A(\mathbf{d}) = 0$, there is an infinite family $(X_\lambda)_{\lambda\in\Lambda}$ of pairwise nonisomorphic indecomposable A-modules with $\mathbf{dim}\, X_\lambda = \mathbf{d}$ for any λ.*

2.4. Sometimes it is convenient to consider an algebra $A = KQ/I$ as a finite K-category whose objects are the vertices of Q, and, for any two vertices $x, y \in Q_0$, the space of morphisms from x to y is the quotient space of the K-vector space $KQ(x, y)$ of all K-linear combinations of paths in Q from x to y by the subspace $I(x, y) = I \cap KQ(x, y)$. A full subcategory C of A is said to be *convex* (in A) if, for any path $a_0 \to a_1 \to \ldots \to a_t$ with a_0 and a_t in C, all vertices a_i, $0 \le i \le t$, belong to C. Clearly, if C is a convex subcategory of A, we may identify mod_C with the full subcategory of mod_A given by all representations with support contained in C. Moreover, if A is triangular and C is a convex subcategory of A, then χ_C and q_C are the restrictions of χ_A and q_A to $K_0(C)$.

By the *support* $\mathrm{supp}(M)$ of an A-module M we mean the support of its dimension-vector. If $\mathrm{supp}(M) = A$ then the module M is said to be *sincere*.

Following [28] an indecomposable A-module M is said to be *directing* if it does not belong to a cycle $M \to M_1 \to \ldots M_r \to M$ of nonzero nonisomorphisms between indecomposable A-modules. Examples of directing modules are provided by indecomposable preprojective and preinjective modules. Following [3] a directing module which is neither preprojective nor preinjective is said to be *internal*.

We end this section with the following consequences of Theorem 2.2, [18, Section 4, 6.2], [24], [28, 4.9, 5.2, p. 375] and the fact that the support of any directing module is convex [4, 3.2].

Corollary. *Let A be a tame quasi-tilted algebra and M an indecomposable A-module. Then there is a convex subcategory B of A such that M is a B-module and one of the following holds*:

(i) *B is tubular or representation-infinite tilted of Euclidean type, and M is a nondirecting module from a tube of Γ_A consisting entirely of B-modules.*

(ii) *B is a tame tilted algebra containing at most two different tame concealed convex subcategories, and M is a sincere directing module lying in a connecting component of Γ_B.*

2.5. Corollary. *Let A be a tame quasi-tilted algebra. Then*

(i) *The support of any stable tube Γ of Γ_A is a tame concealed or tubular convex subcategory of A.*

(ii) *If M is an indecomposable A-module with $\chi_A(\mathbf{dim}\, M) = 0$ then the support of M is a tame concealed or tubular convex subcategory of A.*

3. Geometric preliminaries.

In this section we recall and prove some facts applied in our investigations of module varieties over tame quasi-tilted algebras. For basic background we refer to [12], [16], [19], [20], [29].

3.1. Let $A = KQ/I$ be a triangular algebra (hence $\mathrm{gl.dim}\, A < \infty$) and $\mathbf{d} \in K_0(A) = \mathbb{Z}^{Q_0}$. Given a module $M \in \mathrm{mod}_A(\mathbf{d})$ we denote by $T_M(\mathrm{mod}_A(\mathbf{d}))$ the tangent space to $\mathrm{mod}_A(\mathbf{d})$ at M and by $T_M(\mathcal{O}(M))$ the tangent space to $\mathcal{O}(M)$ at M. Then there is a canonical monomorphism of K-vector spaces

$$T_M(\mathrm{mod}_A(\mathbf{d}))/T_M(\mathcal{O}(M)) \hookrightarrow \mathrm{Ext}_A^1(M, M)$$

(see [19, 2.7]). In particular, if $\mathrm{Ext}_A^1(M, M) = 0$ then $\overline{\mathcal{O}(M)}$ is an irreducible component of $\mathrm{mod}_A(\mathbf{d})$ and $\mathcal{O}(M)$ is an open subset of $\mathrm{mod}_A(\mathbf{d})$.

The *local dimension* $\dim_M \mathrm{mod}_A(\mathbf{d})$ is the maximal dimension of the irreducible components of $\mathrm{mod}_A(\mathbf{d})$ containing M. We have $\dim_K T_M(\mathrm{mod}_A(\mathbf{d})) \geq \dim_M \mathrm{mod}_A(\mathbf{d})$. Further, $M \in \mathrm{mod}_A(\mathbf{d})$ is said to be a *nonsingular point* if $\dim_M \mathrm{mod}_A(\mathbf{d}) = \dim_K T_M(\mathrm{mod}_A(\mathbf{d}))$. If M is a nonsingular point of $\mathrm{mod}_A(\mathbf{d})$ then M belongs to exactly one irreducible component of $\mathrm{mod}_A(\mathbf{d})$ [29, II.2.6]. The nonsingular points in $\mathrm{mod}_A(\mathbf{d})$ form an open nonempty subset. Clearly, $\mathcal{O}(M)$ is irreducible, M is a nonsingular point of $\mathcal{O}(M)$, and hence $\dim \mathcal{O}(M) = \dim_K T_M(\mathcal{O}(M))$. Moreover, $\dim \mathcal{O}(M) = \dim G(\mathbf{d}) - \dim_K \mathrm{End}_A(M)$ (see [20]).

It is also known that $M \in \mathrm{mod}_A(\mathbf{d})$ is nonsingular provided $\mathrm{Ext}_A^2(M, M) = 0$. The standard proof of this fact involves schemes and a result by Voigt (see [13], [26], [32]). For triangular algebras of global dimension at most 2 we shall present an elementary proof of this fact below.

Finally, $\mathrm{mod}_A(\mathbf{d})$ is said to be a *complete intersection* provided the vanishing ideal of $\mathrm{mod}_A(\mathbf{d})$ in the coordinate ring $K[\mathbb{A}(\mathbf{d})]$ of the affine space $\mathbb{A}(\mathbf{d}) = \prod_{\alpha \in Q_1} K^{d_{e(\alpha)} \times d_{s(\alpha)}}$ is generated by $\dim \mathbb{A}(\mathbf{d}) - \dim \mathrm{mod}_A(\mathbf{d})$ polynomials. Observe that this is the case when $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d})$. We also note that the irreducible components of a complete intersection have the same dimension.

3.2. We shall need the following fact.

PROPOSITION. *Let A be a triangular algebra of global dimension at most* 2, *M an A-module with* $\mathrm{Ext}_A^2(M, M) = 0$, *and* $\mathbf{d} = \mathbf{dim}\, M$. *Then M is a nonsingular point of* $\mathrm{mod}_A(\mathbf{d})$, $\dim_M \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d})$, *and* $\dim \mathcal{O}(M) = a(\mathbf{d}) - \dim_K \mathrm{Ext}_A^1(M, M)$.

Proof. For any $M \in \mathrm{mod}_A(\mathbf{d})$ we have

$$\begin{aligned} a(\mathbf{d}) &= \dim G(\mathbf{d}) - q_A(\mathbf{d}) = \dim G(\mathbf{d}) - \chi_A(\mathbf{d}) \\ &= (\dim G(\mathbf{d}) - \dim_K \mathrm{End}_A(M)) \\ &\quad + \dim_K \mathrm{Ext}_A^1(M, M) - \dim_K \mathrm{Ext}_A^2(M, M) \\ &= \dim \mathcal{O}(M) + \dim_K \mathrm{Ext}_A^1(M, M) - \dim_K \mathrm{Ext}_A^2(M, M) \\ &= (\dim_K T_M(\mathcal{O}(M)) + \dim_K \mathrm{Ext}_A^1(M, M)) - \dim_K \mathrm{Ext}_A^2(M, M) \end{aligned}$$

$$\geq \dim_K T_M(\mathrm{mod}_A(\mathbf{d})) - \dim_K \mathrm{Ext}^2_A(M,M).$$

Hence, applying Krull's generalized principal ideal theorem, we get

$$\begin{aligned}\dim_M \mathrm{mod}_A(\mathbf{d}) \geq a(\mathbf{d}) &\geq \dim_K T_M(\mathrm{mod}_A(\mathbf{d})) - \dim_K \mathrm{Ext}^2_A(M,M)\\ &\geq \dim_M \mathrm{mod}_A(\mathbf{d}) - \dim_K \mathrm{Ext}^2_A(M,M).\end{aligned}$$

If $\mathrm{Ext}^2_A(M,M) = 0$, this leads to

$$\dim_M \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d}) = \dim_K T_M(\mathrm{mod}_A(\mathbf{d})).$$

Moreover, $\dim \mathcal{O}(M) = a(\mathbf{d}) - \dim \mathrm{Ext}^1_A(M,M)$. This shows our claims.

3.3. We get the following consequences of the above proposition.

COROLLARY. *Let A be a triangular algebra with* $\mathrm{gl.dim}\, A \leq 2$, *and* $\mathbf{d} \in K_0(A)$ *be a positive vector. Assume that for any maximal $G(\mathbf{d})$-orbit $\mathcal{O}(M)$ in* $\mathrm{mod}_A(\mathbf{d})$ *we have* $\mathrm{Ext}^2_A(M,M) = 0$. *Then* $\mathrm{mod}_A(\mathbf{d})$ *is a complete intersection and* $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d})$.

Proof. Clearly it is enough to show that $a(\mathbf{d}) = \dim \mathrm{mod}_A(\mathbf{d})$. Observe that the closure of any $G(\mathbf{d})$-orbit in $\mathrm{mod}_A(\mathbf{d})$ is an irreducible variety. Moreover, any module N from $\mathrm{mod}_A(\mathbf{d})$ belongs to the closure $\overline{\mathcal{O}(M)}$ of a maximal orbit $\mathcal{O}(M)$ of $\mathrm{mod}_A(\mathbf{d})$. Hence, it follows from our assumption that any irreducible component $\mathcal{Z}$ of $\mathrm{mod}_A(\mathbf{d})$ contains a module M with $\mathrm{Ext}^2_A(M,M) = 0$. Applying now 3.2 we conclude that M is a nonsingular point of $\mathrm{mod}_A(\mathbf{d})$, and so $\mathcal{Z}$ is a unique irreducible component of $\mathrm{mod}_A(\mathbf{d})$ containing M. In particular, applying 3.2 again, we have $\dim \mathcal{Z} = \dim_M \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d})$. Therefore, $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d})$, and this finishes the proof.

3.4. A module variety $\mathrm{mod}_A(\mathbf{d})$ is said to be *normal* if the local ring $\mathcal{O}_M$ of any module $M \in \mathrm{mod}_A(\mathbf{d})$ is integrally closed in its total quotient ring. It is well known that if $\mathrm{mod}_A(\mathbf{d})$ is normal then it is nonsingular in codimension one, that is, the set of singular points in $\mathrm{mod}_A(\mathbf{d})$ is of codimension at least two (see [12, Chapter 11]). We shall need the following consequence of Serre's normality criterion.

THEOREM. *Let A be a triangular algebra,* $\mathbf{d} \in K_0(A)$ *a positive vector, and assume that* $\mathrm{mod}_A(\mathbf{d})$ *is a complete intersection. Then* $\mathrm{mod}_A(\mathbf{d})$ *is normal if and only if* $\mathrm{mod}_A(\mathbf{d})$ *is nonsingular in codimension one.*

Proof. See [16, II.8.23].

3.5. The following theorem shows that the degenerations of finite-dimensional modules over tame quasi-tilted algebras are given by short exact sequences.

THEOREM. *Let A be a tame quasi-tilted algebra*, $\mathbf{d} \in K_0(A)$ *a positive vector*, *and* M, N *two modules in* $\mathrm{mod}_A(\mathbf{d})$. *Then* $M \in \overline{\mathcal{O}(N)}$ *if and only if there exist A-modules* N_i, U_i, V_i *and short exact sequences* $0 \to U_i \to N_i \to V_i \to 0$ *in* mod_A *such that* $N_1 = N$, $N_{i+1} = U_i \oplus V_i$, $1 \le i \le s$, $M = N_{s+1}$ *for some natural number* s.

Proof. It is a direct consequence of [31, Theorem 3], [33, Corollary 3] and the well-known fact (see for example [7, 1.1]) that any short exact sequence $0 \to U \to W \to V \to 0$ of modules gives a degeneration $U \oplus V \in \overline{\mathcal{O}(W)}$.

3.6. The following characterization of maximal orbits in the module varieties of tame quasi-tilted algebras will be crucial in our investigations.

PROPOSITION. *Let A be a tame quasi-tilted algebra*, $\mathbf{d} \in K_0(A)$ *a positive vector*, *and* M *a module in* $\mathrm{mod}_A(\mathbf{d})$. *Then* $\mathcal{O}(M)$ *is a maximal* $G(\mathbf{d})$-*orbit in* $\mathrm{mod}_A(d)$ *if and only if* $\mathrm{Ext}_A^1(M', M'') = 0$ *for any decomposition* $M = M' \oplus M''$ *of* M.

Proof. Suppose $M = M' \oplus M''$ and there exists a nonsplittable short exact sequence
$$0 \to M' \to E \to M'' \to 0.$$
Then $M = M' \oplus M''$ is a proper degeneration of E, and hence the inclusion $\mathcal{O}(M) \subset \overline{\mathcal{O}(E)} \setminus \mathcal{O}(E)$ holds. Therefore, $\mathcal{O}(M)$ is not maximal. Assume now that $\mathrm{Ext}_A^1(M', M'') = 0$ for any decomposition $M = M' \oplus M''$ of M. Let $\mathcal{O}(M) \subset \overline{\mathcal{O}(N)}$ for some module N in $\mathrm{mod}_A(\mathbf{d})$. Applying Theorem 3.5 we conclude that there are A-modules N_i, U_i, V_i and short exact sequences
$$0 \to U_i \to N_i \to V_i \to 0$$
in mod_A such that $N_1 = N$, $N_{i+1} = U_i \oplus V_i$, $1 \le i \le s$, $M = N_{s+1}$ for some natural number s. Invoking now our assumption we infer that the above exact sequences are splittable, and consequently $M = N_{s+1} \simeq N_s \simeq \ldots \simeq N_1 = N$. This shows that $\mathcal{O}(M)$ is a maximal orbit in $\mathrm{mod}_A(\mathbf{d})$.

4. Geometry of stable tubes. Let A be a tame quasi-tilted algebra and $\mathbf{d}$ the dimension-vector of an indecomposable A-module lying in a stable tube of Γ_A. We prove in this section that $\mathrm{mod}_A(\mathbf{d})$ is irreducible, complete intersection, normal, and its open sheet is the union of all maximal $G(\mathbf{d})$-orbits. Invoking 2.5(i), we may assume that A is tame concealed or tubular. We first recall basic information on the structure of Γ_A in both cases.

4.1. Assume A is tame concealed. Then it follows from [28, 4.3] that Γ_A is of the form
$$\Gamma_A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$$

where $\mathcal{P}$ is a preprojective component containing all indecomposable projective modules, $\mathcal{Q}$ is a preinjective component containing all indecomposable injective modules, and $\mathcal{T} = (\mathcal{T}^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ is a family of pairwise orthogonal standard stable tubes separating $\mathcal{P}$ from $\mathcal{Q}$ (in the sense of [28, 3.1]). In particular, we know that all modules in $\mathcal{T}$ have projective dimension and injective dimension one.

4.2. Assume A is tubular. Then by [2.8, 5.2], Γ_A is of the form

$$\Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \Big(\bigvee_{q\in\mathbb{Q}^+} \mathcal{T}_q\Big) \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty$$

where $\mathcal{P}_0$ is a preprojective component, $\mathcal{Q}_\infty$ is a preinjective component, $\mathcal{T}_0 = (\mathcal{T}_0^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ is a family of ray tubes, $\mathcal{T}_\infty = (\mathcal{T}_\infty^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ is a family of coray tubes, and each $\mathcal{T}_q$, $q \in \mathbb{Q}^+$, where $\mathbb{Q}^+$ is the set of all positive rationals, is a family $(\mathcal{T}_q^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ of stable tubes. Further, each $\mathcal{T}_q$, $q \in \mathbb{Q}^+ \cup \{0,\infty\}$, consists of pairwise orthogonal standard tubes, almost all of which are homogeneous (stable tubes of rank 1). The indecomposable projective A-modules lie in $\mathcal{P}_0 \vee \mathcal{T}_0$ while the indecomposable injective A-modules lie in $\mathcal{T}_\infty \vee \mathcal{Q}_\infty$. In particular, we know that all indecomposable A-modules lying in stable tubes of Γ_A have projective dimension and injective dimension one. Finally, each tubular family $\mathcal{T}_q$, $q \in \mathbb{Q}^+ \cup \{0,\infty\}$, separates

$$\mathcal{P}_q = \mathcal{P}_0 \vee \Big(\bigvee_{p<q} \mathcal{T}_q\Big) \ \text{ from } \ \mathcal{Q}_q = \Big(\bigvee_{p>q} \mathcal{T}_q\Big) \vee Q_\infty.$$

In both cases 4.1 and 4.2, the ordering of families from left to right indicates that there are nonzero maps only from each family to itself or to the families to its right.

4.3. Let A be tame concealed or tubular and Γ be a stable tube, say of rank r, in Γ_A. Then for any indecomposable A-module X in Γ there exists a sectional path $X_1 \to X_2 \to \ldots \to X_m = X$ with X_1 lying on the mouth of Γ, and then m is called the *quasi-length* of X, denoted by $\mathrm{ql}(X)$. It is known [28, 3.1(3)] that for any indecomposable module X in Γ we have

(1) $\chi_A(\mathbf{dim}\, X) = 0$ if and only if r divides $\mathrm{ql}(X)$.
(2) $\chi_A(\mathbf{dim}\, X) = 1$ if and only if r does not divide $\mathrm{ql}(X)$.

4.4. We shall also need the following known lemma.

Lemma. *Let Γ be a stable tube of rank r in Γ_A and N an indecomposable A-module not lying in Γ. Then*

(i) *If* $\mathrm{Hom}_A(X,N) \neq 0$ *for some indecomposable module X from Γ, then* $\mathrm{Hom}_A(M,N) \neq 0$ *for any indecomposable module M in Γ with* $\mathrm{ql}(M) \geq r$.
(ii) *If* $\mathrm{Hom}_A(N,X) \neq 0$ *for some indecomposable module X from Γ, then* $\mathrm{Hom}_A(N,M) \neq 0$ *for any indecomposable module M in Γ with* $\mathrm{ql}(M) \geq r$.

4.5. Throughout this section we assume that $\mathbf{d}$ is the dimension-vector of an indecomposable module from a $\mathbb{P}_1(K)$-family $\mathcal{T}$ of pairwise orthogonal standard stable tubes of Γ_A. If A is tubular we may assume that $\mathcal{T} = \mathcal{T}_q$ for some $q \in \mathbb{Q}^+$, and then we put $\mathcal{P} = \mathcal{P}_q$ and $\mathcal{Q} = \mathcal{Q}_q$. Denote by $\mathbf{h}$ the dimension-vector of an indecomposable A-module lying on the mouth of a homogeneous tube of $\mathcal{T}$. Then we have the following characterization of the families $\mathcal{P}$, $\mathcal{T}$, $\mathcal{Q}$.

PROPOSITION. *Let X be an indecomposable A-module. Then X belongs to $\mathcal{P}$* (*respectively*, $\mathcal{T}$, $\mathcal{Q}$) *if and only if* $\langle \mathbf{h}, \mathbf{dim}\, X\rangle_A < 0$ (*respectively*, $\langle \mathbf{h}, \mathbf{dim}\, X\rangle_A = 0$, $\langle \mathbf{h}, \mathbf{dim}\, X\rangle_A > 0$).

Proof. See [28, 4.3, 5.2].

4.6. Our first aim is to describe the maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$. We have two cases to consider: $\chi_A(\mathbf{d}) = 0$ and $\chi_A(\mathbf{d}) = 1$. We call an indecomposable A-module X with $\chi_A(\mathbf{dim}\, X) = 0$ *isotropic*. Clearly, every indecomposable homogeneous A-module (lying in a homogeneous tube of Γ_A) is isotropic. We also note that isotropic modules $X_1, \ldots, X_r$ from $\mathcal{T}$ are pairwise orthogonal if and only if $X_1, \ldots, X_r$ belong to pairwise different tubes of $\mathcal{T}$. Moreover, we abbreviate $\chi = \chi_A$ and $\langle -, -\rangle = \langle -, -\rangle_A$.

PROPOSITION. *Assume $\chi(\mathbf{d}) = 0$. Then a $G(\mathbf{d})$-orbit $\mathcal{O}(M)$ in $\mathrm{mod}_A(\mathbf{d})$ is maximal if and only if $M = M_1 \oplus \ldots \oplus M_t$ for some pairwise orthogonal isotropic modules $M_1, \ldots, M_t$ from $\mathcal{T}$. Moreover, the open sheet of $\mathrm{mod}_A(\mathbf{d})$ is the union of all maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$.*

Proof. Let M be a module in $\mathrm{mod}_A(\mathbf{d})$ and $M = M_1 \oplus \ldots \oplus M_t$ its decomposition into a direct sum of indecomposable A-modules, and $\mathbf{d}_i = \mathbf{dim}\, M_i$ for $1 \le i \le t$. Assume $\mathcal{O}(M)$ is a maximal orbit in $\mathrm{mod}_A(\mathbf{d})$. Applying 3.5 we get $\mathrm{Ext}^1_A(M_i, M_j) = 0$ for all $i \ne j$. Hence, since $\mathrm{gl.dim}\, A \le 2$, we have

$$\begin{aligned}\langle \mathbf{d}_i, \mathbf{d}_j\rangle = {} & \dim_K \mathrm{Hom}_A(M_i, M_j) - \dim_K \mathrm{Ext}^1_A(M_i, M_j) \\ & + \dim_K \mathrm{Ext}^2_A(M_i, M_j) \ge 0\end{aligned}$$

for all $i \ne j$. Moreover, $\langle \mathbf{d}_i, \mathbf{d}_i\rangle = \chi(\mathbf{d}_i) \ge 0$ for any $1 \le i \le t$, because $M_1, \ldots, M_t$ are indecomposable (see 2.3). Therefore,

$$0 = \chi(\mathbf{d}) = \langle \mathbf{d}, \mathbf{d}\rangle = \sum_{1 \le i,j \le t} \langle \mathbf{d}_i, \mathbf{d}_j\rangle$$

implies that $\langle \mathbf{d}_i, \mathbf{d}_j\rangle = 0$ for all $1 \le i, j \le t$. In particular, we get $\langle \mathbf{d}_i, \mathbf{d}_i\rangle = 0$ for $1 \le i \le t$, and so $M_1, \ldots, M_t$ are isotropic. Furthermore, the equalities

$$0 = \langle \mathbf{d}_i, \mathbf{d}_j\rangle = \dim_K \mathrm{Hom}_A(M_i, M_j) + \dim_K \mathrm{Ext}^2_A(M_i, M_j)$$

imply that the modules $M_1, \ldots, M_t$ are also pairwise orthogonal. Finally, observe that, for each $1 \le j \le t$, we have

$$0 = \langle \mathbf{d}, \mathbf{d}_j \rangle = \sum_{1 \le i \le t} \langle \mathbf{d}_i, \mathbf{d}_j \rangle = 0.$$

On the other hand, it follows from our assumption $\chi(\mathbf{d}) = 0$ that $\mathbf{d} = p\mathbf{h}$ for some $p \ge 1$. Therefore, invoking 4.5, we conclude that $M_1, \ldots, M_t$ belong to $\mathcal{T}$. Since these modules are pairwise orthogonal and isotropic, we deduce that they belong to pairwise different tubes of $\mathcal{T}$. Observe also that $\mathbf{d}_i = p_i\mathbf{h}$, $1 \le i \le t$, with $p_1 + \ldots + p_t = p$, and $\dim_K \operatorname{End}_A(M) = p$. Therefore, all maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ have (see 3.1) the same dimension $\dim G(\mathbf{d}) - p$. In particular, the open sheet of $\operatorname{mod}_A(\mathbf{d})$ is the union of all maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$.

Clearly, if $M = M_1 \oplus \ldots \oplus M_t$ is a module in $\operatorname{mod}_A(\mathbf{d})$ with $M_1, \ldots, M_t$ isotropic and pairwise orthogonal then $\dim_K \operatorname{End}_A(M) = p$, and hence the orbit $\mathcal{O}(M)$ is maximal.

4.7. We note that by 2.3 any indecomposable A-module X which is not isotropic is uniquely determined (up to isomorphism) by its dimension-vector.

Proposition. *Assume* $\chi(\mathbf{d}) = 1$. *Then a* $G(\mathbf{d})$*-orbit* $\mathcal{O}(M)$ *in* $\operatorname{mod}_A(\mathbf{d})$ *is maximal if and only if* $M = M_1 \oplus \ldots \oplus M_t$, *where* $M_1, \ldots, M_t$ *are pairwise orthogonal indecomposable modules from* $\mathcal{T}$, M_1 *is not isotropic*, *and* $M_2, \ldots, M_t$ (*if* $t \ge 2$) *are isotropic. Moreover*, *the open sheet of* $\operatorname{mod}_A(\mathbf{d})$ *is the union of all maximal* $G(\mathbf{d})$*-orbits in* $\operatorname{mod}_A(\mathbf{d})$.

Proof. Let $M = M_1 \oplus \ldots \oplus M_t$ be a decomposition of a module M from $\operatorname{mod}_A(\mathbf{d})$ into a direct sum of indecomposable A-modules, and $\mathbf{d}_i = \mathbf{dim}\, M_i$ for $1 \le i \le t$. Assume $\mathcal{O}(M)$ is a maximal orbit in $\operatorname{mod}_A(\mathbf{d})$. As above we deduce that $\langle \mathbf{d}_i, \mathbf{d}_j \rangle \ge 0$ for all $1 \le i, j \le t$. Moreover, $\langle \mathbf{d}_i, \mathbf{d}_i \rangle = \chi(\mathbf{d}_i) \in \{0, 1\}$ for any $1 \le i \le t$, by 2.3. Since $\chi(\mathbf{d}) = 1$ we have

$$1 = \langle \mathbf{d}, \mathbf{d} \rangle = \sum_{1 \le i \le t} \chi(\mathbf{d}_i) + \sum_{i \ne j} \langle \mathbf{d}_i, \mathbf{d}_j \rangle.$$

Suppose $\chi(\mathbf{d}_i) = 0$ for any $1 \le i \le t$. Then there exists a pair (i, j) such that $\langle \mathbf{d}_i, \mathbf{d}_j \rangle = 1$. On the other hand, by our assumption, M_i and M_j are isotropic modules. Hence, A is tubular and M_i, M_j belong to pairwise different tubular families. Invoking $\langle \mathbf{d}_i, \mathbf{d}_j \rangle = 1$ and 4.5 we then get $\langle \mathbf{d}_j, \mathbf{d}_i \rangle < 0$, which contradicts the above inequalities.

Therefore, there exists $i_0 \in \{1, \ldots, t\}$ such that $\chi(\mathbf{d}_{i_0}) = 1$. We may assume that $i_0 = 1$. Assume $t \ge 2$. Then $\chi(\mathbf{d}_i) = 0$ for $2 \le i \le t$, and consequently the modules $M_2, \ldots, M_t$ are isotropic. Moreover,

$$0 = \langle \mathbf{d}_i, \mathbf{d}_j \rangle = \dim_K \operatorname{Hom}_A(M_i, M_j) + \dim_K \operatorname{Ext}^2_A(M_i, M_j)$$

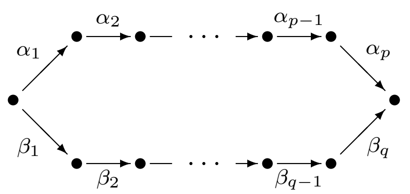
for all $i \neq j$ from $\{1, \ldots, t\}$, and so $M_1, \ldots, M_t$ are pairwise orthogonal and belong to $\mathcal{T}$. Furthermore, $\mathbf{d}_1 = p_1\mathbf{h} + \mathbf{e}$ and $\mathbf{d}_i = p_i\mathbf{h}$, $2 \leq i \leq t$, for some $p_1 \geq 0, p_2 \geq 1, \ldots, p_t \geq 1$ and a connected positive vector $\mathbf{e}$ such that $\mathbf{e} < \mathbf{h}$. Clearly, $\mathbf{d} = p\mathbf{h} + \mathbf{e}$ for $p = p_1 + \ldots + p_t$. Thus $\dim_K \mathrm{End}_A(M_1) = p_1 + 1$, and $\dim_K \mathrm{End}_A(M_i) = p_i$ for $2 \leq i \leq t$ (if $t \geq 2$) (see for example [28, 3.1]). Therefore, $\dim_K \mathrm{End}_A(M) = p + 1$ depends only on $\mathbf{d}$. Hence, all maximal orbits in $\mathrm{mod}_A(\mathbf{d})$ have the same dimension $\dim G(\mathbf{d}) - (p+1)$. In particular, the open sheet of $\mathrm{mod}_A(\mathbf{d})$ is the union of all maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$.

Clearly, if $M = M_1 \oplus \ldots \oplus M_t$ is a module in $\mathrm{mod}_A(\mathbf{d})$ with $M_1, \ldots, M_t$ indecomposable pairwise orthogonal A-modules from $\mathcal{T}$, M_1 not isotropic, and $M_2, \ldots, M_t$ isotropic (if $t \geq 2$), then $\dim_K \mathrm{End}_A(M) = p+1$ and hence $\mathcal{O}(M)$ is a maximal orbit of $\mathrm{mod}_A(\mathbf{d})$. This finishes the proof.

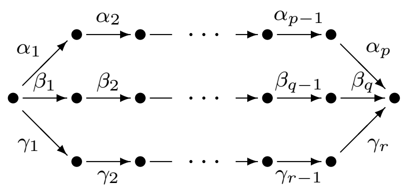
4.8. We can now derive the following consequence of 3.3, 4.6 and 4.7.

PROPOSITION. *The variety* $\mathrm{mod}_A(\mathbf{d})$ *is a complete intersection and has dimension* $a(\mathbf{d})$. *Moreover*, *the union of all maximal* $G(\mathbf{d})$*-orbits in* $\mathrm{mod}_A(\mathbf{d})$ *is its open sheet and consists of nonsingular modules.*

4.9. Our next aim is to prove that the variety $\mathrm{mod}_A(\mathbf{d})$ is irreducible. It is enough to show that the union of all maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$ is its irreducible subset. We shall use some class of canonical algebras introduced in [28]. For integers $q \geq p \geq 1$ we denote by $\Lambda(p,q)$ the path algebra of the Euclidean quiver



For integers $r \geq q \geq p \geq 2$, denote by $\Lambda(p,q,r)$ the bound quiver algebra $\Lambda(p,q,r) = K\Delta(p,q,r)/I(p,q,r)$, where $\Delta(p,q,r)$ is the quiver



and $I(p,q,r)$ is the ideal in $K\Delta(p,q,r)$ generated by $\alpha_p \ldots \alpha_1 + \beta_q \ldots \beta_1 + \gamma_r \ldots \gamma_1$.

Finally, for $a \in K \setminus \{0,1\}$ denote by $\Lambda(2,2,2,2,a)$ the bound quiver algebra $\Lambda(2,2,2,2,a) = K\Delta(2,2,2,2)/I(2,2,2,2,a)$, where $\Delta(2,2,2,2)$ is the quiver

$$\begin{tikzcd}
 & \bullet \arrow[ddr, "\alpha_2"] & \\
 & \bullet \arrow[dr, "\beta_2"] & \\
\bullet \arrow[uur, "\alpha_1"] \arrow[ur, "\beta_1"] \arrow[dr, "\gamma_1"'] \arrow[ddr, "\delta_1"'] & & \bullet \\
 & \bullet \arrow[ur, "\gamma_2"'] & \\
 & \bullet \arrow[uur, "\delta_2"'] &
\end{tikzcd}$$

and $I(2,2,2,2,a)$ is the ideal in $K\Delta(2,2,2,2)$ generated by $\alpha_2\alpha_1 + \beta_2\beta_1 + \gamma_2\gamma_1$ and $\alpha_2\alpha_1 + a\beta_2\beta_1 + \delta_2\delta_1$.

The above canonical algebras $\Lambda(p,q)$, $\Lambda(p,q,r)$, $\Lambda(2,2,2,2,a)$ are quasi-tilted and their Auslander–Reiten quivers admit a canonical separating tubular $\mathbb{P}_1(K)$-family $\mathcal{T}'$ of tubular type (p,q), (p,q,r), $(2,2,2,2)$, respectively. In all these cases the dimension-vector $\mathbf{c}$ of indecomposable modules lying on the mouth of homogeneous tubes (simple homogeneous modules) of $\mathcal{T}'$ has the identity at all vertices of the quiver. Moreover, $\Lambda(p,q)$ and $\Lambda(2,2,2,2,a)$ are tame, and $\Lambda(p,q,r)$ is tame if and only if $1/p+1/q+1/r \ge 1$. Observe that $1/p+1/q+1/r > 1$ (respectively, $1/p+1/q+1/r = 1$) if and only if $(p,q,r) = (2,3,3),(2,3,4),(2,3,5)$ or $(2,2,n-2)$ with $n \ge 4$ (respectively, $(p,q,r) = (3,3,3),(2,4,4),(2,3,6)$). The algebras $\Lambda(p,q)$, $\Lambda(2,2,n-2)$, $\Lambda(2,3,3)$, $\Lambda(2,3,4)$, $\Lambda(2,3,5)$ are tame concealed algebras of type $\widetilde{\mathbb{A}}_{p+q}$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$, respectively. The canonical algebras $\Lambda(3,3,3)$, $\Lambda(2,4,4)$, $\Lambda(2,3,6)$ and $\Lambda(2,2,2,2,a)$ are called *tubular*. We refer to [28] for the representation theory of canonical algebras. We need the following lemma.

Lemma. *Let Λ be one of the above canonical algebras and $\mathbf{c}$ the dimension-vector of simple homogeneous modules in the tubular family $\mathcal{T}'$. Then for any indecomposable Λ-module W with $\mathbf{dim}\, W = \mathbf{c}$ there exists a regular map $\varphi : K \to \mathrm{mod}_\Lambda(\mathbf{c})$ such that $\varphi(K) \cap \mathcal{O}(W) \ne \emptyset$ and $\varphi(K) \cap \mathcal{O}(H) \ne \emptyset$ for any simple homogeneous Λ-module H.*

Proof. This is an easy exercise due to a very simple structure (see [28, 3.7]) of indecomposable Λ-modules of dimension-vector $\mathbf{c}$. We indicate the argument for the algebra $\Lambda(3,3,3)$. In this case, the indecomposable modules of dimension-vector $\mathbf{c}$ lying in one of the three tubes of rank 3 in $\mathcal{T}'$ are of the form

$$\begin{tikzcd}
 & K \arrow[r, "1"] & K \arrow[dr, "1"] & \\
K \arrow[ur, "0"] \arrow[r, "-1"] \arrow[dr, "1"'] & K \arrow[r, "1"] & K \arrow[r, "1"] & K \\
 & K \arrow[r, "1"'] & K \arrow[ur, "1"'] &
\end{tikzcd}, \quad
\begin{tikzcd}
 & K \arrow[r, "0"] & K \arrow[dr, "1"] & \\
K \arrow[ur, "1"] \arrow[r, "-1"] \arrow[dr, "1"'] & K \arrow[r, "1"] & K \arrow[r, "1"] & K \\
 & K \arrow[r, "1"'] & K \arrow[ur, "1"'] &
\end{tikzcd}, \quad
\begin{tikzcd}
 & K \arrow[r, "1"] & K \arrow[dr, "0"] & \\
K \arrow[ur, "1"] \arrow[r, "-1"] \arrow[dr, "1"'] & K \arrow[r, "1"] & K \arrow[r, "1"] & K \\
 & K \arrow[r, "1"'] & K \arrow[ur, "1"'] &
\end{tikzcd}.$$

If W is the first of the above modules then the required regular map $\varphi : K \to \mathrm{mod}_\Lambda(\mathbf{c})$ assigns to each $\mu \in K$ the representation

$$\begin{tikzcd} & K \arrow[r, "1"] & K \arrow[dr, "1"] & \\ K \arrow[ur, "\mu"] \arrow[r, "-1"] \arrow[dr, "1-\mu"'] & K \arrow[r, "1"] & K \arrow[r, "1"] & K \\ & K \arrow[r, "1"'] & K \arrow[ur, "1"'] & \end{tikzcd}$$

Clearly, $\varphi(0) = W$ and $\varphi(\mu)$, $\mu \neq 0, 1$, form (up to isomorphism) the family of all simple homogeneous Λ-modules in $\mathcal{T}'$.

4.10. Let Λ be one of the above tame canonical algebras. Then

$$\Gamma_\Lambda = \mathcal{P}' \vee \mathcal{T}' \vee \mathcal{Q}'$$

where the indecomposable modules in these families may be characterized as follows (see 4.5): an indecomposable Λ-module Z belongs to $\mathcal{P}'$ (respectively, $\mathcal{T}'$, $\mathcal{Q}'$) if and only if $\langle \mathbf{c}, \mathbf{dim}\, Z\rangle_\Lambda < 0$ (respectively, $\langle \mathbf{c}, \mathbf{dim}\, Z\rangle_\Lambda = 0$, $\langle \mathbf{c}, \mathbf{dim}\, Z\rangle_\Lambda > 0$) (see [28, 3.7]). Let $\mathbf{d}'$ be the dimension-vector of an indecomposable Λ-module lying in $\mathcal{T}'$. Then we have the following

PROPOSITION. *The variety* $\mathrm{mod}_\Lambda(\mathbf{d}')$ *is irreducible.*

Proof. We have two cases to consider: $\chi_\Lambda(\mathbf{d}') = 0$ and $\chi_\Lambda(\mathbf{d}') = 1$. Assume first $\chi_\Lambda(\mathbf{d}') = 0$. Then $\mathbf{d}' = p\mathbf{c}$ for some $p \geq 1$. Fix a homogeneous module N_0 of dimension-vector $\mathbf{c}$.

Let $M_1, \ldots, M_t$ be indecomposable Λ-modules from $\mathcal{T}'$ of dimension-vectors $p_1\mathbf{c}, \ldots, p_t\mathbf{c}$, respectively, where $p_1, \ldots, p_t$ are positive integers such that $p = p_1 + \ldots + p_t$. Observe that then there exist indecomposable modules $N_1, \ldots, N_t$ in $\mathcal{T}'$ of dimension-vector $\mathbf{c}$ such that $N_1^{p_1} \oplus \ldots \oplus N_t^{p_t} \in \overline{\mathcal{O}(M_1 \oplus \ldots \oplus M_t)}$ in $\mathrm{mod}_\Lambda(p\mathbf{c}) = \mathrm{mod}_\Lambda(\mathbf{d}')$.

Indeed, for each $1 \leq i \leq t$, there exists an exact sequence $0 \to N_i \to M_i \to L_i \to 0$ where N_i and L_i are indecomposable Λ-modules lying on the same ray as M_i (in the tube of $\mathcal{T}'$ containing M_i), and $\mathbf{dim}\, N_i = \mathbf{c}$, $\mathbf{dim}\, L_i = (p_i - 1)\mathbf{c}$. Clearly, $L_i \oplus N_i \in \overline{\mathcal{O}(M_i)}$ in $\mathrm{mod}_\Lambda(p_i\mathbf{c})$. Hence our claim follows by induction on $p_1, \ldots, p_t$.

Since the closures of orbits are irreducible, we conclude that $M_1 \oplus \ldots \oplus M_t$ and $N_1^{p_1} \oplus \ldots \oplus N_t^{p_t}$ belong to a common irreducible component $\mathcal{Z}_1$ of $\mathrm{mod}_\Lambda(\mathbf{d}')$. On the other hand, by Proposition 4.9, there are regular maps $\varrho_i : K \to \mathrm{mod}_\Lambda(\mathbf{c})$, $1 \leq i \leq t$, such that $\varrho_i(K)$ intersects both $\mathcal{O}(N_i)$ and $\mathcal{O}(N_0)$. For each $1 \leq i \leq t$, we then get a regular map $\xi_i : K^{p_i} \to \mathrm{mod}_\Lambda(p_i\mathbf{c})$ such that $\xi_i(K^{p_i})$ intersects both $\mathcal{O}(N_i^{p_i})$ and $\mathcal{O}(N_0^{p_i})$.

Finally, consider the regular map $\eta : G(\mathbf{d}') \times K^p \to \mathrm{mod}_\Lambda(p\mathbf{c})$ which assigns to each $(g, \mu_1, \ldots, \mu_t) \in G(\mathbf{d}') \times K^p = G(\mathbf{d}') \times K^{p_1} \times \ldots \times K^{p_t}$ the module $g(\bigoplus_{i=1}^{t}\xi_i(\mu_i)) \in \mathrm{mod}_\Lambda(\mathbf{d}')$. Then the image of η is an irreducible subset of $\mathrm{mod}_\Lambda(p\mathbf{c})$ containing both $N_1^{p_1} \oplus \ldots \oplus N_t^{p_t}$ and N_0^p. Therefore, $N_1^{p_1} \oplus \ldots \oplus N_t^{p_t}$ and N_0^p belong to a common irreducible component $\mathcal{Z}_2$ of $\mathrm{mod}_\Lambda(\mathbf{d}')$.

We now note that any module $X \in \mathrm{mod}_\Lambda(\mathbf{d}')$ which is a direct sum of indecomposable modules from $\mathcal{T}'$ is nonsingular, because then X is of projective dimension at most one, and so $\mathrm{Ext}^2_\Lambda(X, X) = 0$. This shows that $\mathcal{Z}_1 = \mathcal{Z}_2$, and it is a unique irreducible component containing $M_1 \oplus \ldots \oplus M_k$ (respectively, N_0^p). Therefore, there exists a unique irreducible component $\mathcal{Z}$ of $\mathrm{mod}_\Lambda(\mathbf{d}')$ containing all maximal $G(\mathbf{d}')$-orbits of $\mathrm{mod}_\Lambda(\mathbf{d}')$, and so $\mathrm{mod}_\Lambda(\mathbf{d}')$ is irreducible.

Assume now $\chi_\Lambda(\mathbf{d}') = 1$. Then $\mathbf{d}' = p\mathbf{c}+\mathbf{e}'$ for some $p \geq 0$ and $\mathbf{e}'$ positive with $\mathbf{e}' < \mathbf{c}$. For each $0 \leq i \leq p$, the vector $\mathbf{d}'_i = i\mathbf{c}+\mathbf{e}'$ is positive, connected, with $\chi_\Lambda(\mathbf{d}'_i) = 1$; let V_i be the unique indecomposable Λ-module (lying in $\mathcal{T}'$) such that $\mathbf{dim}\, V_i = \mathbf{d}'_i$. In fact $V_0, V_1, \ldots, V_p$ are indecomposable modules lying on one ray of a tube in $\mathcal{T}'$. Hence, for each $0 \leq i < p$, we have an exact sequence

$$0 \to V_i \to V_p \to W_i \to 0$$

where W_i is an indecomposable Λ-module of dimension-vector $(p-i)\mathbf{c}$, lying on the same coray as V_p. In particular, $V_i \oplus W_i \in \overline{\mathcal{O}(V_p)}$ in $\mathrm{mod}_\Lambda(\mathbf{d}')$. On the other hand, it follows from the first part of our proof that the set of all modules in $\mathrm{mod}_\Lambda(\mathbf{d}')$ of the form $V_i \oplus W$, for all modules W in $\mathcal{T}'$ with $\mathbf{dim}\, W = (p-i)\mathbf{c}$, is irreducible. Further, the modules V_p, $V_i \oplus W_i$, $0 \leq i < p$, are nonsingular, because they have projective dimension at most one. Invoking now Proposition 4.7 we conclude that there exists an irreducible component of $\mathrm{mod}_\Lambda(\mathbf{d}')$ containing all maximal $G(\mathbf{d}')$-orbits. Consequently, $\mathrm{mod}_\Lambda(\mathbf{d}')$ is irreducible.

4.11. We now return to our considerations concerning the tubular family $\mathcal{T}$ over a tame concealed or tubular algebra A. Applying some results by Bongartz [6] and Lenzing–de la Peña [22] we are now able to prove the following fact.

Proposition. *The variety* $\mathrm{mod}_A(\mathbf{d})$ *is irreducible.*

Proof. It follows from [22, Section 5] that there exists a canonical (tame concealed or tubular) algebra Λ of the same tubular type as A and a tilting Λ-module T such that $A = \mathrm{End}_\Lambda(T)$, $\mathcal{T}'$ is contained in the torsion part $\mathcal{G}_\Lambda$ of mod_Λ defined by the vanishing of $\mathrm{Ext}^1_\Lambda(T, -)$, $\mathcal{T}$ is contained in the torsion-free part $\mathcal{Y}_A$ of mod_A defined by the vanishing of $\mathrm{Tor}_1^A(T, -)$, and $\mathcal{T}$ is the image of $\mathcal{T}'$ via the functor $F = \mathrm{Hom}_\Lambda(T, -)$.

Let M' be an indecomposable Λ-module in $\mathcal{T}'$ such that $\mathbf{dim}\, F(M') = \mathbf{d}$, and let $\mathbf{d}' = \mathbf{dim}\, M'$. Denote by $\mathcal{G}_\Lambda(\mathbf{d}')$ the open subset of $\mathrm{mod}_\Lambda(\mathbf{d}')$ consisting of all torsion modules and by $\mathcal{Y}_A(\mathbf{d})$ the open subset of $\mathrm{mod}_A(\mathbf{d})$ consisting of torsion-free modules. It follows from [6, Section 4] that there is an affine variety $\mathcal{Z}$ and two bundles $\varphi : \mathcal{Z} \to \mathcal{G}_A(\mathbf{d}')$ and $\psi : \mathcal{Z} \to \mathcal{Y}_A(\mathbf{d})$ such that, for each $X \in \mathcal{G}_A(\mathbf{d}')$, we have $\psi\varphi^{-1}(\mathcal{O}(X)) = \mathcal{O}(F(X))$. This establishes a bijection between the $G(\mathbf{d}')$-orbits in $\mathcal{G}_A(\mathbf{d}')$ and the $G(\mathbf{d})$-orbits in $\mathcal{Y}_A(\mathbf{d})$.

Denote by $\mathcal{S}_A(\mathbf{d}')$ the open sheet of $\mathrm{mod}_A(\mathbf{d}')$ and by $\mathcal{S}_A(\mathbf{d})$ the open sheet of $\mathrm{mod}_A(\mathbf{d})$. It follows from 4.8 that $\mathcal{S}_A(\mathbf{d}')$ is contained in $\mathcal{G}_A(\mathbf{d}')$ and $\mathcal{S}_A(\mathbf{d})$ is contained in $\mathcal{Y}_A(\mathbf{d})$. Moreover, $\mathcal{S}_A(\mathbf{d}')$ consists of all maximal $G(\mathbf{d}')$-orbits in $\mathrm{mod}_A(\mathbf{d}')$ and $S_A(\mathbf{d})$ consists of all maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$. Since $\mathrm{mod}_A(\mathbf{d}') = \overline{\mathcal{S}_A(\mathbf{d}')}$ is irreducible (by 4.10), the variety $\mathcal{G}_A(\mathbf{d}')$ is also irreducible, and invoking the definitions of $\mathcal{Z}$ and φ (see [6] for details) we conclude that $\mathcal{Z}$ is irreducible. Hence $\mathcal{Y}_A(\mathbf{d})$ is irreducible and consequently so is $\mathrm{mod}_A(\mathbf{d}) = \overline{\mathcal{Y}_A(\mathbf{d})}$.

4.12. Our final aim in this section is to prove that the variety $\mathrm{mod}_A(\mathbf{d})$ is normal. Since $\mathrm{mod}_A(\mathbf{d})$ is a complete intersection, it is enough to show that $\mathrm{mod}_A(\mathbf{d})$ is nonsingular in codimension one (see 3.4). We also note that since $\mathrm{mod}_A(\mathbf{d})$ is irreducible, the normality of $\mathrm{mod}_A(\mathbf{d})$, as defined in 3.4, is equivalent to the fact that the coordinate ring of $\mathrm{mod}_A(\mathbf{d})$ is integrally closed in its quotient field.

We need some preliminary observations and notations. We may decompose each module X in $\mathrm{mod}_A(\mathbf{d})$ into a direct sum $X = M \oplus N$, where N is a direct sum of indecomposable modules from tubes of rank 1 (homogeneous modules) and M has no homogeneous direct summand. We know by [28, 4.3, 5.2] that there are only finitely many (up to isomorphism) such direct summands M of X. We define $\mathcal{M}(\mathbf{d})$ to be the (finite) set of (isomorphism classes) of A-modules M without homogeneous direct summands such that $M \oplus N \in \mathrm{mod}_A(\mathbf{d})$ for some A-module N which is a direct sum of homogeneous modules.

For each $M \in \mathcal{M}(\mathbf{d})$ consider the set $\mathcal{W}_M(\mathbf{d})$ consisting of all modules $L \oplus N \in \mathrm{mod}_A(\mathbf{d})$ such that $L \simeq M$ and N is a direct sum of homogeneous modules. It follows from 4.6 and 4.11 that $\mathcal{W}_M(\mathbf{d})$ is irreducible. Moreover, $\mathcal{W}_M(\mathbf{d})$, $M \in \mathcal{M}(\mathbf{d})$, are $G(\mathbf{d})$-invariant subsets of $\mathrm{mod}_A(\mathbf{d})$ whose union is $\mathrm{mod}_A(\mathbf{d})$. Finally, for an A-module Z, denote by $\mu(Z)$ the number of homogeneous direct summands (including the multiplicities) of Z. Then $\dim \mathcal{W}_M(\mathbf{d})$ is the maximum of $\dim \mathcal{O}(Z)+\mu(Z)$ for all modules $Z \in \mathcal{W}_M(\mathbf{d})$.

4.13. We first consider the case $\chi(\mathbf{d}) = 0$.

LEMMA. *Assume* $\chi(\mathbf{d}) = 0$. *Then* $\mathrm{mod}_A(\mathbf{d})$ *is normal.*

Proof. As $0=\chi(\mathbf{d})=\dim G(\mathbf{d})-a(\mathbf{d})$, applying 4.8 we get $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d}) = \dim G(\mathbf{d})$. In order to prove that $\mathrm{mod}_A(\mathbf{d})$ is nonsingular in codimension one, it is enough to show that if a set $\mathcal{W}_M(\mathbf{d})$, $M \in \mathcal{M}(\mathbf{d})$, contains a singular module then $\dim \mathcal{W}_M(\mathbf{d}) \le \dim G(\mathbf{d}) - 2$.

Suppose $L \oplus N \in \mathcal{W}_M(\mathbf{d})$, with $L \simeq M$, is a singular point of $\operatorname{mod}_A(\mathbf{d})$. Invoking 3.2 we then have $\operatorname{Ext}^2_A(L \oplus N, L \oplus N) \ne 0$. Since N is a direct sum of homogeneous modules, it follows from 4.1 and 4.2 that $\operatorname{pd}_A N = 1 = \operatorname{id}_A N$. Hence $\operatorname{Ext}^2_A(L, L) \ne 0$. Then there exist indecomposable direct summands L_1, L_2 of L such that $\operatorname{Ext}^2_A(L_2, L_1) \ne 0$. Clearly, $\operatorname{id}_A L_1 = 2$, $\operatorname{pd}_A L_2 = 2$, and so $L_1 \in \mathcal{P}$, $L_2 \in \mathcal{Q}$. In particular, $L_1 \not\simeq L_2$. Therefore, $\dim_K \operatorname{End}_A(L \oplus N) \ge 2 + \mu(N)$, and so

$$\dim \mathcal{O}(L \oplus N) = \dim G(\mathbf{d}) - \dim_K \operatorname{End}_A(L \oplus N) \le G(\mathbf{d}) - 2 - \mu(N).$$

But then $\dim \mathcal{W}_M(\mathbf{d}) \le \dim G(\mathbf{d}) - 2 = \dim \operatorname{mod}_A(\mathbf{d}) - 2$, and this finishes the proof.

4.14. Our next step is to prove the following fact.

Proposition. *Assume $\chi(\mathbf{d}) = 1$ and A is tame concealed. Then the variety $\operatorname{mod}_A(\mathbf{d})$ is normal.*

Proof. As $1 = \chi(\mathbf{d}) = \dim G(\mathbf{d}) - a(\mathbf{d})$, applying 4.8 we get $\dim \operatorname{mod}_A(\mathbf{d}) = a(\mathbf{d}) = \dim G(\mathbf{d}) - 1$. Hence, in order to prove that $\operatorname{mod}_A(\mathbf{d})$ is nonsingular in codimension one, it is enough to show that if $\mathcal{W}_M(\mathbf{d})$, for some $M \in \mathcal{M}(\mathbf{d})$, contains a singular module, then $\dim \mathcal{W}_M(\mathbf{d}) \le \dim G(\mathbf{d}) - 3$.

Since A is tame concealed, by 4.1 we have $\Gamma_A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$, where $\mathcal{P}$ is a preprojective component containing all indecomposable projective modules and Q is a preinjective component containing all indecomposable injective modules. Moreover, for any homogeneous module $H \in \mathcal{T}$, $X \in \mathcal{P}$, $Y \in \mathcal{Q}$, we have $\operatorname{Hom}_A(X, H) \ne 0$ and $\operatorname{Hom}_A(H, Y) \ne 0$. Since $\chi(\mathbf{d}) = 1$, $\mathbf{d}$ is of the form $\mathbf{d} = p\mathbf{h} + \mathbf{e}$ with $0 < \mathbf{e} < \mathbf{h}$. Then it follows from 4.7 that all maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$ have dimension $\dim G(\mathbf{d}) - (p + 1)$.

Suppose now that $L \oplus N \in \mathcal{W}_M(\mathbf{d})$, $M \in \mathcal{M}(\mathbf{d})$, $L \simeq M$, is a singular point of $\operatorname{mod}_A(\mathbf{d})$. Then $\operatorname{Ext}^2_A(L, L) = \operatorname{Ext}^2_A(L \oplus N, L \oplus N) \ne 0$, and there exist indecomposable direct summands L_1, L_2 of L with $\operatorname{Ext}^2_A(L_2, L_1) \ne 0$. Clearly, $\operatorname{id}_A L_1 = 2$, $\operatorname{pd}_A L_2 = 2$, and hence $L_1 \in \mathcal{P}$ and $L_2 \in \mathcal{Q}$.

Assume $N \ne 0$. Then $\operatorname{Hom}_A(L_1, N) \ne 0$, $\operatorname{Hom}_A(N, L_2) \ne 0$, and so we have

$$\begin{aligned}\dim \mathcal{O}(L \oplus N) &= \dim G(\mathbf{d}) - \dim_K \operatorname{End}_A(L \oplus N)\\ &\le \dim G(\mathbf{d}) - 4 - \mu(N) \le (\dim G(\mathbf{d}) - 1) - 3 - \mu(N).\end{aligned}$$

Thus $\dim \mathcal{W}_M(\mathbf{d}) \le \dim \operatorname{mod}_A(\mathbf{d}) - 3$.

Let $N = 0$. In such a case $\mathcal{W}_M(\mathbf{d}) = \mathcal{O}(M)$ and $\dim \mathcal{O}(M) < \dim G(\mathbf{d}) - (p + 1)$, because $\mathcal{O}(M)$ is not maximal (see 4.7). Hence, if $p \ge 1$, we have $\dim \mathcal{W}_M(\mathbf{d}) \le \dim G(\mathbf{d}) - p - 2 \le \operatorname{mod}_A(\mathbf{d}) - 2$.

Suppose $p = 0$. If $\dim_K \mathrm{End}_A(M) \geq 3$, then $\dim \mathcal{O}(M) \leq \dim G(\mathbf{d}) - 3 \leq \dim \mathrm{mod}_A(\mathbf{d}) - 2$, and there is nothing to show. Assume $\dim \mathrm{End}_A(M) = 2$. Then $M = L_1 \oplus L_2$ and $\mathrm{End}_A(L_1) = K$, $\mathrm{End}_A(L_2) = K$ and $\mathrm{Hom}_A(L_1, L_2) = 0$, $\mathrm{Hom}_A(L_2, L_1) = 0$. Further, since $L_1 \in \mathcal{P}$ and $L_2 \in \mathcal{Q}$, we also have $\mathrm{Ext}^1_A(L_1, L_1) = 0$ and $\mathrm{Ext}^1_A(L_2, L_2) = 0$. Moreover, $\dim \mathcal{O}(M) = \dim G(\mathbf{d}) - 2 = \dim \mathrm{mod}_A(\mathbf{d}) - 1$. On the other hand, it follows from the description of maximal orbits in $\mathrm{mod}_A(\mathbf{d})$, given in 4.7, that $\mathrm{mod}_A(\mathbf{d}) = \overline{\mathcal{O}(V)}$ for the unique indecomposable A-module V with $\mathbf{dim}\, V = \mathbf{d} = \mathbf{e}$ (because $p = 0$). Hence L is a minimal degeneration of V, and consequently applying 3.5 we get a nonsplittable exact sequence

$$0 \to L_1 \to V \to L_2 \to 0$$

with $L = L_1 \oplus L_2$. Consider now the induced exact sequence

$$\mathrm{Ext}^1_A(L_1, L_1) \to \mathrm{Ext}^2_A(L_2, L_1) \to \mathrm{Ext}^2_A(V, L_1)$$

induced by the above short exact sequence. Then $\mathrm{Ext}^2_A(V, L_1) = 0$ because $V \in \mathcal{T}$ implies $\mathrm{pd}_A V = 1$, and, as we have seen above, also $\mathrm{Ext}^1_A(L_1, L_1) = 0$. Hence $\mathrm{Ext}^2_A(L_2, L_1) = 0$, a contradiction. Therefore, the set of singular points in $\mathrm{mod}_A(\mathbf{d})$ is of codimension at least 2.

4.15. Our final aim of this section is to prove the following fact which completes the proof that $\mathrm{mod}_A(\mathbf{d})$ is normal.

PROPOSITION. *Assume* $\chi(\mathbf{d}) = 1$ *and* A *is tubular. Then* $\mathrm{mod}_A(\mathbf{d})$ *is normal.*

Proof. Since $\chi(\mathbf{d}) = 1$, we again have $\dim \mathrm{mod}_A(\mathbf{d}) = \dim G(\mathbf{d}) - 1$. Further, by 4.2, Γ_A is of the form

$$\Gamma_A = \mathcal{P}_0 \vee \mathcal{T}_0 \vee \Big(\bigvee_{q \in \mathbb{Q}^+} \mathcal{T}_q \Big) \vee \mathcal{T}_\infty \vee \mathcal{Q}_\infty$$

and, by our assumption 4.5, $\mathcal{T} = \mathcal{T}_q$ for some $q \in \mathbb{Q}^+$. Moreover, $\mathbf{d} = p\mathbf{h} + \mathbf{e}$ with $0 < \mathbf{e} < \mathbf{h}$, and all maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$ have dimension $\dim G(\mathbf{d}) - (p+1) = \dim \mathrm{mod}_A(\mathbf{d}) - p$. Assume that $M \in \mathcal{M}(\mathbf{d})$, and $L \oplus N \in \mathcal{W}_M(\mathbf{d})$ with $L \simeq M$ is a singular point of $\mathrm{mod}_A(\mathbf{d})$. Since $\mathrm{pd}_A N = \mathrm{id}_A N = 1$, there are indecomposable direct summands L_1 and L_2 of L with $\mathrm{Ext}^2_A(L_2, L_1) \neq 0$. Then $\mathrm{id}_A L_1 = 2$, $\mathrm{pd}_A L_2 = 2$, and consequently L_1 belongs either to $\mathcal{P}_0$ or to a tube of $\mathcal{T}_0$ with a projective module, and L_2 belongs either to $\mathcal{Q}_\infty$ or to a tube of $\mathcal{T}_\infty$ containing an injective module.

Suppose now that $L \oplus N = U_1 \oplus U_2 \oplus W$ for some indecomposable direct summands U_1 and U_2 with $\mathrm{Hom}_A(U_1, U_2) \neq 0$ or $\dim_K \mathrm{End}_A(L) \geq 3$. Then

$$\begin{aligned}\dim \mathcal{O}(L \oplus N) &= \dim G(\mathbf{d}) - \dim_K \mathrm{End}_A(L \oplus N) \\ &\leq \dim G(\mathbf{d}) - 3 - \mu(N) \leq \dim \mathrm{mod}_A(\mathbf{d}) - 2 - \mu(N),\end{aligned}$$

and hence $\dim \mathcal{W}_M(\mathbf{d}) \le \dim \operatorname{mod}_A(\mathbf{d}) - 2$. Therefore, we may assume that $L \oplus N$ is a direct sum of pairwise orthogonal indecomposable modules, and $L = L_1 \oplus L_2$ with $\dim_K \operatorname{End}_A(L) = 2$.

We claim that $N = 0$. Suppose $N \neq 0$. Observe that N has no indecomposable direct summands from a family $\mathcal{T}_r$, $r \in \mathbb{Q}^+$, because otherwise $\operatorname{Hom}_A(L_1, N) \neq 0$ and $\operatorname{Hom}_A(N, L_2) \neq 0$, by the separation property of $\mathcal{T}_r$. Next assume that N admits two indecomposable direct summands $N_1 \in \mathcal{T}_0$ and $N_2 \in \mathcal{T}_\infty$. Then the supports C_0 of N_1 and C_∞ of N_2 are unique tame concealed convex subcategories of A, and A is a tubular extension of C_0 (respectively, coextension of C_∞). In particular, C_0 and C_∞ have a common object x. Then there is an indecomposable projective module $P \in \mathcal{P}_0$ and a nonzero map $P \to N_2$ which factorizes through any tube of $\mathcal{T}_0$. Hence $\operatorname{Hom}_A(N_1, N_2) \neq 0$, which contradicts our assumption on $L \oplus N$.

Therefore, by symmetry, we may assume that L_1 belongs to a ray tube of $\mathcal{T}_0$ containing a projective module, the indecomposable direct summands of N lie in homogeneous tubes of $\mathcal{T}_0$, and L_2 is a module in $T_\infty \vee Q_\infty$ whose restriction to C_0 is zero. We shall prove that in fact $N = 0$. Take a module H lying on the mouth of a homogeneous tube of $\mathcal{T}_0$ containing no direct summand of N. Then

$$\begin{aligned}
&\langle \mathbf{d}, \mathbf{dim}\, H\rangle \\
&\quad = \langle \mathbf{dim}(L_1 \oplus N) + \mathbf{dim}\, L_2, \mathbf{dim}\, H\rangle \\
&\quad = \langle \mathbf{dim}(L_1 \oplus N), \mathbf{dim}\, H\rangle + \langle \mathbf{dim}\, L_2, \mathbf{dim}\, H\rangle \\
&\quad = \langle \mathbf{dim}\, L_2, \mathbf{dim}\, H\rangle \\
&\quad = \dim_K \operatorname{Hom}_A(L_2, H) - \dim_K \operatorname{Ext}^1_A(L_2, H) + \dim_K \operatorname{Ext}^2_A(L_2, H) \\
&\quad = -\dim_K \operatorname{Ext}^1_A(L_2, H) = -\dim_K \operatorname{Hom}_A(\tau_A^- H, L_2) \\
&\quad = -\dim_K \operatorname{Hom}_A(H, L_2) = 0.
\end{aligned}$$

Moreover, $\mathbf{d} = p\mathbf{h} + \mathbf{e}$, $0 < \mathbf{e} < \mathbf{h}$, where $\mathbf{e} = \mathbf{dim}\, Z$ for an indecomposable module $Z \in \mathcal{T}$. We also know that $\operatorname{Hom}_A(Z, H) = 0$, $\operatorname{pd}_A Z = 1$, and $\langle \mathbf{h}, \mathbf{dim}\, H\rangle < 0$, by 4.5. Finally, if $p = 0$, then Z is a sincere A-module and so $\operatorname{Hom}_A(H, Z) \neq 0$, or equivalently, $\langle \mathbf{dim}\, Z, \mathbf{dim}\, H\rangle = -\dim_K \operatorname{Ext}^1_A(H, Z) = -\dim_K \operatorname{Hom}_A(Z, H) < 0$. Altogether this gives

$$\begin{aligned}
\langle \mathbf{d}, \mathbf{dim}\, H\rangle &= p\langle \mathbf{h}, \mathbf{dim}\, H\rangle + \langle \mathbf{e}, \mathbf{dim}\, H\rangle \\
&= p\langle \mathbf{h}, \mathbf{dim}\, H\rangle + \langle \mathbf{dim}\, Z, \mathbf{dim}\, H\rangle < 0,
\end{aligned}$$

a contradiction.

Therefore $N = 0$, and $\mathcal{W}_M(\mathbf{d}) = \mathcal{O}(M)$. If $p \ge 1$ then

$$\dim \mathcal{W}_M(\mathbf{d}) = \dim \mathcal{O}(M) \le \dim \operatorname{mod}_A(\mathbf{d}) - p - 1 \le \dim \operatorname{mod}_A(\mathbf{d}) - 2,$$

because the orbit $\mathcal{O}(M)$ is not maximal.

Assume $p = 0$. We claim that $\operatorname{Ext}^1_A(L_1, L_1) = 0$. Indeed, suppose it is not the case. Since $\operatorname{End}_A(L_1) = K$ and $\operatorname{pd}_A L_1 \le 1$, we then have

$$\begin{aligned}\chi(\mathbf{dim}\, L_1) &= \dim_K \operatorname{End}_A(L_1) - \dim_K \operatorname{Ext}^1_A(L_1, L_1) + \dim_K \operatorname{Ext}^2_A(L_1, L_1)\\ &= 1 - \dim_K \operatorname{Ext}^1_A(L_1, L_1),\end{aligned}$$

so $\chi(\mathbf{dim}\, L_1) = 0$. Hence L_1 is an isotropic C_0-module and

$$\begin{aligned}\langle \mathbf{dim}\, L_1, \mathbf{d}\rangle &= \langle \mathbf{dim}\, L_1, \mathbf{dim}\, Z\rangle\\ &= \dim_K \operatorname{Hom}_A(L_1, Z)\\ &= \dim_K \operatorname{Hom}_{C_0}(L_1, Z|_{C_0}) > 0,\end{aligned}$$

because $Z|_{C_0} \neq 0$ and is a direct sum of indecomposable preinjective C_0-modules. But then we get

$$\begin{aligned}\dim_K \operatorname{Hom}_A(L_1, L_2) &= \langle \mathbf{dim}\, L_1, \mathbf{dim}\, L_2\rangle\\ &= \langle \mathbf{dim}\, L_1, \mathbf{dim}\, L_1 + \mathbf{dim}\, L_2\rangle\\ &= \langle \mathbf{dim}\, L_1, \mathbf{d}\rangle > 0,\end{aligned}$$

contrary to our assumption that L_1 and L_2 are orthogonal.

Therefore, $\operatorname{Ext}^1_A(L_1, L_1) = 0$. Repeating now the arguments applied in the final part of 4.14 we conclude that $\operatorname{Ext}^2_A(L_2, L_1) = 0$, again a contradiction. This finishes the proof.

5. Geometry of nonstable tubes. Let A be tame quasi-tilted and $\mathbf{d}$ the dimension-vector of a nonstable nondirecting indecomposable A-module. We shall describe the maximal $G(\mathbf{d})$-orbits in $\operatorname{mod}_A(\mathbf{d})$, the irreducible components of $\operatorname{mod}_A(\mathbf{d})$, discuss the normality of $\operatorname{mod}_A(\mathbf{d})$, and prove that $\operatorname{mod}_A(\mathbf{d})$ is a complete intersection.

5.1. We may assume that A is either tubular or tilted of Euclidean type and $\mathbf{d} = \mathbf{dim}\, X$ for an indecomposable nondirecting A-module X lying in a tube of Γ_A containing a projective module. Moreover, invoking [28, Section 4] (see also [1]) we may assume that additionally the following hold:

(1) There exist a convex tame concealed subcategory C of A, a convex linear category $\Lambda = K\Delta$ (Δ is a quiver of type $\mathbb{A}_n$) and an indecomposable C-module R lying on the mouth of a stable tube $\Gamma^{(0)}$ of Γ_C such that C is obtained from the one-point extension $C[R]$ by identifying its extension vertex with one of the ends of Δ.

(2) The restriction $X|_C$ of X to C is an indecomposable C-module lying on the ray of $\Gamma^{(0)}$ starting at R while the restriction $X|_\Lambda$ of X to Λ is a unique indecomposable sincere Λ-module.

(3) The Auslander–Reiten quiver Γ_A is of the form

$$\Gamma_A = \mathcal{P} \vee \mathcal{T} \vee \mathcal{Q}$$

where $\mathcal{P}$ is the preprojective component of Γ_C, $\mathcal{T}=(\mathcal{T}^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ is a $\mathbb{P}_1(K)$-family of tubes obtained from the unique $\mathbb{P}_1(K)$-family $\Gamma=(\Gamma^{(\lambda)})_{\lambda\in\mathbb{P}_1(K)}$ of stable tubes of Γ_C by ray insertions in the tube $\Gamma^{(0)}$ containing R, and $\mathcal{Q}$ is either a preinjective component (if A is tilted of Euclidean type) or is of the form $\mathcal{Q}=(\bigvee_{q\in\mathbb{Q}^+}\mathcal{T}_q)\vee\mathcal{T}_\infty\vee\mathcal{Q}_\infty$ (if A is tubular), in the notation 4.2. Hence, $\mathcal{T}^{(\lambda)}=\Gamma^{(\lambda)}$ for any $\lambda\in\mathbb{P}_1(K)=K\cup\{\infty\}$ different from 0.

We denote by Σ the unique ray of $\mathcal{T}^{(0)}$ starting from R. Then for any nondirecting indecomposable A-module Z in $\mathcal{T}^{(0)}$ with $Z|_\Lambda\neq 0$, the module $Z|_C$ is indecomposable and lies on Σ. Denote by $\mathbf{h}$ the dimension-vector of a module lying on the mouth of a homogeneous tube of $\mathcal{T}$ (equivalently, of Γ). Then $\mathbf{d}=p\mathbf{h}+\mathbf{e}$ with $\mathbf{d}|_C\neq 0$ and $\mathbf{e}\neq 0$ such that $0\le \mathbf{e}|_C<\mathbf{h}$ and $\mathbf{e}|_\Lambda$ having 1 at each vertex of Δ. Moreover, $\chi_A(\mathbf{d})=1$.

5.2. The following proposition describes the maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$.

PROPOSITION. *Let M be a module in* $\mathrm{mod}_A(\mathbf{d})$. *Then the $G(\mathbf{d})$-orbit $\mathcal{O}(M)$ is maximal in* $\mathrm{mod}_A(\mathbf{d})$ *if and only if $M=M_1\oplus\ldots\oplus M_t$, $t\ge 1$, for some pairwise orthogonal indecomposable A-modules $M_1,\ldots,M_t$ such that $M_2,\ldots,M_t$ (if $t\ge 2$) are isotropic modules from $\mathcal{T}$, and either M_1 lies in the nonstable tube of $\mathcal{T}^{(0)}$ or $M_1|_C=0$, $t\ge 2$, and Σ does not contain any of the modules $M_2,\ldots,M_t$.*

Proof. We know from 3.6 that the orbit $\mathcal{O}(M)$ is maximal if and only if $\mathrm{Ext}^1_A(M',M'')=0$ for any decomposition $M=M'\oplus M''$ of M. Assume $\mathcal{O}(M)$ is maximal. Applying the same arguments as in 4.7 we conclude that $M=M_1\oplus\ldots\oplus M_t$, where $M_1,\ldots,M_t$ are indecomposable A-modules with $\chi(\mathbf{dim}\,M_1)=1$, $\chi(\mathbf{dim}\,M_i)=0$ for $2\le i\le t$ (if $t\ge 2$), $\langle\mathbf{dim}\,M_i,\mathbf{dim}\,M_j\rangle=0$ for all $i\neq j$ from $\{1,\ldots,t\}$, and $\mathbf{dim}\,M_1=p_1\mathbf{h}+\mathbf{e}$, $\mathbf{dim}\,M_i=p_i\mathbf{h}$, $2\le i\le t$, with $p_1\ge 0$, $p_2\ge 1,\ldots,p_t\ge 1$ such that $p=p_1+\ldots+p_t$. Moreover, the modules $M_1,\ldots,M_t$ are pairwise orthogonal, and clearly $M_2,\ldots,M_t$ lie in $\mathcal{T}$.

Further, since $M|_\Lambda=M_1|_\Lambda$ is a sincere indecomposable Λ-module we conclude that M_1 lies in $\mathcal{T}$ (and then in the nonstable tube $\mathcal{T}^{(0)}$) if and only if $M_1|_C\neq 0$. Assume $M_1|_C=0$. Hence $M_1=M|_\Lambda$ lies in $\mathcal{Q}$. Observe also that then $\mathrm{Ext}^1_A(M_1,Z)\neq 0$ for any module Z lying on the ray Σ, and so $M_2,\ldots,M_t$ do not lie on Σ.

We now determine $\dim\mathcal{O}(M)=\dim G(\mathbf{d})-\dim_K\mathrm{End}_A(M)$. We have $\dim_K\mathrm{End}_A(M_1)=p_1+1$ if $\mathbf{dim}\,M_1|_C\neq p_1\mathbf{h}$ with $p_1\ge 1$, and $\dim_K\mathrm{End}_A(M_1)=p_1$ if $\mathbf{dim}\,M_1|_C=p_1\mathbf{h}>0$. Moreover, $\dim_K\mathrm{End}_A(M_i)=p_i$ for $2\le i\le t$. Therefore, we get $\dim_K\mathrm{End}_A(M)=p+1$ if $\mathbf{d}|_C\neq p\mathbf{h}$ or $M_1|_C=0$, and $\dim_K\mathrm{End}_A(M)=p$ if $\mathbf{d}|_C=p\mathbf{h}$ and $M_1|_C\neq 0$.

Hence, if $\mathbf{d}|_C \neq p\mathbf{h}$ then all maximal $G(\mathbf{d})$-orbits in $\mathrm{mod}_A(\mathbf{d})$ have the same dimension $\dim G(\mathbf{d}) - (p+1)$, and their union is the open sheet of $\mathrm{mod}_A(\mathbf{d})$. Moreover, in this case, if $M = M_1 \oplus \ldots \oplus M_t$ with $M_1, \ldots, M_t$ pairwise orthogonal indecomposable lying in $\mathcal{T}$, $M_2, \ldots, M_t$ (if $t \geq 2$) isotropic, and M_1 lying in $\mathcal{T}^{(0)}$, we have $\dim_K \mathrm{End}_A(M) = p+1$, and so $\mathcal{O}(M)$ is a maximal $G(\mathbf{d})$-orbit $\mathrm{mod}_A(\mathbf{d})$.

Assume now that $\mathbf{d}|_C = p\mathbf{h}$. Note that $\mathbf{d}|_C \neq 0$ because $X|_C \neq 0$. It follows from the above discussion that the open sheet of $\mathrm{mod}_A(\mathbf{d})$ is the union of all $G(\mathbf{d})$-orbits of the form $\mathcal{O}(M)$, where $M = M_1 \oplus \ldots \oplus M_t$ with $M_1, \ldots, M_t$ indecomposable A-modules from $\mathcal{T}$ such that M_1 belongs to $\mathcal{T}^{(0)}$ and $M_2, \ldots, M_t$ are isotropic modules lying in pairwise different tubes of $\mathcal{T}$ and different from $\mathcal{T}^{(0)}$.

Conversely, if $M = M_1 \oplus \ldots \oplus M_t$ for $M_1, \ldots, M_t$ pairwise orthogonal indecomposable A-modules such that $M_1|_C = 0, M_2, \ldots, M_t$ are isotropic modules from $\mathcal{T}$ and do not lie on the ray Σ, then $\mathrm{Ext}^1_A(M_i, M_j) = 0$ for all $i \neq j$ from $\{1, \ldots, t\}$, and $\mathcal{O}(M)$ is maximal by 3.6. Clearly, such an orbit $\mathcal{O}(M)$ has dimension $\dim G(\mathbf{d}) - (p+1)$ and hence is not contained in the open sheet of $\mathrm{mod}_A(\mathbf{d})$.

5.3. We may also state the following consequence of the above considerations.

PROPOSITION. *The variety* $\mathrm{mod}_A(\mathbf{d})$ *is a complete intersection and has dimension* $a(\mathbf{d})$. *Moreover*, *the maximal* $G(\mathbf{d})$-*orbits consist of nonsingular modules.*

Proof. Let $\mathcal{O}(M)$ be a maximal $G(\mathbf{d})$-orbit and $M = M_1 \oplus \ldots \oplus M_t$ be a decomposition of M into a direct sum of indecomposable modules. Then for each pair $i, j \in \{1, \ldots, t\}$ we have

$$0 = \langle \mathbf{dim}\, M_i, \mathbf{dim}\, M_j \rangle = \dim_K \mathrm{Hom}_A(M_i, M_j) + \dim_K \mathrm{Ext}^2_A(M_i, M_j)$$

(see 4.7), and so $\mathrm{Ext}^2_A(M_i, M_j) = 0$. Therefore, $\mathrm{Ext}^2_A(M, M) = 0$ and M is a nonsingular point of $\mathrm{mod}_A(\mathbf{d})$, by 3.2. Applying 3.3 we then conclude that $\mathrm{mod}_A(\mathbf{d})$ is a complete intersection and $\dim \mathrm{mod}_A(\mathbf{d}) = a(\mathbf{d})$.

5.4. Our next aim is to prove the following fact.

PROPOSITION. *The variety* $\mathrm{mod}_A(\mathbf{d})$ *has at most two irreducible components. Moreover*, $\mathrm{mod}_A(\mathbf{d})$ *is irreducible if and only if* $\mathbf{d}|_C \neq p\mathbf{h}$ *for any* $p \geq 1$.

Proof. Denote by $\mathcal{X}$ the set of all modules in $\mathrm{mod}_A(\mathbf{d})$ of the form $M_1 \oplus \ldots \oplus M_t$ where $M_1, \ldots, M_t$ are pairwise orthogonal indecomposable A-modules from $\mathcal{T}$, $M_2, \ldots, M_t$ are isotropic, and M_1 lies in $\mathcal{T}^{(0)}$ and is not isotropic. Applying arguments as in 4.10 we infer that $\mathcal{X}$ is irreducible. Moreover, it follows from 5.2 that $\mathcal{X}$ contains the open sheet of $\mathrm{mod}_A(\mathbf{d})$.

Let $\mathcal{Z}_1$ be the closure of $\mathcal{X}$ in $\operatorname{mod}_A(\mathbf{d})$. Clearly, $\mathcal{Z}_1$ is irreducible of dimension $a(\mathbf{d})$, and so is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$. Moreover, if $\mathbf{d}|_C \neq p\mathbf{h}$ for any $p \geq 1$, then $\mathcal{Z}_1 = \operatorname{mod}_A(\mathbf{d})$ and consequently $\operatorname{mod}_A(\mathbf{d})$ is irreducible.

Assume now that $\mathbf{d}|_C = p\mathbf{h}$ for some $p \geq 1$. Denote by $\mathcal{Z}_2$ the set of all modules (representations) M in $\operatorname{mod}_A(\mathbf{d})$ such that $M(\alpha) = 0$ for any arrow connecting the extension vertex of $C[R]$ with C. Obviously, $\mathcal{Z}_1$ is not contained in $\mathcal{Z}_2$.

We claim that $\mathcal{Z}_2$ is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$. Indeed, $\mathcal{Z}_2$ consists of all modules in $\operatorname{mod}_A(\mathbf{d})$ of the form $U \oplus V$, where U is a C-module with $\operatorname{\mathbf{dim}} U = p\mathbf{h}$ and V is a Λ-module with $\operatorname{\mathbf{dim}} V = \mathbf{d} - p\mathbf{h} = \mathbf{e}$. We know that $\mathbf{e}$ is the dimension-vector of a unique sincere indecomposable Λ-module N, and then $\operatorname{mod}_\Lambda(\mathbf{e})$ is the closure of $\mathcal{O}(N)$. Hence $\operatorname{mod}_\Lambda(\mathbf{e})$ is irreducible. Applying 4.11 we conclude that $\operatorname{mod}_C(p\mathbf{h})$ is irreducible. Therefore, $\mathcal{Z}_2$ is also irreducible as a product of $\operatorname{mod}_C(p\mathbf{h})$ and $\operatorname{mod}_\Lambda(\mathbf{e})$. Moreover, $\dim \mathcal{Z}_2 = \dim \operatorname{mod}_A(p\mathbf{h}) + \dim \operatorname{mod}_\Lambda(\mathbf{e}) = a(\mathbf{d})$, because $\mathbf{d} = p\mathbf{h} + \mathbf{e}$ and the supports of $p\mathbf{h}$ and $\mathbf{e}$ are disjoint. This shows that $\mathcal{Z}_2$ is an irreducible component of $\operatorname{mod}_A(\mathbf{d})$. Finally, observe that every maximal $G(\mathbf{d})$-orbit of $\operatorname{mod}_A(\mathbf{d})$ is contained in $\mathcal{Z}_1$ or $\mathcal{Z}_2$, and hence $\operatorname{mod}_A(\mathbf{d}) = \mathcal{Z}_1 \cup \mathcal{Z}_2$. Therefore, $\operatorname{mod}_A(\mathbf{d})$ has exactly two irreducible components $\mathcal{Z}_1$ and $\mathcal{Z}_2$. This finishes the proof.

5.5. The final aim of this section is to prove the following.

PROPOSITION. *The variety* $\operatorname{mod}_A(\mathbf{d})$ *is normal if and only if* $\mathbf{d}|_C \neq p\mathbf{h}$ *for any* $p \geq 1$.

Proof. Assume first that $\mathbf{d}|_C = p\mathbf{h}$ for some $p \geq 1$. We claim that then $\operatorname{mod}_A(\mathbf{d})$ is not normal. We know from 5.4 that $\operatorname{mod}_A(\mathbf{d})$ has two irreducible components $\mathcal{Z}_1$ and $\mathcal{Z}_2$. It is well known that $\mathcal{Z}_1 \cap \mathcal{Z}_2$ consists of singular modules. Hence, in order to prove that $\operatorname{mod}_A(\mathbf{d})$ is not normal it is enough (see 3.4) to find an irreducible subset of $\mathcal{Z}_1 \cap \mathcal{Z}_2$ of dimension $a(\mathbf{d}) - 1$.

Consider the set $\mathcal{Z}$ of all modules in $\operatorname{mod}_A(\mathbf{d})$ of the form $N = N_0 \oplus N_1 \oplus \ldots \oplus N_p$, where N_0 is a unique indecomposable A-module with $N_0|_C = 0$ and $N|_\Lambda = N_0|_\Lambda$, N_1 is a unique indecomposable A-module of dimension-vector $\mathbf{h}$ lying on the ray Σ of $\mathcal{T}^{(0)}$ starting at R, and $N_2, \ldots, N_p$ (if $p \geq 2$) are modules lying on the mouth of pairwise different homogeneous tubes of $\mathcal{T}$. Observe that $\mathcal{Z}$ is contained in $\mathcal{Z}_2$. Further, there exists a short exact sequence

$$0 \to N_1 \to M_1 \to N_0 \to 0$$

where M_1 is a unique indecomposable A-module in $\mathcal{T}^{(0)}$ with $\operatorname{\mathbf{dim}} M_1 = \mathbf{d} - (p-1)\mathbf{h}$. Hence, any module $N = N_0 \oplus N_1 \oplus \ldots \oplus N_p$ of $\mathcal{Z}$ is a degeneration of a module $M = M_1 \oplus N_2 \oplus \ldots \oplus N_p$ from $\mathcal{Z}_1$. This shows that $\mathcal{Z}$ is contained in $\mathcal{Z}_1 \cap \mathcal{Z}_2$. On the other hand, by 4.11, $\mathcal{Z}$ is an irreducible set and $\dim \mathcal{Z} = a(\mathbf{d}) - 1$. Therefore, $\operatorname{mod}_A(\mathbf{d})$ is not normal.

Assume now that $\mathbf{d}|_C \neq p\mathbf{h}$ for any $p \geq 1$. We know from 5.4 that then $\operatorname{mod}_A(\mathbf{d})$ is irreducible. We claim that $\operatorname{mod}_A(\mathbf{d})$ is also normal. Since $\operatorname{mod}_A(\mathbf{d})$ is a complete intersection, it is enough to show that $\operatorname{mod}_A(\mathbf{d})$ is nonsingular in codimension one.

We proceed as in 4.14 and 4.15, and use notation introduced in 4.12. Assume $M \in \mathcal{M}(\mathbf{d})$ and $L \oplus N \in \mathcal{M}(\mathbf{d})$, with $L \simeq M$, is a singular point of $\operatorname{mod}_A(\mathbf{d})$. We claim that $\dim \mathcal{W}_M(\mathbf{d}) \leq \dim \operatorname{mod}_A(\mathbf{d}) - 2$. As in 4.15 we reduce the consideration to the case where $L \oplus N$ is a direct sum of pairwise orthogonal indecomposable A-modules, $L = L_1 \oplus L_2$, $\dim_K \operatorname{End}_A(L) = 2$, $\operatorname{Ext}_A^2(L_2, L_1) \neq 0$, and by definition N is a direct sum of homogeneous A-modules. Since $\operatorname{pd}_A L_1 = 2$, we have $L_1 \in \mathcal{P}$ or $L_1 \in \mathcal{T}^{(0)}$. Similarly, since $\operatorname{id}_A L_2 = 2$, we infer that L_2 belongs to a component $\mathcal{Q}$ (coray tube or preinjective) containing an injective module.

We prove that $N = 0$. Suppose it is not the case. We claim that then N is a direct sum of homogeneous modules from $\mathcal{T}$. Indeed, assume there exists an indecomposable direct summand H of N which does not belong to $\mathcal{T}$. Then

$$\langle \mathbf{d}, \mathbf{dim}\, H \rangle = \langle \mathbf{dim}\, X, \mathbf{dim}\, H \rangle = \dim_K \operatorname{Hom}_A(X, H) > 0$$

where X is a unique indecomposable module with $\mathbf{dim}\, X = \mathbf{d}$. On the other hand, $H = N \oplus W$ for some indecomposable A-module W and

$$\begin{aligned}\langle \mathbf{d}, \mathbf{dim}\, H \rangle &= \langle \mathbf{dim}(L \oplus W) + \mathbf{dim}\, H, \mathbf{dim}\, H \rangle = \langle \mathbf{dim}(L \oplus W), \mathbf{dim}\, H \rangle \\ &= \dim_K \operatorname{Hom}_A(L \oplus W, H) - \dim_K \operatorname{Ext}_A^1(L \oplus H)\end{aligned}$$

because $\chi(\mathbf{dim}\, H) = 0$ and $\operatorname{pd}_A H = 1$. Finally, we get $\operatorname{Hom}_A(L \oplus W, H) \neq 0$, which contradicts our assumption on $L \oplus N$.

Thus, N is a direct sum of homogeneous modules from $\mathcal{T}$, and consequently $L_2|_C = 0$. Take a module H lying on the mouth of a homogeneous tube of $\mathcal{T}$ containing no direct summand of N. Then

$$\begin{aligned}\langle \mathbf{dim}\, H, \mathbf{d} \rangle &= \langle \mathbf{dim}\, H, \mathbf{dim}\, L + \mathbf{dim}\, N \rangle \\ &= \dim_K \operatorname{Hom}_A(H, L_2) + \dim_K \operatorname{Hom}_A(H, N)\end{aligned}$$

because $\operatorname{pd}_A H = 1$ and $\operatorname{Hom}_A(L \oplus N, \tau_A H) = \operatorname{Hom}_A(L \oplus N, H) = 0$. On the other hand, $\mathbf{d} = \mathbf{dim}\, X$ for a unique indecomposable A-module X lying in $\mathcal{T}^{(0)}$, and so $\langle \mathbf{dim}\, H, \mathbf{d} \rangle = 0$. Hence, $\operatorname{Hom}_A(H, L_2) = 0$ and $\operatorname{Hom}_A(H, N) = 0$. In particular, $L_2|_C = 0$ and N is a direct sum of homogeneous modules from $\mathcal{T}$. We know that if $\tau_A^- L_1|_\Lambda \neq 0$, then it is an indecomposable Λ-module. Observe that $L_2 = L_2|_\Lambda$ is also an indecomposable Λ-module. Since

Λ is a hereditary algebra of type $\mathbb{A}_n$, we then get the inequality

$$\begin{aligned}\dim_K \operatorname{Ext}^1_A(L_2, L_1) &= \dim_K \mathrm{D}\, \underline{\operatorname{Hom}}_A(\tau_A^- L_1, L_2)\\ &\le \dim_K \operatorname{Hom}_\Lambda(\tau_A^- L_1|_\Lambda, L_2) \le 1.\end{aligned}$$

But then we obtain

$$\begin{aligned}1 &= \langle \mathbf{d}, \mathbf{d}\rangle = \langle \mathbf{dim}(L_1 \oplus L_2 \oplus N), \mathbf{dim}(L_1 \oplus L_2 \oplus N)\rangle\\ &= \chi(\mathbf{dim}\, L_1) + \chi(\mathbf{dim}\, L_2) - \dim_K \operatorname{Ext}^1_A(L_2, L_1) + \dim_K \operatorname{Ext}^2_A(L_2, L_1)\\ &= 2 - \dim_K \operatorname{Ext}^1_A(L_2, L_1) + \dim_K \operatorname{Ext}^2_A(L_2, L_1)\\ &\ge 1 + \dim_K \operatorname{Ext}^2_A(L_2, L_1),\end{aligned}$$

and so $\operatorname{Ext}^2_A(L_2, L_1) = 0$, a contradiction.

Hence, $N = 0$. If $\mathbf{d} = p\mathbf{h}+\mathbf{e}$ with $p \ge 1$, then $\dim \mathcal{W}_M(\mathbf{d}) = \dim \mathcal{O}(M) \le \dim \operatorname{mod}_A(\mathbf{d}) - p - 1 \le \dim \operatorname{mod}_A(\mathbf{d}) - 2$, because $\mathcal{O}(M)$ is not a maximal orbit in $\operatorname{mod}_A(\mathbf{d})$. Finally, as in 4.14, $p = 0$, $\dim_K \operatorname{End}_A(L_1 \oplus L_2) = 2$ and $\operatorname{pd}_A X \le 1$ again leads to $\operatorname{Ext}^2_A(L_2, L_1) = 0$.

This finishes the proof that $\operatorname{mod}_A(\mathbf{d})$ is nonsingular in codimension one, and consequently is normal, in the case $\mathbf{d}|_C \ne p\mathbf{h}$ for any $p \ge 1$.

6. Proofs of the main results. The aim of this section is to sum up the results of Sections 4, 5 and our paper [3] and prove the results stated in Section 1.

6.1. Theorem 1 is a direct consequence of 4.8, 4.11, 5.3, 5.4 and [3, Theorem 1].

6.2. Theorem 2 follows from 2.3, 2.4, 2.5, 4.11, 4.13, 4.14, 4.15, 5.4, 5.5 and [3, Theorem 2].

6.3. Corollary 3 is a direct consequence of 2.3, 2.5 and Theorems 1 and 2.

6.4. We now prove Corollary 4. Let A be a tame quasi-tilted algebra. First observe that if Γ_A admits a nonstable tube Γ then, by 5.4, 5.5 and their duals, there exists an indecomposable A-module X in Γ such that $\operatorname{mod}_A(\mathbf{dim}\, X)$ is neither irreducible nor normal. In particular, it is the case if A is not tilted (see 2.2).

Therefore, assume that A is tilted and all tubes of Γ_A are stable. Then any nondirecting indecomposable A-module M lies in a stable tube and it follows from 4.11, 4.13, 4.14 and 4.15 that $\operatorname{mod}_A(\mathbf{dim}\, M)$ is both irreducible and normal. Let M be a directing (indecomposable) A-module. We know from [3, Theorem 2] that $\operatorname{mod}_A(\mathbf{dim}\, M)$ is not irreducible (equivalently, is not normal) if and only if M is an internal directing A-module and $\mathbf{dim}\, M = \mathbf{h}_1 + \mathbf{h}_2$ with connected positive vectors $\mathbf{h}_1 = \mathbf{dim}\, H_1$, $\mathbf{h}_2 = \mathbf{dim}\, H_2$ of $K_0(A)$ such that $\chi_A(\mathbf{h}_1) = 0$, $\chi_A(\mathbf{h}_2) = 0$ and $\langle \mathbf{h}_1, \mathbf{h}_2\rangle_A = 1$, for some indecomposable modules H_1 and H_2 lying in homogeneous tubes of

Γ_A and with the property $\dim_K \mathrm{Hom}_A(H_1, H_2) = 1$. Moreover, we then also have $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle = 0$, $\mathrm{Hom}_A(H_2, H_1) = 0$, and hence $-\dim_K \mathrm{Ext}^1_A(H_2, H_1) + \dim_K \mathrm{Ext}^2_A(H_2, H_1) = \langle \mathbf{h}_2, \mathbf{h}_1 \rangle - \dim_K \mathrm{Hom}_A(H_2, H_1) = 0$.

Conversely, suppose that H_1, H_2 are indecomposable modules from homogeneous tubes of Γ_A such that

$$\dim_K \mathrm{Hom}_A(H_1, H_2) = 1, \qquad \dim_K \mathrm{Ext}^1_A(H_2, H_1) = \dim_K \mathrm{Ext}^2_A(H_2, H_1).$$

Then it follows from the structure of Γ_A, described in [18], that H_1 is a torsion-free module and H_2 is a torsion module, in the torsion theory given by the tilting module defining the tilted algebra A, and hence $\mathrm{Hom}_A(H_2, H_1) = 0$. Clearly, $\chi_A(\mathbf{dim}\, H_1) = 0$ and $\chi_A(\mathbf{dim}\, H_2) = 0$, because H_1, H_2 are from homogeneous tubes. Then we get the equalities

$$\begin{aligned}\langle \mathbf{dim}\, H_1, \mathbf{dim}\, H_2 \rangle = {} & \dim_K \mathrm{Hom}_A(H_1, H_2) - \dim_K \mathrm{Ext}^1_A(H_1, H_2) \\ & + \dim_K \mathrm{Ext}^2_A(H_1, H_2) = \dim_K \mathrm{Hom}_A(H_1, H_2) = 1\end{aligned}$$

and

$$\begin{aligned}\langle \mathbf{dim}\, H_2, \mathbf{dim}\, H_1 \rangle = {} & \dim_K \mathrm{Hom}_A(H_2, H_1) - \dim_K \mathrm{Ext}^1_A(H_2, H_1) \\ & + \dim_K \mathrm{Ext}^2_A(H_2, H_1) = \dim_K \mathrm{Hom}_A(H_2, H_1) = 0.\end{aligned}$$

Hence, $\chi_A(\mathbf{dim}\, H_1 + \mathbf{dim}\, H_2) = \langle \mathbf{dim}\, H_1 + \mathbf{dim}\, H_2, \mathbf{dim}\, H_1 + \mathbf{dim}\, H_2 \rangle = 1$. Moreover, $\mathbf{dim}\, H_1 + \mathbf{dim}\, H_2$ is connected, because $\mathrm{Hom}_A(H_1, H_2) \neq 0$. Therefore, applying 2.3 we conclude that $\mathbf{dim}\, H_1 + \mathbf{dim}\, H_2 = \mathbf{dim}\, M$ for a unique indecomposable A-module M. In fact then M lies in the connecting component of Γ_A, and so is an internal directing A-module. In particular, $\mathrm{mod}_A(\mathbf{dim}\, M)$ is neither irreducible nor normal. Finally, we recall that an indecomposable A-module H lies in a homogeneous tube of Γ_A if and only if it is τ_A-invariant, that is, $\tau_A H = H$. This finishes the proof of Corollary 4.

6.5. Let A be a quasi-tilted algebra. It has been proved in [31, Theorem 3] that A is tame if and only if for any modules M, N such that $\mathbf{dim}\, M = \mathbf{dim}\, N$ and $N \in \overline{\mathcal{O}(M)} \setminus \mathcal{O}(M)$ in $\mathrm{mod}_A(\mathbf{d})$, the module N is decomposable. Therefore, A is tame if and only if the orbit $\mathcal{O}(M)$ of any indecomposable A-module M is maximal. In particular, if every indecomposable A-module belongs to the open sheet of $\mathrm{mod}_A(\mathbf{dim}\, M)$ then A is tame.

Conversely, if A is tame and $\mathbf{d}$ is the dimension-vector of an indecomposable A-module then it follows from 4.6, 4.7, 5.2 and [3, Theorem 1] that the $G(\mathbf{d})$-orbits of indecomposable modules in $\mathrm{mod}_A(\mathbf{d})$ have maximal dimension, and so belong to the open sheet of $\mathrm{mod}_A(\mathbf{d})$. This proves Corollary 5.

7. Examples. We shall illustrate our considerations by some examples.

7.1. Let $A = KQ/I$ be the bound quiver algebra given by the quiver

$$Q\colon \quad 1 \xleftarrow{\alpha} 2 \xleftarrow{\beta} 3$$

and the ideal I in KQ generated by $\alpha\beta$. It is well known that A is a tilted algebra of type $\mathbb{A}_3$ and the Auslander–Reiten quiver Γ_A is of the form

$$\begin{tikzcd}[column sep=small, row sep=small] & P_2 \arrow[dr] & & & \\ S_1 \arrow[ur] & & S_2 \arrow[dr] & & S_3 \\ & & & I_2 \arrow[ur] & \end{tikzcd}$$

where $\mathbf{dim}\, S_1 = (1,0,0)$, $\mathbf{dim}\, P_2 = (1,1,0)$, $\mathbf{dim}\, S_2 = (0,1,0)$, $\mathbf{dim}\, I_2 = (0,1,1)$, $\mathbf{dim}\, S_3 = (0,0,1)$. In particular, we have $\mathrm{Ext}^1_A(X,X) = 0$ for any indecomposable A-module X. Then for any dimension-vector $\mathbf{d} \in K_0(A) = \mathbb{Z}^3$, the irreducible components of $\mathrm{mod}_A(\mathbf{d})$ are the closures of the orbits of modules M in $\mathrm{mod}_A(\mathbf{d})$ with $\mathrm{Ext}^1_A(M,M) = 0$ (see [5, Lemma 4]).

Take now $\mathbf{d} = (2,2,2)$. Then it is easy to see that the irreducible components of $\mathrm{mod}_A(\mathbf{d})$ are the closures of orbits of the following three modules:

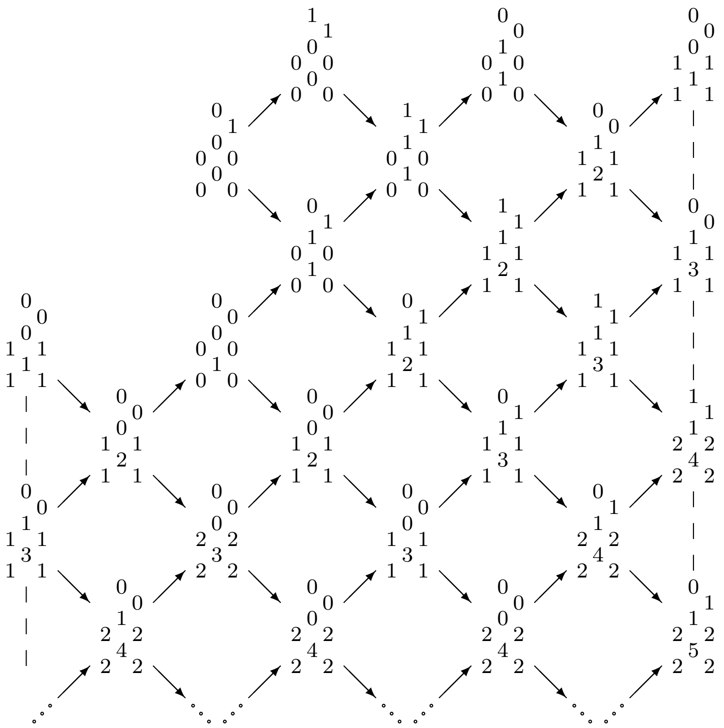
$$\begin{aligned} M_1 &= S_1 \oplus P_2 \oplus I_2 \oplus S_3, \\ M_2 &= P_2 \oplus P_2 \oplus S_3 \oplus S_3, \\ M_3 &= S_1 \oplus S_1 \oplus I_2 \oplus I_2. \end{aligned}$$

Moreover, $\dim_K \mathrm{End}_A(M_1) = 7$, $\dim_K \mathrm{End}_A(M_2) = 8$ and $\dim_K \mathrm{End}_A(M_3) = 8$. Since $\dim G(\mathbf{d}) = 12$, we get $\dim \overline{\mathcal{O}(M_1)} = 5$, $\dim \overline{\mathcal{O}(M_2)} = 4$ and $\dim \overline{\mathcal{O}(M_3)} = 4$. Hence, the irreducible components of $\mathrm{mod}_A(\mathbf{d})$ are not equidimensional. Therefore, $\mathrm{mod}_A(\mathbf{d})$ is not Cohen–Macaulay and so is not a complete intersection. On the other hand, note that $\mathbf{d}$ is not the dimension-vector of an indecomposable A-module.

7.2. Let $A = KQ/I$ be the bound quiver algebra given by the quiver

$$Q\colon\quad \begin{tikzcd}[column sep=small, row sep=small] & 8 & \\ & & 7 \arrow[ul, "\varrho"'] \arrow[dl, "\eta"] \\ 1 \arrow[dr, "\alpha"'] & 6 \arrow[d, "\xi"] & 4 \\ & 3 \arrow[ur, "\gamma"'] \arrow[dr, "\sigma"] & \\ 2 \arrow[ur, "\beta"] & & 5 \end{tikzcd}$$

and I generated by $\gamma\xi$ and $\sigma\xi$. Then A is a tilted algebra of Euclidean type $\widetilde{\mathbb{D}}_7$, being the domestic tubular extension of the tame concealed (even hereditary) algebra of Euclidean type $\widetilde{\mathbb{D}}_4$ given by the vertices 1, 2, 3, 4, 5. The Auslander–Reiten quiver Γ_A of A consists of a preprojective component, a preinjective component and a $\mathbb{P}_1(K)$-family of ray tubes, all of them stable with the exception of one tube which is of the form



where the indecomposable modules are replaced by their dimension-vectors and the dotted lines have to be identified in order to obtain the (ray) tube. Then according to 5.4 and 5.5,

$$\begin{smallmatrix} 0 & & \\ & & 0 \\ & 1 & \\ n & & n \\ & 2n & \\ n & & n \end{smallmatrix} \qquad \begin{smallmatrix} 0 & & \\ & & 1 \\ & 1 & \\ n & & n \\ & 2n & \\ n & & n \end{smallmatrix} \qquad \begin{smallmatrix} 1 & & \\ & & 1 \\ & 1 & \\ n & & n \\ & 2n & \\ n & & n \end{smallmatrix}$$

form a complete list of the dimension-vectors $\mathbf{d}$ of indecomposable A-modules for which the module variety $\operatorname{mod}_A(\mathbf{d})$ is not irreducible (equivalently, is not normal).

7.3. Let $A = KQ/I$ be the bound quiver algebra given by the quiver

$$Q\colon \quad 1 \underset{\gamma}{\overset{\alpha}{\leftleftarrows}} 2 \underset{\sigma}{\overset{\beta}{\leftleftarrows}} 3$$

and the ideal I in KQ generated by $\alpha\beta$, $\gamma\sigma$ and $\gamma\beta - \alpha\sigma$. Then A is a tame tilted algebra obtained by glueing two Kronecker algebras $C_1 = KQ^{(1)}$ and $C_2 = KQ^{(2)}$, where $Q^{(1)}$ (respectively, $Q^{(2)}$) is the full subquiver of A given by vertices 1 and 2 (respectively, 2 and 3). Moreover, Γ_A is of the form

$$\Gamma_A = \mathcal{P}_1 \vee \mathcal{T}_1 \vee \mathcal{C} \vee \mathcal{T}_2 \vee \mathcal{Q}_2$$

where $\mathcal{P}_1$ is the preprojective component of Γ_{C_1}, $\mathcal{T}_1$ is a $\mathbb{P}_1(K)$-family of stable (homogeneous) tubes of Γ_{C_1}, $\mathcal{T}_2$ is a $\mathbb{P}_1(K)$-family of stable (homogeneous) tubes of Γ_{C_2}, $\mathcal{Q}_2$ is the preinjective component of Γ_{C_2}, and $\mathcal{C}$ is the connecting component of the form

$$\begin{tikzcd}[column sep=small]
 & & & & {\scriptstyle 1\,2\,1} \arrow[dr] & & & & \\
 & {\scriptstyle 3\,4\,0} \arrow[dr, shift left] \arrow[dr, shift right] & & {\scriptstyle 1\,2\,0} \arrow[ur] \arrow[dr, shift left] \arrow[dr, shift right] & & {\scriptstyle 0\,2\,1} \arrow[dr, shift left] \arrow[dr, shift right] & & {\scriptstyle 0\,4\,3} \arrow[dr, shift left] \arrow[dr, shift right] & \\
\cdots \arrow[ur, shift left] \arrow[ur, shift right] & & {\scriptstyle 2\,3\,0} \arrow[ur, shift left] \arrow[ur, shift right] & & {\scriptstyle 0\,1\,0} \arrow[ur, shift left] \arrow[ur, shift right] & & {\scriptstyle 0\,3\,2} \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots
\end{tikzcd}$$

obtained by glueing the preinjective component of Γ_{C_1} and the preprojective component of Γ_{C_2} using the indecomposable projective-injective A-module P of dimension-vector $\mathbf{d} = (1, 2, 1)$. Observe that $\mathbf{d} = \mathbf{h}_1 + \mathbf{h}_2$, where $\mathbf{h}_1 = (1, 1, 0)$ and $\mathbf{h}_2 = (0, 1, 1)$ are generators of the radicals of χ_{C_1} and χ_{C_2}, respectively. Moreover, $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle_A = 1$ and $\langle \mathbf{h}_2, \mathbf{h}_1 \rangle_A = 0$, and it follows from Theorem 2 that $\operatorname{mod}_A(\mathbf{d})$ is neither normal nor irreducible.

On the other hand, any indecomposable A-module nonisomorphic to P is either a C_1-module or a C_2-module. Hence, $\operatorname{mod}_A(\mathbf{e})$ is normal and irreducible for the remaining dimension-vectors $\mathbf{e}$ of indecomposable A-modules.

7.4. Finally, let $A = KQ/I$ be the bound quiver algebra given by the quiver

$$Q\colon\quad \begin{tikzcd}
1 \arrow[dr, "\alpha"] & & & 5 \arrow[dr, "\xi"] & \\
 & 3 \arrow[r, "\gamma"] & 4 \arrow[ur, "\sigma"] \arrow[dr, "\delta"'] & & 7 \\
2 \arrow[ur, "\beta"'] & & & 6 \arrow[ur, "\eta"'] &
\end{tikzcd}$$

and the ideal I generated by $\xi\sigma\gamma - \eta\delta\gamma$ and $\xi\sigma\gamma\alpha$. Denote by C_1 the tame hereditary algebra of type $\widetilde{\mathbb{A}}_4$ given by vertices 4, 5, 6, 7, and by C_2 the tame hereditary algebra of type $\widetilde{\mathbb{D}}_5$ given by vertices 1, 2, 3, 4, 5, 6. Moreover, let B_1 be the convex subcategory of A given by all vertices except 1. Then B_1 is a tilted algebra of type $\widetilde{\mathbb{D}}_5$ which is the tubular extension of C_1 using the simple homogeneous module

$$\begin{tikzcd}
 & K \arrow[dr, "1"] & \\
K \arrow[ur, "1"] \arrow[dr, "1"'] & & K \\
 & K \arrow[ur, "1"'] &
\end{tikzcd}$$

Moreover, A is the one-point extension $B_1[R]$ of B_1 by the indecomposable preinjective B_1-module R of the form

$$\begin{tikzcd}
 & & & K \arrow[dr] & \\
 & K \arrow[r, "1"'] & K \arrow[ur, "1"] \arrow[dr, "1"'] & & 0 \\
0 \arrow[ur] & & & K \arrow[ur] &
\end{tikzcd}$$

Further, the radicals of χ_{C_1} and χ_{C_2} are generated respectively by

$$\mathbf{h}_1 = {}^{0}_{0}0\,1\,{}^{1}_{1}1 \qquad \text{and} \qquad \mathbf{h}_2 = {}^{1}_{1}2\,2\,{}^{1}_{1}0$$

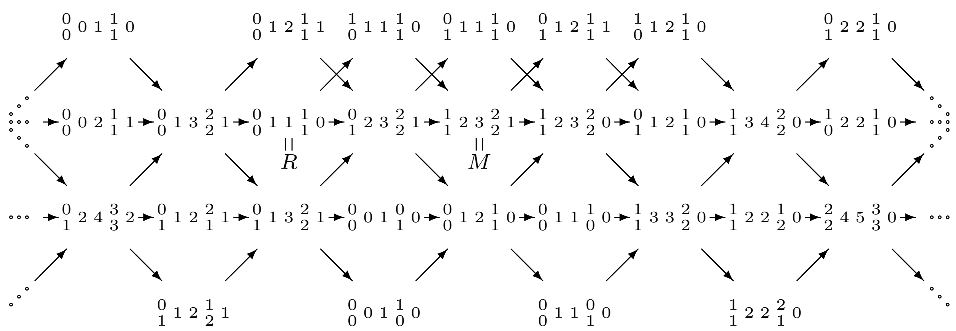
Then it follows (see also [25, 1.5]) that A is a tame tilted algebra whose Auslander–Reiten quiver is of the form

$$\Gamma_A = \mathcal{P}_1 \vee \mathcal{T}_1 \vee \mathcal{C} \vee \mathcal{T}_2 \vee \mathcal{Q}_2$$

where $\mathcal{P}_1$ is the preprojective component of Γ_{C_1}, $\mathcal{T}_1$ is the $\mathbb{P}_1(K)$-family of ray tubes of Γ_{B_1}, $\mathcal{T}_2$ is the $\mathbb{P}_1(K)$-family of stable tubes of Γ_{C_2}, $\mathcal{Q}_2$ is the preinjective component of Γ_{C_2}, and the connecting component $\mathcal{C}$ contains a unique sincere indecomposable (directing) module M and

$$\mathbf{dim}\, M = {}^{1}_{1}2\,3\,{}^{2}_{2}1 = {}^{0}_{0}0\,1\,{}^{1}_{1}1 + {}^{1}_{1}2\,2\,{}^{1}_{1}0 = \mathbf{h}_1 + \mathbf{h}_2.$$

In fact, $\mathcal{C}$ is a glueing of the preinjective component of Γ_{B_1} with the preprojective component of Γ_{C_2}, and the neighbourhood of M in $\mathcal{C}$ is as follows:



Moreover, it is easy to check that $\langle \mathbf{h}_1, \mathbf{h}_2 \rangle_A = 2$, and hence $\mathrm{mod}_A(\mathbf{dim}\, M)$ is irreducible and normal. Observe that if N is an arbitrary indecomposable directing A-module nonisomorphic to M then N is either a B_1-module or a C_2-module, and consequently $\mathrm{mod}_A(\mathbf{dim}\, N)$ is irreducible, normal and a complete intersection.

On the other hand, we note that the tubular family $\mathcal{T}_1$ has one nonstable tube containing indecomposable modules with dimension-vectors

$${}^{0}_{0}1\,n\,{}^{n}_{n}n = {}^{0}_{0}1\,0\,{}^{0}_{0}0 + n\mathbf{h}_1, \quad {}^{0}_{1}1\,n\,{}^{n}_{n}n = {}^{0}_{1}1\,0\,{}^{0}_{0}0 + n\mathbf{h}_1, \quad n \geq 1,$$

for which the associated modules varieties are, according to 5.4 and 5.5, neither irreducible nor normal.

REFERENCES

[1] I. Assem, A. Skowroński and B. Tomé, *Coil enlargements of algebras*, Tsukuba J. Math. 19 (1995), 453–479.
[2] M. Auslander, I. Reiten and S. O. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1994.
[3] G. Bobiński and A. Skowroński, *Geometry of directing modules over tame algebras*, preprint, Toruń, 1998.
[4] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. 28 (1983), 461–469.
[5] —, *A geometric version of the Morita equivalence*, J. Algebra 139 (1991), 159–171.
[6] —, *Minimal singularites of Dynkin quivers*, Comment. Math. Helv. 69 (1994), 575–611.
[7] —, *On degenerations and extensions of finite dimensional modules*, Adv. Math. 121 (1996), 245–287.
[8] —, *Some geometric aspects of representation theory*, in: Proc. Workshop ICRA VIII (Trondheim 1996), CMS Conf. Proc., in press.
[9] C. de Concini and E. Strickland, *On the variety of complexes*, Adv. Math. 41 (1981), 57–77.
[10] W. W. Crawley-Boevey, *On tame algebras and bocses*, Proc. London Math. Soc. 56 (1988), 451–483.
[11] Yu. A. Drozd, *Tame and wild matrix problems*, in: Lecture Notes in Math. 832, Springer, 1980, 242–258.
[12] D. Eisenbud, *Commutative Algebra with a View toward Algebraic Geometry*, Grad. Texts in Math. 150, Springer, 1996.
[13] P. Gabriel, *Finite representation type is open*, in: Lecture Notes in Math. 488, Springer, 1975, 132–155.
[14] —, *Auslander–Reiten sequences and representation-finite algebras*, in: Lecture Notes in Math. 831, Springer, 1979, 1–71.
[15] D. Happel, I. Reiten and S. O. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575 (1996).
[16] R. Hartshorne, *Introduction to Algebraic Geometry*, Springer, 1977.
[17] V. G. Kac, *Infinite root systems, representations of graphs and invariant theory*, Invent. Math. 56 (1980), 57–92.
[18] O. Kerner, *Tilting wild algebras*, J. London Math. Soc. 39 (1989), 29–47.
[19] H. Kraft, *Geometrische Methoden in der Invariantentheorie*, Vieweg, 1984.
[20] —, *Geometric methods in representation theory*, in: Lecture Notes in Math. 944, Springer, 1981, 180–258.
[21] H. Kraft and C. Procesi, *Closures of conjugacy classes of matrices are normal*, Invent. Math. 53 (1978), 227–247.
[22] H. Lenzing and J. A. de la Peña, *Concealed-canonical algebras and separating tubular families*, Proc. London Math. Soc., in press.
[23] J. A. de la Peña, *On the dimension of the module-varieties of tame and wild algebras*, Comm. Algebra 19 (1991), 1795–1807.
[24] —, *Tame algebras with sincere directing modules*, J. Algebra 161 (1993), 171–185.
[25] —, *The families of two-parametric tame algebras with sincere directing modules*, in: CMS Conf. Proc. 14 (1993), 361–392.
[26] J. A. de la Peña and A. Skowroński, *Geometric and homological characterizations of polynomial growth strongly simply connected algebras*, Invent. Math. 126 (1996), 287–296.

[27] C. M. Ringel, *The rational invariants of the tame quivers*, Invent. Math. 58 (1980), 217–239.
[28] —, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, 1984.
[29] I. R. Shafarevich, *Basic Algebraic Geometry*, Grad. Texts in Math. 213, Springer, 1977.
[30] A. Skowroński, *Tame quasi-tilted algebras*, J. Algebra 203 (1998), 470–490.
[31] A. Skowroński and G. Zwara, *Degenerations for indecomposable modules and tame algebras*, Ann. Sci. École Norm. Sup. 31 (1998), 153–180.
[32] D. Voigt, *Induzierte Darstellungen in der Theorie der endlichen, algebraischen Gruppen*, Lecture Notes in Math. 336, Springer, 1977.
[33] G. Zwara, *Degenerations of finite dimensional modules are given by extensions*, preprint, Toruń, 1998.

Faculty of Mathematics and Informatics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: gregbob@mat.uni.torun.pl
skowron@mat.uni.torun.pl

Received 9 April 1998;
revised 26 May 1998