C O L L O Q U I U M   M A T H E M A T I C U M

VOL. 79 1999 NO. 1

ON SELFINJECTIVE ALGEBRAS OF EUCLIDEAN TYPE

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The aim of this note is to complete the description of domestic selfinjective algebras having simply connected Galois coverings, given by the second named author in [6], and derive some consequences.

Throughout the paper K will denote a fixed algebraically closed field. By an *algebra* we mean a finite-dimensional associative K-algebra with an identity, which we shall assume to be basic and connected. For an algebra A, we denote by $\operatorname{mod} A$ the category of finite-dimensional (over K) right A-modules and by $D : \operatorname{mod} A \to \operatorname{mod} A^{\mathrm{op}}$ the standard duality $\operatorname{Hom}_K(-, K)$. An algebra A is called *selfinjective* if $A \simeq D(A)$ in $\operatorname{mod} A$, that is, A_A is injective. Moreover, A is called *symmetric* if A and $D(A)$ are isomorphic as A-A-bimodules. An important class of selfinjective algebras is formed by the algebras of the form $\widehat{B}/G$, where $\widehat{B}$ is the *repetitive algebra* (locally finite-dimensional, without identity)

$$\widehat{B} = \begin{bmatrix} \ddots & \ddots & & & & 0 & \\ & B_{m-1} & Q_{m-1} & & & & \\ & & B_m & Q_m & & & \\ & & & B_{m+1} & Q_{m+1} & & \\ & 0 & & & \ddots & \ddots \end{bmatrix}$$

of an algebra B. Here $B_m = B$ and $Q_m = {}_B D(B)_B$ for all $m \in \mathbb{Z}$, all the remaining entries are zero, the matrices in $\widehat{B}$ have only finitely many nonzero elements, addition is the usual addition of matrices, multiplication is induced from the canonical maps $B \otimes_B D(B) \to D(B)$, $D(B) \otimes_B B \to D(B)$, and the zero map $D(B) \otimes_B D(B) \to 0$, and G is an admissible group of K-linear automorphisms of $\widehat{B}$. Recall that a group G of K-linear automorphisms of $\widehat{B}$ is called *admissible* if its action on the objects of $\widehat{B}$ is free and has finitely

1991 *Mathematics Subject Classification*: 16G20, 16G60, 16D50, 16E30, 14H60.

Research of the second author supported by the Polish Scientific Grant KBN No. 2 P03A 012 14.

many orbits. Denote by $\nu_{\widehat{B}}$ the Nakayama automorphism of $\widehat{B}$ shifting B_m to B_{m+1} and Q_m to Q_{m+1} for all $m \in \mathbb{Z}$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the *trivial extension* $B \ltimes D(B)$ of B by $D(B)$, and so is symmetric. We note that if B is of finite global dimension then the stable module category $\underline{\mathrm{mod}}\,\widehat{B}$ of $\mathrm{mod}\,\widehat{B}$ is equivalent, as a triangulated category, to the derived category $D^{\mathrm{b}}(\mathrm{mod}\,B)$ of bounded complexes over $\mathrm{mod}\,B$ (see [4]). Following [6] an algebra A is called *standard* if it admits a Galois covering $R \to R/G = A$ with R simply connected (in the sense of [2]). Further, A is called *domestic* if there are a finite number of $K[x]$-A-bimodules M_i, $1 \le i \le n$, which are finitely generated free left modules over the polynomial algebra $K[x]$ in one variable, and, for each dimension d, all but a finite number of indecomposable right A-modules of dimension d are of the form $V \otimes_{K[x]} M_i$ for some i and some indecomposable $K[x]$-module V. The algebra A is called *n-parametric* if the minimal number of such bimodules is n. Finally, A is said to be *representation-infinite* provided the number of isomorphism classes of indecomposable finite-dimensional A-modules is infinite.

It has been proved in [6, Theorem 1.5] that a selfinjective algebra A is standard and representation-infinite domestic if and only if A is isomorphic to $\widehat{B}/G$, where B is a representation-infinite tilted algebra of Euclidean type $\Delta \in \{\widetilde{\mathbb{A}}_{p,q}, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and G is an admissible infinite cyclic group of K-linear automorhisms of $\widehat{B}$. An algebra A of the above form $\widehat{B}/G$ is said to be a *selfinjective algebra of Euclidean type Δ*. The admissible infinite cyclic groups G have been described in [6, Section 2] with the exception of the case $\Delta = \widetilde{\mathbb{E}}_7$. We may now complete this case and state the following theorem.

THEOREM. *Let A be a selfinjective algebra of Euclidean type $\Delta \in \{\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$. Then $A = \widehat{B}/(\varphi\nu_{\widehat{B}}^m)$, where B is a representation-infinite tilted algebra of type Δ, m is a positive integer, and φ is an automorphism of $\widehat{B}$ induced by an automorphism of B. In particular, A is $2m$-parametric.*

We note that if $\Delta = \widetilde{\mathbb{A}}_{p,q}$ or $\widetilde{\mathbb{D}}_n$ then for any $r \ge 1$ there exist r-parametric selfinjective algebras of type Δ (see [6, (2.6), (2.7)]).

The following fact is a direct consequence of the above theorem.

COROLLARY 1. *Let A be a symmetric algebra of Euclidean type $\Delta \in \{\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$. Then $A \simeq B \ltimes D(B)$ for some representation-infinite tilted algebra of type Δ.*

Applying [7, Theorem 5.5] and [8, Theorem 1, Corollary] we also get the following consequence of the above theorem.

COROLLARY 2. *Let A be a selfinjective algebra of Eulidean type $\Delta \in \{\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$ and Λ a selfinjective algebra which is stably equivalent to A.*

Then Λ is of Euclidean type Δ and has the same number of pairwise non-isomorphic simple modules as A.

Let B be a representation-infinite tilted algebra of Euclidean type $\Delta \in \{\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$. Then the Auslander–Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ is of the form

$$\Gamma_{\widehat{B}} = \bigvee_{p\in\mathbb{Z}} (\mathcal{X}_p \vee \mathcal{R}_q)$$

where, for each $p \in \mathbb{Z}$, $\mathcal{X}_p$ is a component whose stable part is of the form $\mathbb{Z}\Delta$ and $\mathcal{R}_q$ is a $\mathbb{P}_1(K)$-family of components whose stable parts are tubes, and $\nu_{\widehat{B}}(\mathcal{X}_p) = \mathcal{X}_{p+2}$, $\nu_{\widehat{B}}(\mathcal{R}_p) = \mathcal{R}_{p+2}$ (see [1], [6]). It was shown in [6, Section 2] that any admissible group G of K-linear automorphisms of $\widehat{B}$ is infinite cyclic generated by a K-linear automorphism g of $\widehat{B}$ such that $g(\mathcal{X}_p) = \mathcal{X}_{p+m}$, $g(\mathcal{R}_p) = \mathcal{R}_{p+m}$ for a fixed positive integer m. Following [6] a K-linear automorphism g of $\widehat{B}$ such that $g(\mathcal{X}_p) = \mathcal{X}_p$ and $g(\mathcal{R}_p) = \mathcal{R}_p$ for all $p \in \mathbb{Z}$ is said to be *rigid*.

In order to prove the theorem it is enough to show (see [6, Proposition 2.13]) that $\widehat{B}$ does not admit an automorphism σ such that $\sigma(\mathcal{X}_p) = \mathcal{X}_{p+1}$, $\sigma(\mathcal{R}_p) = \mathcal{R}_{p+1}$ for any $p \in \mathbb{Z}$, or equivalently, $\sigma^2 = \varrho\nu_{\widehat{B}}$ for some rigid K-linear automorphism ϱ of $\widehat{B}$. This condition implies that Δ has an even number of vertices (see [6, Corollary 2.5]), and so there is no such σ if $\Delta = \widetilde{\mathbb{E}}_6$ or $\widetilde{\mathbb{E}}_8$. We shall present below a unified argument showing that such a σ does not exist for $\Delta \in \{\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8\}$.

In fact, our argument will apply to an even wider class of algebras. It is known [3] that a representation-infinite tilted algebra B of Euclidean type can be realized as the endomorphism algebra of a tilting object T in a category $\operatorname{coh}\mathbb{X}$ of coherent sheaves on a weighted projective line $\mathbb{X}$ of weight type $\mathbf{p} = (p_1, \ldots, p_t)$, where additionally the *discriminant*

$$\delta(\mathbf{p}) = p\left[(t-2) - \sum_{i=1}^{t} \frac{1}{p_i}\right], \qquad p = \operatorname{lcm}(p_1, \ldots, p_t),$$

of $\mathbf{p}$ is < 0. Combined with Happel's theorem [4] this yields triangle-equivalences

$$\underline{\operatorname{mod}}\, \widehat{B} \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{mod} B) \simeq \mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X}),$$

which we treat as identifications. Passing to the stable level each automorphism σ of $\operatorname{mod}\widehat{B}$ induces an automorphism $\underline{\sigma}$ of $\underline{\operatorname{mod}}\,\widehat{B}$, hence of $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$. We are next discussing basic features of $\operatorname{coh}\mathbb{X}$ and of automorphisms α of $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$.

Recall first that the Grothendieck groups of $\operatorname{coh}\mathbb{X}$ (with respect to short exact sequences) and of $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ (with respect to distinguished triangles) are naturally isomorphic; they are denoted by $K_0\mathbb{X}$ from now on. Moreover,

$K_0\mathbb{X}$ is equipped with the *Euler form*, given on classes of coherent sheaves by

$$\langle [X],[Y]\rangle = \dim_K \operatorname{Hom}(X,Y) - \dim_K \operatorname{Ext}^1(X,Y).$$

The Auslander–Reiten translation $\tau_{\mathbb{X}}$ for $\operatorname{coh}\mathbb{X}$ (resp. $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$) induces on the K-theoretic level the *Coxeter transformation* $\tau : K_0\mathbb{X} \to K_0\mathbb{X}$ preserving the Euler form and characterized by the property

$$\langle x, \tau y\rangle = -\langle y, x\rangle \quad \text{for all } x, y \in K_0\mathbb{X}.$$

We will also need the *virtual Euler form* on $K_0\mathbb{X}$ given by

$$\langle\langle x, y\rangle\rangle = \sum_{j=0}^{p-1} \langle \tau^j x, y\rangle.$$

From now on we assume that the discriminant $\delta(\mathbf{p})$ is non-zero. Notice that this just excludes the case where $\operatorname{coh}\mathbb{X}$ is tame tubular and leaves the cases where $\operatorname{coh}\mathbb{X}$ has either a tame domestic or a wild classification problem (see [3, Remark 5.4]).

The *radical* $\operatorname{rad} K_0\mathbb{X}$ of the quadratic form $q_{\mathbb{X}}(x) = \langle x, x\rangle$ equals the fixed point set of τ and is a cyclic direct factor $\mathbb{Z}\mathbf{w}$ of $K_0\mathbb{X}$. Moreover, we choose the generator $\mathbf{w}$ so that $\langle \mathbf{a}, \mathbf{w}\rangle = 1$, where $\mathbf{a} = [\mathcal{O}]$ is the class of the structure sheaf $\mathcal{O}$ on $\mathbb{X}$. For the following facts we refer to [3] or [5]. Define *rank* and *degree* as the linear forms given by $\operatorname{rk}(x) = \langle x, \mathbf{w}\rangle$ and $\deg(x) = \langle\langle \mathbf{a}, x\rangle\rangle - \langle\langle \mathbf{a}, \mathbf{a}\rangle\rangle \operatorname{rk}(x)$, respectively. Then each indecomposable $X \in \mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ has a well-defined *slope*

$$\mu X = \frac{\deg X}{\operatorname{rk} X} \in \mathbb{Q} \cup \{\infty\}.$$

Note that for an automorphism of $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ we use the same symbol to denote the induced map on $K_0\mathbb{X}$.

PROPOSITION. *Let $\mathbb{X}$ be a weighted projective line of weight type $\mathbf{p}$ such that $\delta(\mathbf{p}) \neq 0$. Then for each automorphism α of $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ the integer $d(\alpha) = \langle\langle \mathbf{a}, \alpha\mathbf{a}\rangle\rangle - \langle\langle \mathbf{a}, \mathbf{a}\rangle\rangle$ satisfies*

$$\mu(\alpha X) = \mu(X) + d(\alpha)$$

for each indecomposable $X \in \mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$.

We note that the assertion does not hold in the tubular case $\delta(\mathbf{p}) = 0$.

Proof. Since $\alpha : K_0\mathbb{X} \to K_0\mathbb{X}$ preserves the radical, we get $\alpha(\mathbf{w}) = \pm\mathbf{w}$. Let T denote the translation functor for the derived category. Switching from α to $T \circ \alpha$, if necessary, we may thus assume that α preserves the rank. Now the linear form

$$\lambda(x) = \langle\langle \mathbf{a}, \alpha x\rangle\rangle - \langle\langle \mathbf{a}, \alpha\mathbf{a}\rangle\rangle \operatorname{rk}(x)$$

has the following two properties:

(1) $\lambda(\mathbf{a}) = 0$,
(2) $\lambda([S]) = p/p'$ if S is a simple sheaf of τ-period p'.

Property (1) is obvious, and (2) follows from the fact that, up to translation in $\mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$, the automorphism α, being rank-preserving, sends simple sheaves to simple sheaves and moreover preserves their multiplicities (= τ-periods).

Since the classes of the structure sheaf and of the simple sheaves generate the Grothendieck group of $\operatorname{coh}\mathbb{X}$, the degree is the only linear form satisfying (1) and (2) (see [3, Proposition 2.8]), and hence $\lambda = \deg$. Invoking the definition of the degree, we get the assertion.

In the special situation $\alpha = \tau_{\mathbb{X}}$ we obtain $d(\tau_{\mathbb{X}}) = \delta(\mathbf{p})$, hence (see [3], [5])

$$\mu(\tau_{\mathbb{X}}X) = \mu(X) + \delta(\mathbf{p}) \quad \text{for all indecomposable } X \in \mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X}).$$

COROLLARY 3. *If α is an automorphism of* $\operatorname{mod}\widehat{B}$ *such that*

$$\alpha^n = \varrho\nu_{\widehat{B}}$$

for some rigid automorphism of $\widehat{B}$, then

$$n\cdot d(\underline{\alpha}) = \delta(\mathbf{p}),$$

where $\underline{\alpha}$ is the automorphism of $\underline{\operatorname{mod}}\,\widehat{B} = \mathcal{D}^{\mathrm{b}}(\operatorname{coh}\mathbb{X})$ *induced by α.*

Proof. Passing to the stable level we obtain

$$\underline{\alpha}^n = \underline{\varrho}\underline{\nu}_{\widehat{B}} = \underline{\varrho}\tau_{\mathbb{X}}T^2,$$

then passing to slopes we get

$$nd(\underline{\alpha}) = d(\underline{\varrho}) + d(\tau_{\mathbb{X}})$$

because $d(T) = 0$. Since ϱ preserves the finite full convex subcategory B of $\widehat{B}$, we deduce that ϱ has finite order, hence $d(\underline{\varrho}) = 0$ as a torsion element of $\mathbb{Z}$. Thus $nd(\underline{\alpha}) = d(\tau_{\mathbb{X}}) = \delta(\mathbf{p})$.

We now recall that the Euclidean types $\widetilde{\mathbb{A}}_{r,s}$, $\widetilde{\mathbb{D}}_m$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ correspond to the weight types (r,s), $(2,2,m-2)$, $(2,3,3)$, $(2,3,4)$ and $(2,3,5)$ (see [3]). The assertion of the theorem now follows by inspection of the table

weight type $\mathbf{p}$	discriminant $\delta(\mathbf{p})$
(np, nq), $(p,q) = 1$	$-n(p+q)$
$(2,2,2n+1)$	-2
$(2,2,2n)$	-1
$(2,3,3)$	-1
$(2,3,4)$	-1
$(2,3,5)$	-1

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Received 10 March 1998;
revised 15 May 1998