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UNIFORM BOUNDARY STABILIZATION OF A THERMOELASTIC BAR WITH A NONLINEAR WEAK DAMPING

BY

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1. Introduction. In this paper we shall prove exponential decay of the energy of a one-dimensional homogeneous thermoelastic bar of unit length. Let u be the displacement and θ be the temperature deviation from the reference temperature. Then u and θ satisfy the following linear one-dimensional thermoelastic system:

$$u_{tt} - u_{xx} + b\theta_x = 0 \quad \text{in } (0,1)\times(0,\infty), \tag{1.1}$$

$$\theta_t - \theta_{xx} + bu_{xt} = 0 \quad \text{in } (0,1)\times(0,\infty), \tag{1.2}$$

with initial conditions

$$u(x,0) = u_0(x), \qquad u_t(0,x) = u_1(x), \qquad \theta(0,x) = \theta_0(x), \tag{1.3}$$

where $b \neq 0$ is a real number.

We assume that u and θ satisfy the boundary conditions

$$\theta(0,t) = \theta(1,t) = 0, \qquad t > 0, \tag{1.4}$$

$$u(0,t) = 0, \quad u_x(1,t) = -g(u_t(1,t)), \qquad t > 0. \tag{1.5}$$

Since the pioneering work of Dafermos [5] on linear thermoelasticity, significant progress has been made on the mathematical aspect of thermoelasticity (see [2, 4, 6, 7, 9–11, 14–19] among others). Most studies focused on the existence, regularity, and asymptotic behavior of solutions. More precisely, Dafermos [5] has shown that if $(u_0, u_1, \theta_0) \in H^1 \times L^2 \times L^2$, then the energy function of the system defined as

$$E(t) = \|u_x\|^2 + \|u_t\|^2 + \|\theta\|^2 \tag{1.6}$$

converges to zero as time goes to infinity. However, no decay rate was given. In 1981, Slemrod [19] used the energy method to prove that for the system (1.1)–(1.3) if u, θ satisfy Dirichlet and Neumann boundary conditions at both ends and if $(u_0, u_1, \theta_0) \in H^2 \times H^1 \times H^2$ satisfy the compatibility

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conditions, then there are positive constants M and α such that

$$\begin{aligned}(1.7)\quad &\|u_t(x)\|^2 + \|u_x(x)\|^2 + \|u_{tt}(x)\|^2 + \|u_{xt}(x)\|^2 + \|u_{xx}(x)\|^2 \\ &\quad + \|\theta(t)\|^2 + \|\theta_t(t)\|^2 + \|\theta_x(t)\|^2 + \|\theta_{xx}(t)\|^2 \\ &\leq M(\|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2 + \|\theta_0\|_{H^2}^2)e^{-\alpha t}, \quad t > 0,\end{aligned}$$

where $\|\cdot\|$ denotes the L^2 norm in $(0,1)$ and H^s is the usual Sobolev space.

In 1992, Muñoz Rivera [15] proved that the estimate (1.7) still holds if u and θ both satisfy the Dirichlet boundary condition at both ends (clamped, constant temperature). The problem of establishing an energy estimate of the form

$$E(t) \leq ME(0)e^{-\alpha t}, \quad \forall t > 0, \tag{1.8}$$

has remained open for some time now.

When u and θ satisfy the Dirichlet and Neumann boundary conditions, respectively (or vice versa), Hansen [7] in 1992 succeeded in establishing (1.8) using the Fourier series expansion method and a decoupling technique. We refer to Gibson–Rosen–Tao [6] for another approach, a combination of semigroup theory and the energy method. When u and θ both satisfy the Dirichlet boundary conditions, Kim [11] and Liu–Zheng [14] independently proved that the estimate (1.8) still holds. The methods of these two papers are quite different. Kim's method is based on a control theory approach and a unique continuation theorem by Lions. In [14], Liu–Zheng used a spectral theorem due to Huang [8].

Quite recently, in 1996, Ammar Khodja–Benabdallah–Teniou [3] proved that if the function g appearing in (1.5) is linear, then (1.8) still holds. They used the method based on the construction of energy functionals developed by Komornik–Zuazua [13]. However, their result has a serious drawback from the point of view of physical applications: the feedback $g(x) = x$ is never bounded. Motivated by this problem, we are interested here in the decay property of the solutions of the problem (1.1)–(1.5) with $g(x)$ such that

$$-\infty < \lim_{x\to-\infty} g(x) < \lim_{x\to\infty} g(x) < \infty. \tag{1.9}$$

If g satisfies at most (1.9) the dissipative effect by $g(u_t)$ is weak as $|u_t|$ is large and for convenience we call such a term *weak dissipation*.

Hereafter, we consider the most typical example $g(x) = x/\sqrt{1+x^2}$, which is increasing, globally Lipschitz continuous, satisfies $xg(x) \geq 0$ for all $x \in \mathbb{R}$, and $\lim_{x\to\pm\infty} g(x) = \pm 1$.

In this paper we shall prove that (1.8) still holds for solutions of (1.1)–(1.5). Our main tool is an integral inequality, combined with a multiplier technique.

The paper is organized as follows. In Section 2, we state the main theorem. In Section 3, we give the proof of the main result.

2. Statement of the main theorem. First, let us introduce some notations. We denote by Ω the interval $(0,1)$, and

$$H^1_{\Gamma_0}(\Omega) = \{u \in H^1(\Omega) : u(0,t) = 0\}, \tag{2.1}$$

$$H^1_0(\Omega) = \{u \in H^1(\Omega) : u(0,t) = u(1,t) = 0\}. \tag{2.2}$$

The problem (1.1)–(1.5) is well-posed and dissipative. Indeed, we can write it in the first order form

$$U' + \mathcal{A}U = 0, \tag{2.3}$$

$$U(0) = U_0, \tag{2.4}$$

where $U = (u, u_t, \theta)$, $U_0 = (u_0, u_1, \theta_0)$ and the operator $\mathcal{A}$ is given by

$$\mathcal{A}(u, u_t, \theta) = (-u_t, -u_{xx} + b\theta_x, -\theta_{xx} + bu_{tx}), \tag{2.5}$$

$$\begin{aligned} D(\mathcal{A}) = \{(u, u_t, \theta) \in H^1_{\Gamma_0} \times L^2 \times L^2 : u_{xx} \in L^2,\ u_t \in H^1_{\Gamma_0}, \\ \theta \in H^2 \cap H^1_0,\ u_x(1,t) = -g(u_t(1,t))\}. \end{aligned} \tag{2.6}$$

For all given initial data $(u_0, u_1, \theta_0) \in H^1_{\Gamma_0} \times L^2 \times L^2$, by the standard semigroup theory, there exists a unique weak solution (u, θ) such that

$$u \in C(\mathbb{R}_+, H^1_{\Gamma_0}(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega)), \tag{2.7}$$

$$\theta \in C(\mathbb{R}_+, L^2(\Omega)). \tag{2.8}$$

Moreover, if $(u_0, u_1, \theta_0) \in D(\mathcal{A})$ then we have the following regularity property:

$$u \in C(\mathbb{R}_+, H^2 \cap H^1_{\Gamma_0}) \cap C^1(\mathbb{R}_+, H^1_{\Gamma_0}) \cap C^2(\mathbb{R}_+, L^2), \tag{2.9}$$

$$\theta \in C(\mathbb{R}_+, H^2 \cap H^1_0) \cap C^1(\mathbb{R}_+, L^2); \tag{2.10}$$

we say in this case that (u, θ) is a *strong solution.*

We define the energy of the solutions by the formula

$$E(t) := \frac{1}{2} \int_\Omega (u_t^2 + u_x^2 + \theta^2)\, dx. \tag{2.11}$$

If (u, θ) is a strong solution, then we have by a simple computation

$$E'(t) = -\Big\{ \int_\Omega \theta_x^2\, dx + u_t(1,t) g(u_t(1,t)) \Big\} \le 0, \tag{2.12}$$

and for all $0 \le S < T < \infty$,

$$E(S) - E(T) = \int_S^T \int_\Omega \theta_x^2\, dx\, dt + \int_S^T u_t(1,t) g(u_t(1,t))\, dt. \tag{2.13}$$

This identity remains valid for all mild solutions by an easy density argument. Hence, the energy is non-increasing and our main result is the following

MAIN THEOREM. *There exist two constants $M > 0$, $\omega > 0$ such that*

$$E(t) \le ME(0)e^{-\omega t}, \qquad \forall t > 0, \tag{2.14}$$

for all initial data $(u_0, u_1, \theta_0) \in D(\mathcal{A})$.

For the proof, we need the following lemma.

LEMMA 2.1 ([12], Lemma 8.1). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-increasing function and assume that there exists a constant $T > 0$ such that*

$$\int_t^\infty E(s)\,ds \le TE(t), \qquad \forall t \in \mathbb{R}_+. \tag{2.15}$$

Then

$$E(t) \le E(0)e^{1-t/T}, \qquad \forall t \ge T. \tag{2.16}$$

3. Proof of the main theorem. From now on we denote by c various positive constants which may be different at different occurrences.

First, we multiply the equation (1.1) with u and integrate over $(0,T)$ to obtain

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (u_{tt}u - u_{xx}u + b\theta_x u)\,dx\,dt \\ &= \Big[\int_\Omega uu_t\Big]_0^T - \int_0^T \int_\Omega u_t^2 + \int_0^T \int_\Omega u_x^2 \\ &\quad + b\int_0^T \int_\Omega u\theta_x + \int_0^T u(1,t)g(u_t(1,t))\,dt. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^T \int_\Omega (u_t^2 + u_x^2 + \theta^2) = {} & -\Big[\int_\Omega uu_t\Big]_0^T + 2\int_0^T \int_\Omega u_t^2 + \int_0^T \int_\Omega (\theta^2 - b\theta_x u) \\ & - \int_0^T u(1,t)g(u_t(1,t))\,dt. \end{aligned}$$

That is,

$$\begin{aligned} 2\int_0^T E(t)\,dt = {} & -\Big[\int_\Omega uu_t\Big]_0^T + 2\int_0^T \int_\Omega u_t^2 + \int_0^T \int_\Omega (\theta^2 - b\theta_x u) \\ & - \int_0^T u(1,t)g(u_t(1,t))\,dt. \end{aligned} \tag{3.1}$$

Next, we multiply (1.1) with $2xu_x$ and integrate over $(0,T)$ to obtain

$$\begin{aligned}0 &= \int_0^T\int_\Omega (2xu_xu_{tt} - 2xu_xu_{xx} + 2bxu_x\theta_x)\,dx\,dt \\ &= \Big[\int_\Omega 2xu_xu_t\Big]_0^T - \int_0^T\int_\Omega (2xu_tu_{tx} + 2xu_xu_{xx})\,dx\,dt \\ &\quad + 2b\int_0^T\int_\Omega xu_x\theta_x.\end{aligned} \tag{3.2}$$

Since we have

$$\begin{aligned}\int_\Omega (2xu_tu_{tx} + 2xu_xu_{xx}) &= \int_\Omega (x(u_t^2)_x - u_x(2xu_x)_x)\,dx + [2xu_x^2]_0^1 \\ &= -\int_\Omega (u_x^2 + u_t^2)\,dx + u_t^2(1,t) + u_x^2(1,t),\end{aligned}$$

we conclude from (3.2) that

$$\begin{aligned}0 &= \Big[\int_\Omega 2xu_xu_t\Big]_0^T + \int_0^T\int_\Omega (u_x^2 + u_t^2)\,dx\,dt + 2b\int_0^T\int_\Omega xu_x\theta_x\,dx\,dt \\ &\quad - \int_0^T (u_t^2(1,t) + g^2(u_t(1,t)))\,dt.\end{aligned} \tag{3.3}$$

Hence the relations (3.1)–(3.3) give

$$\begin{aligned}2\int_0^T E(t)\,dt &\le \Big[\int_\Omega (uu_t + 4xu_xu_t)\Big]_T^0 + \int_0^T\int_\Omega (\theta^2 - b\theta_x u) - 4b\int_0^T\int_\Omega xu_x\theta_x \\ &\quad + \int_0^T (2g^2(u_t(1,t)) - u(1,t)g(u_t(1,t)) + 2u_t^2(1,t))\,dt.\end{aligned}$$

Now we want to majorize the right hand side of the above inequality. We have

$$\Big|\int_\Omega (uu_t + 4xu_xu_t)\Big| \le \int_\Omega u^2 + \int_\Omega u_t^2 + 4\int_\Omega u_x^2 + 4\int_\Omega u_t^2 \le cE(0), \tag{3.4}$$

$$\begin{aligned}\int_\Omega (\theta^2 - bu\theta_x) &\le \int_\Omega (c\theta_x^2 + \varepsilon u^2 + c(\varepsilon)\theta_x^2)\,dx \\ &\le c(\varepsilon)\int_\Omega \theta_x^2\,dx + c\varepsilon\int_\Omega u_x^2\,dx \le -c(\varepsilon)E' + c\varepsilon E(0)\end{aligned}$$

and hence

$$\int_0^T\int_\Omega(\theta^2 - b\theta_x u)\,dx\,dt \le c(\varepsilon)E(0) + c\varepsilon\int_0^T E(t)\,dt, \tag{3.5}$$

$$\begin{aligned}\Big|4b\int_0^T\int_\Omega xu_x\theta_x\,dx\Big| &\le \varepsilon\int_0^T\int_\Omega u_x^2\,dx\,dt + c(\varepsilon)\int_0^T\int_\Omega \theta_x^2\,dx\,dt \\ &\le \varepsilon\int_0^T E(t)\,dt + c(\varepsilon)E(0),\end{aligned} \tag{3.6}$$

and finally,

$$\begin{aligned}|-u(1,t)g(u_t(1,t))| &\le \varepsilon u^2(1,t) + c(\varepsilon)g^2(u_t(1,t)) \\ &\le \varepsilon\int_\Omega u_x^2\,dx + c(\varepsilon)g^2(u_t(1,t)) \\ &\le 2\varepsilon E(t) + c(\varepsilon)g^2(u_t(1,t)).\end{aligned} \tag{3.7}$$

We deduce from (3.4)–(3.7) that

$$(2-c\varepsilon)\int_0^T E(t)\,dt \le c(\varepsilon)E(0) + c(\varepsilon)\int_0^T(u_t^2(1,t) + g^2(u_t(1,t)))\,dt. \tag{3.8}$$

As the function $g(x) = x/\sqrt{1+x^2}$ satisfies

$$\frac{1}{\sqrt{2}}|x| \le |g(x)| \le |x| \qquad \text{if } |x| \le 1, \tag{3.9}$$

$$\frac{1}{\sqrt{2}} \le |g(x)| \le |x| \qquad \text{if } |x| > 1, \tag{3.10}$$

we conclude from (3.8) that

$$(2-c\varepsilon)\int_0^T E(t)\,dt \le c(\varepsilon)E(0) + c(\varepsilon)\int_0^T u_t^2(1,t)\,dt. \tag{3.11}$$

If $|u_t(1,t)| \le 1$, then from (3.9) and (3.11) we obtain

$$\begin{aligned}(2-c\varepsilon)\int_0^T E(t)\,dt &\le c(\varepsilon)E(0) + c(\varepsilon)\int_0^T u_t(1,t)g(u_t(1,t))\,dt \\ &\le c(\varepsilon)E(0) + c(\varepsilon)\int_0^T -E'(t)\,dt \\ &\le c(\varepsilon)E(0).\end{aligned}$$

Choosing $\varepsilon = 1/c$, we obtain the desired result by applying Lemma 2.1.

If $|u_t(1,t)| \geq 1$, then from the trace theorem $H^1(\Omega) \hookrightarrow C(\overline{\Omega}) \hookrightarrow L^\infty(\Gamma)$ and (3.10) we obtain

$$(2 - c\varepsilon) \int_0^T E(t)\,dt \leq c(\varepsilon)E(0) + \|u_t\|_\infty c(\varepsilon) \int_0^T u_t g(u_t)\,dt$$
$$\leq c(\varepsilon)E(0),$$

and hence, the choice $\varepsilon = 1/c$ with Lemma 2.1 yields the desired decay estimate.

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