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CHARGE TRANSFER SCATTERING IN A CONSTANT ELECTRIC FIELD

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We prove the asymptotic completeness of the quantum scattering for a Stark Hamiltonian with a time dependent interaction potential, created by N classical particles moving in a constant electric field.

1. Introduction. We consider a model describing the quantum dynamics of a light particle (such as an electron) in collisions with some heavy particles (such as some ions) obeying the laws of classical dynamics. Thus only the light particle is considered a quantum particle, while the heavy particles follow some classical trajectories $\mathbb{R} \ni t \mapsto \chi_k(t) \in \mathbb{R}^d$. If V_k denotes the quantum interaction potential between the quantum particle and the kth classical particle, the total quantum time-dependent interaction $V(t)$ is the operator of multiplication by

$$V(t,x) = \sum_{1 \le k \le N} V_k(x - \chi_k(t)), \tag{1.1}$$

and the total time-dependent Hamiltonian $H(t)$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$,

$$H(t) = H_0 + V(t,x), \tag{1.2}$$

where H_0 denotes the free motion Hamiltonian. The subject of scattering theory is to describe the large time behaviour of the evolution propagator $\{U(t,t_0)\}_{t \ge t_0}$ of $H(t)$, that is, the family of unitary operators in $L^2(\mathbb{R}^d)$ satisfying

$$i\frac{d}{dt}U(t,t_0)\varphi = H(t)U(t,t_0)\varphi, \qquad U(t_0,t_0)\varphi = \varphi, \tag{1.3}$$

for φ from the domain of H_0.

The first papers describing such a model considered the case of linear classical trajectories and H_0 the Laplace operator [10, 25, 26]. The papers

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[7, 29] deal with classical trajectories which are only asymptotically linear and the papers [30, 31, 32] deal with the dispersive case when H_0 is a more general elliptic operator. We note that all these papers consider the hypothesis that different classical trajectories have different asymptotic velocities $\lim_{t\to\infty}\chi_k'(t)$, which implies the separation of trajectories: $|\chi_k(t)-\chi_{k'}(t)|\geq ct$ with $c>0$ if $k\neq k'$.

The aim of this paper is to consider the situation arising in the presence of a constant electric field $E\in\mathbb{R}^d\setminus\{0\}$, when the free motion Hamiltonian for a particle of mass $m>0$ and charge $q\neq 0$ has the form

$$h_0(x,p)=\frac{p^2}{2m}-qE\cdot x$$

and the Hamilton equations $\dot p(t)=qE$, $\dot x(t)=p(t)/m$ have the solutions of the form

$$p(t)=qEt+mv,\qquad x(t)=\frac{qE}{2m}t^2+vt+\omega,$$

where $v=p(0)/m\in\mathbb{R}^d$ and $\omega=x(0)\in\mathbb{R}^d$. Thus the above solutions of the Hamilton equations describe the motion that is free in the directions orthogonal to the constant field E and uniformly accelerated in the direction parallel to E.

We shall consider only the simplest situation when different classical trajectories have different asymptotic accelerations $\lim_{t\to\infty}\chi_k''(t)$. More precisely we begin by assuming the following separation condition: there exist constants T_0, $c>0$, such that for $t\geq T_0$,

$$|\chi_k(t)-\chi_{k'}(t)|\geq ct^2\quad\text{ if } 1\leq k<k'\leq N. \tag{1.4}$$

Let m_k, q_k be the mass and the charge of the kth classical particle and assume that $\chi(t)=(\chi_1(t),\ldots,\chi_N(t))$ is a solution of the Newton equations

$$m_k\chi_k''(t)=q_kE-\sum_{k'\in\{1,\ldots,N\}\setminus\{k\}}\nabla V_{k,k'}(\chi_k(t)-\chi_{k'}(t)), \tag{1.5}$$

where the classical interaction potentials $V_{k,k'}$ satisfy the decay condition

$$|\nabla V_{k,k'}(x)|\leq C_0|x|^{-1-\mu_0}\quad\text{ for } |x|\geq C_0 \tag{1.6}$$

with $C_0,\mu_0>0$.

It is clear that (1.4)–(1.6) imply

$$\chi_k''(t)=z_k+O(t^{-2(1+\mu_0)})\quad\text{ with } z_k=\frac{q_k}{m_k}E \tag{1.7}$$

as $t\to\infty$, i.e. $z_k=(q_k/m_k)E=\lim_{t\to\infty}\chi_k''(t)$ is the asymptotic acceleration of the trajectory χ_k. Since (1.7) means that $\frac{d}{dt}(\chi_k'(t)-z_kt)=O(t^{-2-2\mu_0})$, the limit

$$v_k=\lim_{t\to\infty}(\chi_k'(t)-z_kt)$$

exists and introducing $\widetilde{\chi}_k$ by the relation

$$\chi_k(t) = \tfrac{1}{2} z_k t^2 + \upsilon_k t + \widetilde{\chi}_k(t), \tag{1.8}$$

we have

$$\widetilde{\chi}_k''(t) = O(t^{-2-2\mu_0}), \quad \widetilde{\chi}_k'(t) = O(t^{-1-2\mu_0}) \quad \text{as } t \to \infty. \tag{1.9}$$

The Hamiltonian of the free motion for a quantum particle of mass $m_0 > 0$ and charge $q_0 \neq 0$ has the form

$$H_0 = \frac{p^2}{2m_0} - q_0 E \cdot x, \tag{1.10}$$

where $p = (p_1, \dots, p_d) = (-i\partial_{x_1}, \dots, -i\partial_{x_d})$.

For quantum interactions V_k we assume that for some constants $C, \widehat{C}, \varepsilon_0 > 0$,

$$V_k(x)(1+p^2)^{-1+\varepsilon_0} \text{ is a compact operator in } L^2(\mathbb{R}^d), \tag{1.11a}$$

$$|\partial_x^\alpha V_k(x)| \le C \quad \text{for } |x \cdot E| \ge \widehat{C} \text{ and } |\alpha| \le 2, \tag{1.11b}$$

and $V_k = V_k^l + V_k^s$ with real valued functions V_k^l, V_k^s, such that for some $\mu > 0$ we have

$$|\partial_x^\alpha V_k^l(x)| \le C(1+|x|)^{-\mu-|\alpha|} \quad \text{for } x \in \mathbb{R}^d \text{ and } |\alpha| \le 1, \tag{1.11c}$$

$$|\partial_x^\alpha V_k^s(x)| \le C(1+|x|)^{-\mu+(|\alpha|-1)/2} \quad \text{for } |x \cdot E| \ge \widehat{C} \text{ and } |\alpha| \le 1. \tag{1.11d}$$

THEOREM 1. *Let* $U(t,t_0)$ *be defined by* (1.3) *with* $H(t)$ *given by* (1.1), (1.2), (1.10). *For* $k = 0, 1, \dots, N$, *let* $z_k = q_k E/m_k$ *be such that* $z_k \neq z_{k'}$ *if* $0 \le k < k' \le N$. *Assume that the trajectories* $\chi_k(t)$ *have the form* (1.8) *with* $\widetilde{\chi}_k(t)$ *satisfying* (1.9) *for some* $\mu_0 > 0$. *If* $V_k = V_k^l + V_k^s$ *satisfy* (1.11a–d) *for some* $\mu > 0$, $\varepsilon_0 > 0$, *then the limit*

$$\begin{aligned} \Omega(t_0)\psi &= \lim_{t\to\infty} U(t,t_0)^* e^{-itH_0 - iS(t)}\psi \quad \text{with} \\ S(t) &= \int_1^t d\tau \sum_{1\le k\le N} V_k^l\big(\tfrac{1}{2} z_0 \tau^2 - \chi_k(\tau)\big), \end{aligned} \tag{1.12}$$

exists in the norm of $L^2(\mathbb{R}^d)$ *for every* $\psi \in L^2(\mathbb{R}^d)$. *Moreover, the asymptotic completeness holds, i.e. the wave operator* $\Omega(t_0)$ *defined by* (1.12) *is unitary.*

We recall the result of I. M. Sigal [20] (cf. also [3, 4, 5]) which guarantees the absence of eigenvalues for 2-body Stark Hamiltonians $H_k = H_0 + V_k(x)$. This allows us to neglect bound states and the asymptotic completeness formulated in Theorem 1 implies that for every $\varphi \in L^2(\mathbb{R}^d)$ there exists $\psi \in L^2(\mathbb{R}^d)$ such that $\varphi = \Omega(t_0)\psi$. Thus $U(t,t_0)\varphi - e^{-itH_0 - iS(t)}\psi \to 0$ as $t \to \infty$, which means that the asymptotic behaviour of $U(t,t_0)\varphi$ is asymptotically the same as for the free evolution (modulo a phase factor $e^{-iS(t)}$).

We note that the approach used in the proof below comes from recent developments of scattering theory of N-body systems ([6, 8, 21]). We also mention the references [9, 12, 15–17, 19, 23, 24, 27, 28, 33] concerning Stark scattering in the 2-body case and [1, 2, 13, 14, 18, 22] in the N-body case.

In Section 2 we begin by describing in Lemma 2.1 asymptotic concentration of the free evolution trajectories $e^{-itH_0}\varphi$ on classical Stark trajectories. Then it is easy to prove the existence of the wave operator $\Omega(t_0)$ given by (1.12). Clearly $\Omega(t_0)$ is an isometric injection and in order to prove the asymptotic completeness it suffices to prove the existence of the limit

$$\Omega(t_0)^*\varphi = \lim_{t\to\infty} e^{itH_0+iS(t)}U(t,t_0)\varphi \tag{1.12$'$}$$

for every $\varphi \in L^2(\mathbb{R}^d)$. Indeed, if $\Omega(t_0)^*$ given by (1.12′) exists, then applying the chain rule we get $\Omega(t_0)\Omega(t_0)^*\varphi = \varphi$, that is, $\Omega(t_0)$ is surjective and hence unitary.

To begin the proof of the existence of (1.12′) we assume for simplicity $V_k^s = 0$ and introduce the auxiliary observable η_t. This observable is used in Proposition 3.2 to introduce an energy cut-off, similarly to the "boosted Hamiltonian" of Graf [7]. However, instead of Enss approach used in Graf [7], our next step is based on the existence of the wave operators $\Omega_k(t)$ of Proposition 3.7 (similar to the Deift–Simon operators of the N-body theory developed in Graf [8]). Then Proposition 3.7 allows us to localize and "distinguish" interactions of different classical charges, reducing the problem to the 2-body problem when the number of classical charges is $N = 1$.

The situation $N = 1$ is studied in Section 4 using the ideas of the Mourre estimate. More precisely, knowing that $z_0\cdot p$ is the conjugate operator for H_0 (i.e. we have the positive commutator $[iH_0, z_0\cdot p] = z_0^2 I$), we find the propagation estimate of Proposition 4.3 using a suitable cut-off $g_1(z_0\cdot p/t)$ instead of $z_0\cdot p$. Finally, in Section 5 we sketch the idea allowing one to modify the observable η_t in order to recover all the previous results in the case of interaction potentials with singularities, $V_k^s \neq 0$.

2. Preliminary estimates. For $\mathcal{U} \subset \mathbb{R}^d$, $C_0^\infty(\mathcal{U})$ is the set of smooth functions with compact support in $\mathcal{U}$. We write $a_t = O(f(t))$ if there is a constant $C > 0$ such that $\|a_t\| \le Cf(t)$, where $\|\cdot\|$ is the norm of $L^2(\mathbb{R}^d)$ or the norm of bounded operators $B(L^2(\mathbb{R}^d))$. The analogous notation will be used when $a_t = (a_t^1,\ldots,a_t^d)$ assuming $\|a_t\| = (\|a_t^1\|^2 + \ldots + \|a_t^d\|^2)^{1/2}$. Moreover, $a_t = b_t + O(f(t))$ means $a_t - b_t = O(f(t))$. For $Z \subset \mathbb{R}$, $\mathbf{1}_Z$ denotes the characteristic function of Z on $\mathbb{R}$.

Assume that V_0 is a real function satisfying

$$|\partial_t^n \partial_x^\alpha V_0(t,x)| \le C t^{-2\mu-2|\alpha|-n} \quad \text{for } |\alpha|+n \le 1, \tag{2.1}$$

and denote by $U_0(t,t_0)$ the evolution propagator of the Hamiltonian

$$H_0(t) = H_0 + V_0(t,x), \tag{2.2}$$

where H_0 is given by (1.10). By rescaling we may assume further on that $m_0 = 1$.

Let $y_t = (y_t^1, \ldots, y_t^d)$, $w_t = (w_t^1, \ldots, w_t^d)$ be systems of d commuting self-adjoint operators,

$$y_t = \frac{2x}{t^2} - z_0, \qquad w_t = \frac{p}{t} - z_0. \tag{2.3}$$

LEMMA 2.1. *Let $U_0(t,t_0)$, y_t, w_t be as above and $\varphi \in C_0^\infty(\mathbb{R}^d)$. Then*

$$w_t U_0(t,t_0)\varphi = O(t^{-1}), \qquad y_t U_0(t,t_0)\varphi = O(t^{-1}) \tag{2.4}$$

and for every $\kappa > 0$ and $j = 1, \ldots, d$ one has $\mathbf{1}_{[\kappa;\infty[}(|y_t^j|)U_0(t,t_0)\varphi = O(t^{-1})$.

Proof. Define $U_0(t,t_0) = U_t^0$, $f(t) = U_t^{0*} p U_t^0 \varphi$ and $g(t) = U_t^{0*} x U_t^0 \varphi$. Then

$$\begin{aligned} f'(t) &= U_t^{0*}[iH_0(t), p]U_t^0\varphi = z_0\varphi + O(t^{-2(1+\mu)}),\\ g'(t) &= U_t^{0*}[iH_0(t), x]U_t^0\varphi = f(t)\\ &= f(t_0) + \int_{t_0}^{t} f'(\tau)\,d\tau = t z_0 \varphi + O(1), \end{aligned}$$

hence $w_t U_t^0 \varphi = t^{-1} U_t^0 (f(t) - z_0 t\varphi) = O(t^{-1})$. Moreover,

$$g(t) = g(t_0) + \int_{t_0}^{t} g'(\tau)\,d\tau = \tfrac{1}{2} z_0 t^2 \varphi + O(t),$$

and $\big(x - \frac{1}{2} z_0 t^2\big) U_t^0 \varphi = U_t^0 \big(g(t) - \frac{1}{2} z_0 t^2 \varphi\big) = O(t)$ implies the second estimate (2.4). Finally, using $\kappa^2 \mathbf{1}_{[\kappa;\infty[}(|\lambda|) \le \lambda^2$ and the second estimate (2.4) we obtain

$$(\kappa^2 \mathbf{1}_{[\kappa;\infty[}(|y_t^j|) U_t^0\varphi, U_t^0\varphi) \le ((y_t^j)^2 U_t^0\varphi, U_t^0\varphi) = \|y_t^j U_t^0 \varphi\|^2 = O(t^{-2}). \ ■$$

Note that (1.9) implies the existence of

$$\lim_{t\to\infty} \widetilde{\chi}_k(t) = \omega_k \quad \text{with} \quad \widetilde{\chi}_k(t) = \omega_k + O(t^{-2\mu_0}), \tag{2.5}$$

hence

$$\chi_k'(t) = z_k t + \upsilon_k + O(t^{-1-2\mu_0}), \qquad \chi_k(t) = \tfrac{1}{2} z_k t^2 + \upsilon_k t + O(1). \tag{2.5$'$}$$

By rotation of the coordinate system we may assume further on that $E = (E_1, 0, \ldots, 0)$ with $E_1 \in \mathbb{R} \setminus \{0\}$, hence $z_k = (z_k^1, 0, \ldots, 0)$ with $z_k^1 =$

$E_1 q_k/m_k$. Further, we set

$$\tau = \tfrac{1}{16}\min\{|z_k^1 - z_{k'}^1| : 0 \le k < k' \le N\}. \tag{2.6}$$

Fix $J^0 \in C_0^\infty(]-4\tau; 4\tau[)$ such that $0 \le J^0 \le 1$, $J^0 = 1$ on $[-2\tau; 2\tau]$, define $\overline{J}^0 = 1 - J^0$ and let

$$\begin{aligned} V_{0k}(t,x) &= \overline{J}^0(4x_1/t^2 - 2z_k^1)V_k^l(x - \chi_k(t)) \\ &= \overline{J}^0(2y_t^1 - 2\widetilde{z}_k)V_k^l(x - \chi_k(t)) \end{aligned} \tag{2.7}$$

where we have set $\widetilde{z}_k = z_k^1 - z_0^1$. Then we have

PROPOSITION 2.2. *Let* $V_0 = \sum_{1\le k\le N} V_{0k}$, *where* V_{0k} *is given by* (2.7). *Then* (2.1) *holds and for every* $\varphi \in L^2(\mathbb{R}^d)$ *the following limits exist*:

$$\begin{aligned} \widetilde{\Omega}(t_0)^*\varphi &= \lim_{t\to\infty} e^{itH_0 + iS(t)}U_0(t,t_0)\varphi, \\ \widetilde{\Omega}(t_0)\varphi &= \lim_{t\to\infty} U_0(t,t_0)^* e^{-itH_0 - iS(t)}\varphi. \end{aligned} \tag{2.8}$$

Proof. Since $\chi_k(t) = \frac{1}{2}z_k t^2 + O(t)$ there is T_0 such that for $t \ge T_0$ we have

$$\begin{aligned} \overline{J}^0(4x_1/t^2 - 2z_k^1) \ne 0 &\Rightarrow |4x_1/t^2 - 2z_k^1| \ge 2\tau \\ &\Rightarrow |x - \chi_k(t)| \ge \left|x_1 - \tfrac{1}{2}z_k^1 t^2\right| - \left|\tfrac{1}{2}z_k t^2 - \chi_k(t)\right| \\ &\qquad \ge \tfrac{1}{2}\tau t^2 - C't \ge \tfrac{1}{4}\tau t^2 \end{aligned}$$

and applying (1.11) we find

$$|x - \chi_k(t)| \ge \tfrac{1}{4}\tau t^2 \Rightarrow |(\partial^\alpha V_k^l)(x - \chi_k(t))| \le Ct^{-2(\mu+|\alpha|)} \quad \text{if } |\alpha| \le 1. \tag{2.9}$$

We conclude that V_0 satisfies (2.1) noting that

$$\frac{\partial}{\partial x_1}(\overline{J}^0(4x_1/t^2 - 2z_k^1)) = O(t^{-2}), \qquad \frac{\partial}{\partial t}(\overline{J}^0(4x_1/t^2 - 2z_k^1)) = O(t^{-1}).$$

Since $C_0^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, to obtain the existence of $\widetilde{\Omega}(t_0)^*\varphi$ it suffices to consider $\varphi \in C_0^\infty(\mathbb{R}^d)$ and to check that

$$\begin{aligned} &\frac{d}{dt}(e^{itH_0+iS(t)}U_0(t,t_0)\varphi) \\ &\qquad = e^{itH_0+iS(t)}i(S'(t) - V_0(t,x))U_0(t,t_0)\varphi = O(t^{-1-2\mu}). \end{aligned} \tag{2.10}$$

However, for $1 \le k \le N$ we have $|z_k^1 - z_0^1| \ge 16\tau$, hence $\overline{J}^0(2z_0^1 - 2z_k^1) = 1$ and

$$V_0\big(t, \tfrac{1}{2}z_0t^2\big) = \sum_{1\le k\le N} \overline{J}^0(2z_0^1 - 2z_k^1)V_k^l\big(\tfrac{1}{2}z_0t^2 - \chi_k(t)\big) = S'(t). \tag{2.11}$$

Thus we may write

$$V_0(t,x) - S'(t) = V_0(t,x) - V_0\big(t, \tfrac{1}{2}z_0t^2\big) = \gamma_t \cdot \big(x - \tfrac{1}{2}z_0t^2\big) = \tfrac{1}{2}\gamma_t \cdot t^2 y_t$$

with

$$\gamma_t = \int_0^1 d\theta\, \nabla_x V_0\big(t, (1-\theta)x + \tfrac{1}{2}\theta z_0 t^2\big)$$

and (2.1) implies $t^2\gamma_t = O(t^{-2\mu})$. Therefore

$$\begin{aligned}(2.12)\qquad \|(S'(t) - V_0(t,x))U_0(t,t_0)\varphi\| &= \big\|\tfrac{1}{2}t^2\gamma_t \cdot y_t U_0(t,t_0)\varphi\big\| \\ &\le Ct^{-2\mu}\|y_t U_0(t,t_0)\varphi\|\end{aligned}$$

and by (2.4) the right hand side of (2.12) is $O(t^{-1-2\mu})$, i.e. (2.10) follows.

We may use $V_0(t,x) = 0$ in Lemma 2.1, hence it is clear that e^{-itH_0} satisfies the same estimates as $U_0(t,t_0)$, and we obtain the existence of the second limit (2.8) as above with e^{-itH_0} and $U_0(t,t_0)$ interchanged. ∎

Proof of the existence of $\Omega(t_0)$. Using the chain rule and the existence of (2.8), we note that it suffices to prove the existence of $\lim_{t\to\infty} U(t,t_0)^* \times U_0(t,t_0)\varphi$, where as before we may assume $\varphi \in C_0^\infty(\mathbb{R}^d)$. Let $J \in C_0^\infty(\mathbb{R}^d)$ be such that $J(x) = 1$ for $|x| \le \tau$, $J(x) = 0$ for $|x| \ge 2\tau$, $0 \le J \le 1$. Then Lemma 2.1 implies

$$\|(1-J)(y_t)U_0(t,t_0)\varphi\| \le \|\mathbf{1}_{[\tau;\infty[}(|y_t|)U_0(t,t_0)\varphi\| = O(t^{-1}),$$

i.e.

$$\lim_{t\to\infty} U(t,t_0)^* J(y_t)U_0(t,t_0)\varphi = \lim_{t\to\infty} U(t,t_0)^* U_0(t,t_0)\varphi$$

and it suffices to show that

$$\begin{aligned}(2.13)\qquad &\frac{d}{dt}(U(t,t_0)^* J(y_t)U_0(t,t_0)\varphi) \\ &= U(t,t_0)^*(\mathbb{D}_{H_0} J(y_t) + iJ(y_t)(V(t,x) - V_0(t,x)))U_0(t,t_0)\varphi \\ &= O(t^{-1-2\mu}) + O(t^{-2}),\end{aligned}$$

where $\mathbb{D}_{a_t} b_t = [ia_t, b_t] + \frac{d}{dt} b_t$ denotes the Heisenberg derivative.

However, a simple calculation gives

$$(2.14)\qquad \mathbb{D}_{H_0} J(y_t) = \frac{2}{t}\sum_{1\le j\le d} \partial_j J(y_t)(w_t^j - y_t^j) + O(t^{-3})$$

and using (2.4) we obtain $(\mathbb{D}_{H_0} J(y_t))U_0(t,t_0)\varphi = O(t^{-2})$.

Next for $1 \le k \le N$ we have

$$\begin{aligned}J^0(2y_t^1 - 2\widetilde{z}_k) \ne 0 &\Rightarrow |y_t^1 - \widetilde{z}_k| < 2\tau \\ &\Rightarrow |y_t^1| \ge |\widetilde{z}_k| - 2\tau = |z_k^1 - z_0^1| - 2\tau \ge 14\tau \Rightarrow J(y_t) = 0,\end{aligned}$$

hence $J(y_t)\overline{J}^0(2y_t^1 - 2\widetilde{z}_k) = J(y_t)$ and

$$J(y_t)(V - V_0)(t,x) = \sum_{1\le k\le N} J(y_t)\overline{J}^0(2y_t^1 - 2\widetilde{z}_k)V_k^s(x - \chi_k(t)).$$

If T_0 is as at the beginning of the proof of Proposition 2.2, then for $t \geq T_0$ we have

$$\overline{J}^0(2y_t^1 - 2\widetilde{z}_k) \neq 0 \Rightarrow |x - \chi_k(t)| \geq \tfrac{1}{4}\tau t^2 \Rightarrow |V_k^s(x - \chi_k(t))| \leq Ct^{-1-2\mu}. \ \blacksquare$$

Until the end of Section 4 we assume that $V_k^s = 0$, that is, $V_k = V_k^l$. We now introduce

$$\eta_t^0 = \frac{1}{2}\left(\frac{p_1}{t} - \frac{2x_1}{t^2}\right)^2 + \frac{1}{4}\left(\frac{2x_1}{t^2} - z_0^1\right)^2 + \frac{1}{2}\sum_{2\leq j\leq d}\frac{p_j^2}{t^2} + I, \tag{2.15}$$

$$\eta_t = \eta_t^0 + \frac{V(t,x)}{t^2}. \tag{2.16}$$

LEMMA 2.3. *If η_t^0, η_t are given by* (2.15)–(2.16) *and $\mathbb{D}$ is defined as below* (2.13), *then $\mathbb{D}_{H(t)}\eta_t = \mathbb{D}_{H_0}\eta_t^0 + r_t$ with*

$$r_t = \frac{d}{dt}\left(\frac{V(t,x)}{t^2}\right) - \left[iV(t,x), \frac{x_1p_1 + p_1x_1}{t^3}\right]. \tag{2.17}$$

Proof. A simple transformation of the expression (2.15) gives

$$\begin{aligned}
\eta_t^0 &= \frac{1}{2}\left(\frac{p_1^2}{t^2} - 2\frac{x_1p_1 + p_1x_1}{t^3} + \frac{4x_1^2}{t^4}\right) \\
&\quad + \frac{1}{4}\left(4\frac{x_1^2}{t^4} - 4\frac{z_0^1x_1}{t^2} + (z_0^1)^2\right) + \frac{1}{2}\sum_{2\leq j\leq d}\frac{p_j^2}{t^2} + I \\
&= \frac{1}{2}\frac{p_1^2}{t^2} - \frac{x_1p_1 + p_1x_1}{t^3} + \left(\frac{1}{2}\cdot 4 + \frac{1}{4}\cdot 4\right)\frac{x_1^2}{t^4} \\
&\quad - \frac{z_0^1x_1}{t^2} + \frac{(z_0^1)^2}{4} + \frac{1}{2}\sum_{2\leq j\leq d}\frac{p_j^2}{t^2} + I \\
&= \frac{1}{t^2}\left(\frac{1}{2}p^2 - z_0^1x_1\right) - \frac{x_1p_1 + p_1x_1}{t^3} + \frac{3x_1^2}{t^4} + \frac{(z_0^1)^2}{4} + I.
\end{aligned}$$

Therefore we may express η_t^0 in the following way:

$$\eta_t^0 = \frac{H_0}{t^2} - \frac{x_1p_1 + p_1x_1}{t^3} + \frac{3x_1^2}{t^4} + \frac{(z_0^1)^2}{4} + I \tag{2.15$'$}$$

and compute

$$\begin{aligned}
\mathbb{D}_{H(t)}\eta_t &= \mathbb{D}_{H(t)}\left(\eta_t^0 + \frac{V(t,x)}{t^2}\right) = \mathbb{D}_{H(t)}\left(\frac{H(t)}{t^2} - \frac{x_1p_1 + p_1x_1}{t^3} + \frac{3x_1^2}{t^4}\right) \\
&= \mathbb{D}_{H(t)}\left(\frac{H(t)}{t^2}\right) - \left[iV(t,x), \frac{x_1p_1 + p_1x_1}{t^3}\right] \\
&\quad + \mathbb{D}_{H_0}\left(-\frac{x_1p_1 + p_1x_1}{t^3} + \frac{3x_1^2}{t^4}\right)
\end{aligned}$$

$$= \frac{d}{dt}\left(\frac{H_0}{t^2} + \frac{V(t,x)}{t^2}\right) - \left[iV(t,x), \frac{x_1p_1 + p_1x_1}{t^3}\right]$$
$$+ \mathbb{D}_{H_0}\left(- \frac{x_1p_1 + p_1x_1}{t^3} + \frac{3x_1^2}{t^4}\right)$$
$$= \mathbb{D}_{H_0}\left(\frac{H_0}{t^2}\right) + \mathbb{D}_{H_0}\left(- \frac{x_1p_1 + p_1x_1}{t^3} + \frac{3x_1^2}{t^4}\right) + r_t = \mathbb{D}_{H_0}\eta_t^0 + r_t. \ \blacksquare$$

LEMMA 2.4. *If r_t is given by* (2.17) *then* $r_t = O(t^{-2})$.

Proof. First note that

$$\frac{d}{dt}(t^{-2}V(t,x)) = t^{-2}\partial_t V(t,x) - 2t^{-3}V(t,x) = t^{-2}\partial_t V(t,x) + O(t^{-3}).$$

Thus setting $\chi_k'(t) = (\dot\chi_k^1(t), \dot\chi_k^\perp(t)) \in \mathbb{R}\times\mathbb{R}^{d-1}$ and using $\dot\chi_k^\perp(t) = O(1)$, we have

$$t^2 r_t = \partial_t V(t,x) - \left[iV(t,x), \frac{x_1p_1 + p_1x_1}{t}\right] + O(t^{-1})$$
$$= \sum_{1\le k\le N} \partial_{x_1} V_k(x - \chi_k(t))\left(\frac{2x_1}{t} - \dot\chi_k^1(t)\right) + O(1).$$

But $2x_1/t - \dot\chi_k^1(t) = (2/t)(x_1 - \chi_k^1(t)) + O(1)$ by (2.5′) and we complete the proof noting that $\partial_{x_1} V_k(x - \chi_k(t))(x_1 - \chi_k^1(t)) = O(1)$. ■

PROPOSITION 2.5. *If η_t is given by* (2.16) *and $\mathbb{D}$ as below* (2.13), *then*

$$\mathbb{D}_{H(t)}\eta_t = -\frac{3}{t}\left(\frac{p_1}{t} - \frac{2x_1}{t^2}\right)^2 - \sum_{2\le j\le d}\frac{p_j^2}{t^3} + O(t^{-2}). \tag{2.18}$$

Proof. By Lemmas 2.3 and 2.4 it suffices to check that

$$\mathbb{D}_{H_0}\eta_t^0 = -\frac{3}{t}\left(\frac{p_1}{t} - \frac{2x_1}{t^2}\right)^2 - \sum_{2\le j\le d}\frac{p_j^2}{t^3}. \tag{2.19}$$

Now we note that formally

$$\mathbb{D}_{a_t}(b_t\widetilde{b}_t) = (\mathbb{D}_{a_t}b_t)\widetilde{b}_t + (b_t\mathbb{D}_{a_t}\widetilde{b}_t). \tag{2.20}$$

If a_t and b_t are self-adjoint, then

$$\mathbb{D}_{a_t}(b_t)^2 = b_t(\mathbb{D}_{a_t}b_t) + (\mathbb{D}_{a_t}b_t)b_t = 2b_t(\mathbb{D}_{a_t}b_t) + hc, \tag{2.20′}$$

where $m_t + hc = \frac{1}{2}(m_t + m_t^*)$ denotes the Hermitian symmetrization of the operator m_t. In particular, using

$$\mathbb{D}_{H_0}w_t = -\frac{w_t}{t}, \qquad \mathbb{D}_{H_0}y_t^1 = \frac{2}{t}(w_t^1 - y_t^1) \tag{2.21}$$

[where w_t, y_t are given by (2.3)], we obtain

$$\frac{1}{4}\mathbb{D}_{H_0}(y_t^1)^2 = \frac{1}{2}y_t^1\mathbb{D}_{H_0}y_t^1 + hc = \frac{1}{t}y_t^1(w_t^1 - y_t^1) + hc,$$

$$\begin{aligned}\frac{1}{2}\mathbb{D}_{H_0}(w_t^1 - y_t^1)^2 &= (w_t^1 - y_t^1)\mathbb{D}_{H_0}(w_t^1 - y_t^1) + hc\\ &= \frac{1}{t}(w_t^1 - y_t^1)(2y_t^1 - 3w_t^1) + hc.\end{aligned}$$

Introducing $w_t^\perp = (w_t^2, \ldots, w_t^d) = (p_2/t, \ldots, p_d/t)$ we may express (2.15) in the form

$$\eta_t^0 = \tfrac{1}{2}(w_t^1 - y_t^1)^2 + \tfrac{1}{4}(y_t^1)^2 + \tfrac{1}{2}|w_t^\perp|^2 + I$$

and it is clear that $\frac{1}{2}\mathbb{D}_{H_0}|w_t^\perp|^2 = -\frac{1}{t}|w_t^\perp|^2$. To complete the proof we compute

$$\begin{aligned}\frac{1}{2}\mathbb{D}_{H_0}&(w_t^1 - y_t^1)^2 + \frac{1}{4}\mathbb{D}_{H_0}(y_t^1)^2\\ &= \frac{1}{t}(w_t^1 - y_t^1)(2y_t^1 - 3w_t^1) + \frac{1}{t}(w_t^1 - y_t^1)y_t^1 + hc\\ &= \frac{1}{t}(w_t^1 - y_t^1)(3y_t^1 - 3w_t^1) + hc = -\frac{3}{t}(w_t^1 - y_t^1)^2. \ \blacksquare\end{aligned}$$

3. Propagation estimates. We denote by $\mathcal{G}(H)$ the set of operator-valued functions $t \mapsto M(t) \in B(L^2(\mathbb{R}^d))$ satisfying

$$\int_1^T dt\, \mathrm{Re}(M(t)U(t,t_0)\varphi, U(t,t_0)\varphi) \le C\|\varphi\|^2 \tag{3.1}$$

for all $\varphi \in L^2(\mathbb{R}^d)$, all $T \ge 1$ and for some constant $C > 0$.

Sometimes we write $M(t) \in \mathcal{G}(H(t))$ instead of $M \in \mathcal{G}(H)$. We note that

(3.2) if $M(t) = O(t^{-1-\varepsilon})$ with $\varepsilon > 0$, then $M \in \mathcal{G}(H)$,

(3.3) if ($\widetilde{M} \in \mathcal{G}(H)$ and $M(t) \le \widetilde{M}(t)$ for all $t \ge 1$), then $M \in \mathcal{G}(H)$.

If $\mathbb{D}_{H(t)}M(t)$ is well defined, then writing $U(t,t_0)\varphi = \varphi_t$ we have

$$\int_1^T dt\,((\mathbb{D}_{H(t)}M(t))\varphi_t, \varphi_t) = \int_1^T dt\,\frac{d}{dt}(M(t)\varphi_t, \varphi_t) = [(M(t)\varphi_t, \varphi_t)]_1^T \tag{3.4}$$

and if $M(t) = O(1)$, then $\mathbb{D}_{H(t)}M(t) \in \mathcal{G}(H(t))$.

Note that $\eta_t^0 \ge I$ and $\eta_t = \eta_t^0 + O(t^{-2})$, hence for $n \ge 1$, $t \ge T_0$, $\widetilde{\eta}_{n,t} = (1 + \eta_t/n)^{-1}$ is well defined and satisfies $0 \le \widetilde{\eta}_{n,t} \le I$. Introducing

$$M_0(t) = \frac{1}{t}\widetilde{\eta}_{n,t}(3(w_t^1 - y_t^1)^2 + |w_t^\perp|^2)\widetilde{\eta}_{n,t}, \tag{3.5}$$

we find that Proposition 2.5 gives

$$n\mathbb{D}_{H(t)}\widetilde{\eta}_{n,t} = -\widetilde{\eta}_{n,t}(\mathbb{D}_{H(t)}\eta_t)\widetilde{\eta}_{n,t} = M_0(t) + O(t^{-2}). \tag{3.6}$$

It is clear that (3.4), (3.2) and (3.6) give

COROLLARY 3.1. *If* M_0 *is given by* (3.5), *then* $M_0 \in \mathcal{G}(H)$.

PROPOSITION 3.2. *For every* $\varphi \in L^2(\mathbb{R}^d)$ *we have*

$$\lim_{n\to\infty} \sup_{t\geq T_0} \|(I - \widetilde{\eta}^2_{n,t})U(t,t_0)\varphi\| = 0.$$

Proof. First we set $U(t,t_0)\varphi = \varphi_t$ and note that $0 \leq \lambda \leq 1 \Rightarrow (1-\lambda^2)^2 \leq 4(1-\lambda)$, hence

$$\|(I - \widetilde{\eta}^2_{n,t})\varphi_t\|^2 = ((I - \widetilde{\eta}^2_{n,t})^2\varphi_t, \varphi_t) \leq 4((I - \widetilde{\eta}_{n,t})\varphi_t, \varphi_t).$$

It remains to note that $\widetilde{\eta}_{n,T_0}\varphi_{T_0} \to \varphi_{T_0}$ as $n \to \infty$, and $-n\mathbb{D}_{H(t)}\widetilde{\eta}_{n,t} \leq -M_0(t) + Ct^{-2} \leq Ct^{-2}$ allows us to estimate

$$[((I - \widetilde{\eta}_{n,t})\varphi_t, \varphi_t)]^T_{T_0} = -\int_{T_0}^{T} dt\,((\mathbb{D}_{H(t)}\widetilde{\eta}_{n,t})\varphi_t, \varphi_t) \leq \int_{T_0}^{T} dt\,Ct^{-2}/n \leq C/n. \ \blacksquare$$

Further on in this section we assume $n \geq 1$ fixed and write simply $\widetilde{\eta}_t = \widetilde{\eta}_{n,t}$. As below (2.20′), $M(t)+hc$ denotes the symmetrization $\frac{1}{2}(M(t)+M(t)^*)$.

LEMMA 3.3. *Let* $J_0 \in C_0^\infty(\mathbb{R})$. *Then* $M_1 \in \mathcal{G}(H)$ *if*

$$M_1(t) = \frac{1}{t}\widetilde{\eta}_t(y_t^1 - w_t^1)J_0(y_t^1)\widetilde{\eta}_t + hc. \tag{3.7}$$

Proof. Let $J \in C^\infty(\mathbb{R})$ be such that the derivative $J' = -J_0$, and set

$$M_{1,0}(t) = \widetilde{\eta}_t J(y_t^1)\widetilde{\eta}_t.$$

Then $\mathbb{D}_{H(t)}M_{1,0} = M_{1,1} + M_{1,2}$ with

$$M_{1,1}(t) = \widetilde{\eta}_t(\mathbb{D}_{H(t)}J(y_t^1))\widetilde{\eta}_t = 2M_1(t) + O(t^{-3}),$$
$$M_{1,2}(t) = 2\widetilde{\eta}_t J(y_t^1)\mathbb{D}_{H(t)}\widetilde{\eta}_t + hc.$$

From (3.4) we have $\mathbb{D}_{H(t)}M_{1,0} \in \mathcal{G}(H)$ and it is clear that in order to show $M_1 \in \mathcal{G}(H)$ it suffices to check that $-M_{1,2} \in \mathcal{G}(H)$.

Noting that

$$w_t^\perp\widetilde{\eta}_t = O(1), \quad y_t^1\widetilde{\eta}_t = O(1), \quad (w_t^1 - y_t^1)\widetilde{\eta}_t = O(1),$$

it is easy to estimate the commutators

$$n[\widetilde{\eta}_t, w_t^\perp] = -\widetilde{\eta}_t[\eta_t^0 + O(t^{-2}), w_t^\perp]\widetilde{\eta}_t = O(t^{-2}),$$
$$n[\widetilde{\eta}_t, w_t^1 - y_t^1] = \widetilde{\eta}_t[\eta_t^0 + O(t^{-2}), y_t^1 - w_t^1]\widetilde{\eta}_t$$
$$= \widetilde{\eta}_t\big[\tfrac{1}{4}(y_t^1)^2, y_t^1 - w_t^1\big]\widetilde{\eta}_t + O(t^{-2}) = O(t^{-2}),$$

$$n[\widetilde{\eta}_t, J(y_t^1)] = -\widetilde{\eta}_t[\eta_t^0, J(y_t^1)]\widetilde{\eta}_t = O(t^{-2}).$$

Using (2.18) to express $\mathbb{D}_{H(t)}\widetilde{\eta}_t$ in $M_{1,2}(t)$ it is easy to see that the above commutator estimates allow us to write

$$-M_{1,2}(t) = \frac{2}{t}\widetilde{\eta}_t(3(w_t^1 - y_t^1)a_t(w_t^1 - y_t^1) + w_t^\perp a_t w_t^\perp)\widetilde{\eta}_t + O(t^{-2})$$

with $a_t = -n^{-1}J(y_t^1)\widetilde{\eta}_t + hc$, and it is clear that the inequality $a_t \le CI$ implies

$$-M_{1,2}(t) \le 2CM_0(t) + Ct^{-2} \tag{3.8}$$

where M_0 is given by (3.5). By Lemma 3.3 the right hand side of (3.8) belongs to $\mathcal{G}(H)$ and consequently $-M_{1,2} \in \mathcal{G}(H)$. ∎

PROPOSITION 3.4. *Let* $J_0 \in C_0^\infty(\mathbb{R} \setminus \{\widetilde{z}_1, \ldots, \widetilde{z}_N\})$ *where* $\widetilde{z}_k = z_k^1 - z_0^1$. *Then* $M_2 \in \mathcal{G}(H)$ *if*

$$M_2(t) = \frac{1}{t}\widetilde{\eta}_t J_0(y_t^1)y_t^1\widetilde{\eta}_t. \tag{3.9}$$

Proof. If M_1 is given by (3.7), then $M_1 \in \mathcal{G}(H)$ and $M_2 = 3M_1 + M_3$ with

$$M_3(t) = \frac{1}{t}\widetilde{\eta}_t(3w_t^1 - 2y_t^1)J_0(y_t^1)\widetilde{\eta}_t + hc.$$

Thus it remains to show that $M_3 \in \mathcal{G}(H)$. But for $1 \le k \le N$, $\widetilde{z}_k \notin \operatorname{supp} J_0$ and

$$\begin{aligned} J_0(y_t^1) \ne 0 &\Rightarrow |y_t^1 - \widetilde{z}_k| = |2x_1/t^2 - z_k^1| \ge c > 0 \\ &\Rightarrow |x - \chi_k(t)| \ge \left|x_1 - \tfrac{1}{2}z_k^1 t^2\right| - C't \ge \tfrac{1}{2}ct^2 - C't \end{aligned}$$

implies

$$[iV(t,x), w_t^1]J_0(y_t) = -\partial_x V(t,x)J_0(y_t)t^{-1} = O(t^{-3}).$$

Therefore introducing

$$M_{3,0}(t) = \widetilde{\eta}_t(y_t^1 - w_t^1)J_0(y_t)\widetilde{\eta}_t + hc,$$

we find that $\mathbb{D}_{H(t)}M_{3,0} = M_{3,1} + M_{3,2} + M_{3,3}$ with

$$\begin{aligned} M_{3,1}(t) &= \widetilde{\eta}_t(\mathbb{D}_{H(t)}(y_t^1 - w_t^1))J_0(y_t^1)\widetilde{\eta}_t = M_3(t) + O(t^{-3}), \\ M_{3,2}(t) &= \widetilde{\eta}_t(y_t^1 - w_t^1)(\mathbb{D}_{H(t)}J_0(y_t^1))\widetilde{\eta}_t + hc, \\ M_{3,3}(t) &= 2\widetilde{\eta}_t(y_t^1 - w_t^1)J_0(y_t^1)\mathbb{D}_{H(t)}\widetilde{\eta}_t + hc. \end{aligned}$$

As before, (3.4) gives $\mathbb{D}_{H(t)}M_{3,0} \in \mathcal{G}(H)$ and $M_3 \in \mathcal{G}(H)$ follows if we know that $-M_{3,2}, -M_{3,3} \in \mathcal{G}(H)$. To show $-M_{3,3} \in \mathcal{G}(H)$ we note that we may replace $M_{1,2}$ by $M_{3,3}$ in (3.8) using $a_t = n^{-1}J_0(y_t^1)(w_t^1 - y_t^1)\widetilde{\eta}_t + hc \le CI$ to express $-M_{3,3}$ similarly to $-M_{1,2}$. Also

$$-M_{3,2}(t) = -\frac{2}{t}\widetilde{\eta}_t(y_t^1 - w_t^1)J_0'(y_t^1)(y_t^1 - w_t^1)\widetilde{\eta}_t + O(t^{-3})$$

$$\leq CM_0(t) + Ct^{-3} \in \mathcal{G}(H(t)). \ \blacksquare$$

We keep the notations $J^0, \widetilde{z}_k, V_{0k}, V_0, H_0(t), U_0(t,t_0)$ introduced in Section 2. Moreover, for $1 \leq k \leq N$ we denote by $U_k(t,t_0)$ the evolution propagator of the Hamiltonian

$$\begin{aligned} &H_k(t) = H_0 + V^k(t,x) \quad \text{with} \\ &V^k(t,x) = V_k(x - \chi_k(t)) + \sum_{k' \in \{1,\ldots,N\}\setminus\{k\}} V_{0k'}(t,x). \end{aligned} \tag{3.10}$$

COROLLARY 3.5. *If M_0, M_2, H_k are as above, then $M_0, M_2 \in \mathcal{G}(H_k)$.*

Proof. Define η_t^k by using $V^k(t,x)$ instead of $V(t,x)$ in (2.16). As before we obtain

$$M_0^k(t) = \frac{1}{t}\widetilde{\eta}_t^k(3(w_t^1 - y_t^1)^2 + |w_t^\perp|^2)\widetilde{\eta}_t^k \in \mathcal{G}(H_k(t))$$

with $\widetilde{\eta}_t^k = (1 + \eta_t^k/n)^{-1}$. We recall that $|\partial_t^n \partial_x^\alpha V_{0k'}(t,x)| \leq Ct^{-2\mu-2|\alpha|-n}$ for $|\alpha| + n \leq 1$, and reasoning as in the proof of Proposition 3.4 we find

$$M_2^k(t) = \frac{1}{t}\widetilde{\eta}_t^k J_0(y_t^1) y_t^1 \widetilde{\eta}_t^k \in \mathcal{G}(H_k(t))$$

for $J_0 \in C_0^\infty(\mathbb{R} \setminus \{\widetilde{z}_1, \ldots, \widetilde{z}_N\})$. However, $\eta_t = \eta_t^k + O(t^{-2})$ implies

$$((w_t^1 - y_t^1)^2 + |w_t^\perp|^2)(\widetilde{\eta}_t - \widetilde{\eta}_t^k) = ((w_t^1 - y_t^1)^2 + |w_t^\perp|^2)\widetilde{\eta}_t^k(\eta_t - \eta_t^k)\widetilde{\eta}_t/n = O(t^{-2}),$$

hence

$$M_j(t) = M_j^k(t) + O(t^{-2}) \in \mathcal{G}(H_k(t)), \qquad j = 0, 2. \ \blacksquare$$

The following well known lemma is the basic tool allowing us to obtain the existence of wave operators (we give its proof in the Appendix):

LEMMA 3.6. *Let $U(t,t_0)$ and $\widetilde{U}(t,t_0)$ be the evolution propagators of $H(t) = H_0 + V(t)$ and $\widetilde{H}(t) = H_0 + \widetilde{V}(t)$ respectively. Assume that for $M(t) \in B(L^2(\mathbb{R}^d))$ we may define $\mathbb{D}_{H_0} M(t)$ as bounded operators with*

$$\begin{aligned} &(\widetilde{V}(t) - V(t))M(t) = O(t^{-1-\varepsilon}) \quad \text{and} \\ &\mathbb{D}_{H_0} M(t) = \widetilde{M}(t) + O(t^{-1-\varepsilon}) \end{aligned} \tag{3.11}$$

where $\varepsilon > 0$, and that there exists $\widetilde{M}_0 \in \mathcal{G}(H) \cap \mathcal{G}(\widetilde{H})$ satisfying the estimates

$$-\widetilde{M}_0(t) \leq \widetilde{M}(t) \leq \widetilde{M}_0(t) \quad \text{and} \quad \widetilde{M}_0(t) \geq 0 \quad \text{for all } t \geq 1. \tag{3.11'}$$

If $\varphi \in L^2(\mathbb{R}^d)$ and $\Omega_t = \widetilde{U}(t,t_0)^* M(t) U(t,t_0)$, then the limit $\lim_{t\to\infty} \Omega_t \varphi$ exists.

PROPOSITION 3.7. *Set $\bar{J}(y_t^1) = 1 - \sum_{1\leq k\leq N} J^0(y_t^1 - \widetilde{z}_k)^2$ and define*

$$\begin{aligned} &\Omega_0(t,t_0) = U_0(t,t_0)^* \bar{J}(y_t^1) U(t,t_0), \\ &\Omega_k(t,t_0) = U_k(t,t_0)^* J^0(y_t^1 - \widetilde{z}_k) U(t,t_0) \quad \text{for } k = 1, \ldots, N. \end{aligned} \tag{3.12}$$

Then for every $\varphi \in L^2(\mathbb{R}^d)$, $k = 0, 1, \ldots, N$, *the following limits exist*:

$$\Omega_k(t_0)\varphi = \lim_{t\to\infty} \Omega_k(t, t_0)\varphi. \tag{3.12$'$}$$

P r o o f. Consider first the case $k = 0$. By Proposition 3.2 it suffices to show that

$$\lim_{t\to\infty} U_0(t, t_0)^* \bar{J}(y_t^1)\widetilde{\eta}_{n,t}^2 U(t, t_0)\varphi$$

exists for every $n \geq 1$. Further on n is fixed, we write $\widetilde{\eta}_t = \widetilde{\eta}_{n,t}$ and we apply Lemma 3.6 with $\widetilde{H}(t) = H_0(t)$ and $M(t) = \bar{J}(y_t^1)\widetilde{\eta}_t^2$.

We begin by noting that the first condition of (3.11) follows from

$$(H(t) - H_0(t))\bar{J}(y_t^1) = \sum_{1\leq k\leq N} J^0(2y_t^1 - 2\widetilde{z}_k)V_k^l(x - \chi_k(t))\bar{J}(y_t^1) = 0. \tag{3.13}$$

To check (3.13) we note that $J^0(y_t^1 - \widetilde{z}_k) \neq 0 \Rightarrow |y_t^1 - \widetilde{z}_k| < 4\tau$ and for $k' \neq k$ we have $|\widetilde{z}_k - \widetilde{z}_{k'}| = |z_k^1 - z_{k'}^1| \geq 16\tau$, hence $J^0(y_t^1 - \widetilde{z}_k) \neq 0 \Rightarrow J^0(y_t^1 - \widetilde{z}_{k'}) = 0$ for $k' \neq k$. Thus it is clear that $J^0(2y_t^1 - 2\widetilde{z}_k) \neq 0 \Rightarrow |y_t^1 - \widetilde{z}_k| < 2\tau \Rightarrow J^0(y_t^1 - \widetilde{z}_k) = 1 \Rightarrow \bar{J}(y_t^1) = 1 - J^0(y_t^1 - \widetilde{z}_k)^2 = 0$.

Next we find that $\mathbb{D}_{H_0} M = \widetilde{M}_1 + \widetilde{M}_2$ with

$$\widetilde{M}_1(t) = (\mathbb{D}_{H_0}\bar{J}(y_t^1))\widetilde{\eta}_t^2 = \frac{2}{t}\widetilde{\eta}_t(w_t^1 - y_t^1)\bar{J}'(y_t^1)\widetilde{\eta}_t + hc + O(t^{-2}), \tag{3.14}$$

$$\widetilde{M}_2(t) = 2\widetilde{\eta}_t\bar{J}(y_t^1)\mathbb{D}_{H(t)}\widetilde{\eta}_t + hc + O(t^{-2}). \tag{3.15}$$

Next for $k = 1, \ldots, N$, we have $|y_t^1| \leq 2\tau \Rightarrow |y_t^1 - \widetilde{z}_k| \geq 14\tau \Rightarrow J^0(2y_t^1 - 2\widetilde{z}_k) = 0$. Therefore $\bar{J} = 1$ on $[-2\tau; 2\tau]$ and $0 \notin \operatorname{supp} \bar{J}'$ allows us to define $J_0 \in C_0^\infty(\mathbb{R} \setminus \{\widetilde{z}_1, \ldots, \widetilde{z}_N\})$ satisfying $J_0(\lambda)\lambda = \bar{J}'(\lambda)^2$ and to estimate

$$\begin{aligned} \pm\,(w_t^1 - y_t^1)\bar{J}'(y_t^1) + hc &\leq 2(w_t^1 - y_t^1)^2 + 2J_0(y_t^1)y_t^1 \\ &\Rightarrow \pm\widetilde{M}_1 \leq 4M_0 + 4M_2 \end{aligned} \tag{3.16}$$

with M_0, M_2 given by (3.5), (3.9). Then similarly to the proof of Lemma 3.3 we find $\pm\widetilde{M}_2(t) \leq CM_0(t) + Ct^{-2}$, hence it is clear that the hypotheses of Lemma 3.6 hold with $\widetilde{M}_0 = C_0M_0 + 4M_2 \in \mathcal{G}(H) \cap \mathcal{G}(H_k)$ by Corollary 3.1, 3.5 and Proposition 3.4.

In the case $k = 1, \ldots, N$, we apply Lemma 3.6 with $\widetilde{H}(t) = H_k(t)$ and $M(t) = \widetilde{J}(y_t^1)\widetilde{\eta}_t^2$, where $\widetilde{J}(\lambda) = J^0(\lambda - \widetilde{z}_k)$. As before we have

$$\begin{aligned} &(H(t) - H_k(t))\widetilde{J}(y_t^1) \\ &\quad = \sum_{k'\in\{1,\ldots,N\}\setminus\{k\}} J^0(2y_t^1 - 2\widetilde{z}_{k'})V_{k'}^l(x - \chi_{k'}(t))\widetilde{J}(y_t^1) = 0. \end{aligned} \tag{3.17}$$

Indeed, $\widetilde{J}(y_t^1) \neq 0 \Rightarrow |y_t^1 - \widetilde{z}_k| < 4\tau \Rightarrow |y_t^1 - \widetilde{z}_{k'}| \geq 2\tau$ for $k' \neq k \Rightarrow J^0(2y_t^1 - 2\widetilde{z}_{k'}) = 0$ for $k' \neq k$. We complete the proof noting that $\widetilde{J} = 0$ on $[-2\tau; 2\tau]$ and (3.14)–(3.16) still hold if $\bar{J}$ is replaced by $\widetilde{J}$. ∎

4. Asymptotic completeness. In order to obtain the asymptotic completeness it remains to prove

PROPOSITION 4.1. *If* $k = 1, \dots, N$ *and* $\varphi \in L^2(\mathbb{R}^d)$, *then*

$$\lim_{t\to\infty} J^0(y_t^1 - \widetilde{z}_k) U_k(t, t_0)\varphi = 0.$$

Indeed, using Propositions 2.2, 3.7 and 4.1, we can see that via the chain rule,

$$\begin{aligned} e^{itH_0+iS(t)} U(t,t_0)\varphi &= e^{itH_0+iS(t)} \Big(\overline{J}(y_t^1) + \sum_{1\le k\le N} J^0(y_t^1 - \widetilde{z}_k)^2 \Big) U(t,t_0)\varphi \\ &= e^{itH_0+iS(t)} U_0(t,t_0) \Omega_0(t,t_0)\varphi \\ &\quad + \sum_{1\le k\le N} e^{itH_0+iS(t)} J^0(y_t^1 - \widetilde{z}_k) U_k(t,t_0) \Omega_k(t,t_0)\varphi \end{aligned}$$

converges to $\widetilde{\Omega}_0(t_0)^* \Omega_0(t_0)\varphi$, i.e. the limit (1.12′) exists.

Before starting the proof of Proposition 4.1 we introduce more notation. We set

$$H_{0k} = \tfrac{1}{2}p^2 + \widetilde{z}_k x_1, \qquad \widetilde{H}_k = H_{0k} + V_k(x - \omega_k), \tag{4.1}$$

where $k = 1, \dots, N$ and ω_k is as in (2.5). We define

$$\chi_k^0(t) = \tfrac{1}{2} z_k t^2 + \upsilon_k t, \qquad \dot{\chi}_k^0(t) = z_k t + \upsilon_k, \tag{4.2}$$

$$\widetilde{H}_k(t) = H_{0k} + \widetilde{V}_k(t, x) \tag{4.3}$$

with

$$\begin{aligned} \widetilde{V}_k(t,x) &= V^k(t, x + \chi_k^0(t)) \\ &= V_k(x - \widetilde{\chi}_k(t)) + \sum_{k' \in \{1,\dots,N\}\setminus\{k\}} V_{0k'}\big(t, x + \tfrac{1}{2} z_k t^2 + \upsilon_k t\big). \end{aligned}$$

It is easy to see that $V_{0k'}\big(t, x + \frac{1}{2} z_k t^2 + \upsilon_k t\big)$ satisfies estimates (2.1) similarly to $V_{0k'}$. The following lemma allows us to compare $\widetilde{H}_k$ and $\widetilde{H}_k(t)$.

LEMMA 4.2. (a) *We have* $V_k(x - \widetilde{\chi}_k(t)) = V_k(x - \omega_k) + O(t^{-2\mu_0})$ *and*

$$\frac{d}{dt} V_k(x - \widetilde{\chi}_k(t)) = -\widetilde{\chi}_k'(t) \cdot \nabla V_k(x - \widetilde{\chi}_k(t)) = O(t^{-1-2\mu_0}).$$

(b) *If* $h \in C_0^\infty(\mathbb{R})$ *then* $h(\widetilde{H}_k(t)) = h(\widetilde{H}_k) + O(t^{-2\mu_0}) + O(t^{-2\mu})$ *and*

$$\mathbb{D}_{\widetilde{H}_k(t)} h(\widetilde{H}_k(t)) = \frac{d}{dt} h(\widetilde{H}_k(t)) = O(t^{-1-2\mu_0}) + O(t^{-1-2\mu}).$$

(c) *If* $g, \widetilde{h} \in C_0^\infty(\mathbb{R})$ *then* $[\widetilde{h}(\widetilde{H}_k), g(\widetilde{w}_t)] = O(t^{-1})$.

We note that our assumptions $\nabla V_k = \nabla V_k^l = O(1)$ and (1.9) give immediately the indicated estimate of $\frac{d}{dt} V_k(x - \widetilde{\chi}_k(t))$, while the first estimate of

Lemma 4.2(a) follows by integration. The proof of estimates in (b) and (c) is given in the Appendix.

PROPOSITION 4.3. *Let* $g \in C_0^\infty\big(]-\frac{3}{4}|\widetilde{z}_k|; \frac{3}{4}|\widetilde{z}_k|[\big)$ *and* $h \in C_0^\infty(\mathbb{R})$. *Then*

$$\widetilde{M}_h(t) = \frac{1}{t} h(\widetilde{H}_k(t)) g(\widetilde{w}_t)^2 h(\widetilde{H}_k(t)) \in \mathcal{G}(\widetilde{H}_k(t)), \tag{4.4}$$

where we have set $\widetilde{w}_t = p_1/t$.

Proof. Let $n \in \mathbb{N}$ be such that $h \in C_0^\infty(]-n; n[)$. Since $(\widetilde{M}_h(t)\varphi, \varphi) = t^{-1}\|g(\widetilde{w}_t)h(\widetilde{H}_k(t))\varphi\|^2$, it is clear that $\widetilde{M}_{h_1+h_2}(t) \le 2\widetilde{M}_{h_1}(t) + 2\widetilde{M}_{h_2}(t)$. Thus it suffices to show that for every $\lambda \in [-n; n]$ there is $\delta > 0$ such that $\widetilde{M}_h(t) \in \mathcal{G}(\widetilde{H}_k(t))$ with $h \in C_0^\infty(]\lambda - \delta; \lambda + \delta[)$, $|h| \le 1$.

Let $g_1 \in C^\infty(\mathbb{R})$ satisfy $g_1' = -g^2$ and set

$$M_0(t) = \widetilde{z}_k h(\widetilde{H}_k(t)) g_1(\widetilde{w}_t) h(\widetilde{H}_k(t)).$$

Let $\varepsilon = \min\{1, 2\mu_0, 2\mu\}$. Then Lemma 4.2 allows us to write

$$\begin{aligned} \mathbb{D}_{\widetilde{H}_k(t)} M_0(t) &= \widetilde{z}_k h(\widetilde{H}_k(t))(\mathbb{D}_{\widetilde{H}_k(t)} g_1(\widetilde{w}_t)) h(\widetilde{H}_k(t)) + O(t^{-1-\varepsilon}) \\ &= \widetilde{z}_k h(\widetilde{H}_k)(\mathbb{D}_{\widetilde{H}_k} g_1(\widetilde{w}_t)) h(\widetilde{H}_k) + O(t^{-1-\varepsilon}). \end{aligned}$$

We now show that choosing $\delta > 0$ small enough we have

$$\widetilde{z}_k h(\widetilde{H}_k)[iV_k(x-\omega_k), g_1(\widetilde{w}_t)]h(\widetilde{H}_k) \ge -\frac{\widetilde{z}_k^2}{8t} h(\widetilde{H}_k) g(\widetilde{w}_t)^2 h(\widetilde{H}_k) - Ct^{-2}. \tag{4.5}$$

Using (1.11b) and the standard pseudo-differential expansion [(A.1) of Appendix with $n = 2$ and then with $n = 1$] we find the following expression of the commutator:

$$\begin{aligned} [iV_k(x - \omega_k), g_1(\widetilde{w}_t)] &= -\frac{1}{t}\partial_{x_1} V_k(x - \omega_k) g_1'(\widetilde{w}_t) + O(t^{-2}) \\ &= \frac{1}{t} g(\widetilde{w}_t)\partial_{x_1} V_k(x - \omega_k) g(\widetilde{w}_t) + O(t^{-2}), \end{aligned} \tag{4.6}$$

and since $\widetilde{H}_k$ has no eigenvalues (cf. [20]), $\mathbf{1}_{[\lambda-2\delta;\lambda+2\delta]}(\widetilde{H}_k) \to 0$ strongly as $\delta \to 0$. As $\partial_{x_1} V_k(x - \omega_k)\mathbf{1}_{[-n;n]}(\widetilde{H}_k)$ is compact, for $\delta > 0$ small enough we have

$$\widetilde{z}_k \widetilde{h}(\widetilde{H}_k)\partial_{x_1} V_k(x - \omega_k)\widetilde{h}(\widetilde{H}_k) \ge -\tfrac{1}{8}\widetilde{z}_k^2 \tag{4.7}$$

if $\widetilde{h} \in C_0^\infty(]\lambda-2\delta; \lambda+2\delta[)$, $0 \le \widetilde{h} \le 1$. Using $\widetilde{h}$ such that $h = h\widetilde{h}$ and Lemma 4.2(c) we obtain (4.5) from (4.6)–(4.7). Next we note that $\mathbb{D}_{H_{0k}} g_1(\widetilde{w}_t) = -t^{-1}(\widetilde{z}_k + \widetilde{w}_t)g_1'(\widetilde{w}_t) = t^{-1}(\widetilde{z}_k + \widetilde{w}_t)g(\widetilde{w}_t)^2$ and since $\lambda \in \operatorname{supp} g \Rightarrow |\lambda| \le \frac{3}{4}|\widetilde{z}_k| \Rightarrow \widetilde{z}_k(\widetilde{z}_k + \lambda) \ge \frac{1}{4}\widetilde{z}_k^2$, it is clear that

$$\widetilde{z}_k h(\widetilde{H}_k)(\mathbb{D}_{H_{0k}} g_1(\widetilde{w}_t))h(\widetilde{H}_k) \ge \frac{1}{4t}\widetilde{z}_k^2 h(\widetilde{H}_k) g(\widetilde{w}_t)^2 h(\widetilde{H}_k). \tag{4.8}$$

Let M_1, M_2 denote the left hand sides of (4.5) and (4.8). Then (4.4) follows from

$$\begin{aligned}\tfrac{1}{8}\widetilde{z}_k^2\widetilde{M}_h(t) &\le (M_1+M_2)(t) + Ct^{-1-\varepsilon}\\ &= \mathbb{D}_{\widetilde{H}_k(t)}M_0(t) + O(t^{-1-\varepsilon}) \in \mathcal{G}(\widetilde{H}_k(t)). \ \blacksquare\end{aligned}$$

Proof of Proposition 4.1. STEP 1. Introduce

$$G_k(t) = e^{-i\Phi_k(t)}e^{-ix\cdot\dot{\chi}_k^0(t)}e^{ip\cdot\chi_k^0(t)} \quad \text{where}$$

$$\Phi_k(t) = \int_1^t d\tau\,\big(z_0\cdot\chi_k^0(\tau) + \tfrac{1}{2}\dot{\chi}_k^0(\tau)^2\big). \tag{4.9}$$

Since $e^{-ix\cdot\dot{\chi}_k^0(t)}p = (p+\dot{\chi}_k^0(t))e^{-ix\cdot\dot{\chi}_k^0(t)}$ and $e^{ip\cdot\chi_k^0(t)}x = (x+\chi_k^0(t))e^{ip\cdot\chi_k^0(t)}$, we compute

$$\begin{aligned}G_k'(t) &= \big(-z_0\cdot\chi_k^0(t) - \tfrac{1}{2}\dot{\chi}_k^0(t)^2 - x\cdot z_k + (p+\dot{\chi}_k^0(t))\cdot\dot{\chi}_k^0(t)\big)iG_k(t),\\ iG_k(t)H_k(t) &= \big(\tfrac{1}{2}(p+\dot{\chi}_k^0(t))^2 - z_0\cdot(x+\chi_k^0(t)) + V^k(t,x+\chi_k^0(t))\big)iG_k(t)\\ &= \big(\widetilde{H}_k(t) + p\cdot\dot{\chi}_k^0(t) + \tfrac{1}{2}\dot{\chi}_k^0(t)^2 - z_k\cdot x - z_0\cdot\chi_k^0(t)\big)iG_k(t)\\ &= i\widetilde{H}_k(t)G_k(t) + G_k'(t).\end{aligned}$$

Thus we have

$$\begin{aligned}&\frac{d}{dt}(\widetilde{U}_k(t,t_0)^*G_k(t)U_k(t,t_0)\varphi)\\ &\quad = \widetilde{U}_k(t,t_0)^*(G_k'(t) + i\widetilde{H}_k(t)G_k(t) - iG_k(t)H_k(t))U_k(t,t_0)\varphi = 0,\end{aligned}$$

which implies

$$\widetilde{U}_k(t,t_0) = G_k(t)U_k(t,t_0)G_k(t_0)^{-1}. \tag{4.10}$$

We write $\widetilde{y}_t = 2x_1/t^2$. Then

$$G_k(t)J^0(y_t^1 - \widetilde{z}_k) = J^0(\widetilde{y}_t + 2v_k^1/t)G_k(t) = J^0(\widetilde{y}_t)G_k(t) + O(t^{-1})$$

and using (4.10) we obtain

$$\lim_{t\to\infty}\|J^0(y_t^1-\widetilde{z}_k)U_k(t,t_0)\varphi\| = \lim_{t\to\infty}\|J^0(\widetilde{y}_t)\widetilde{U}_k(t,t_0)G_k(t_0)\varphi\|. \tag{4.11}$$

STEP 2. It suffices to show that for every $h\in C_0^\infty(\mathbb{R})$ we have

$$\liminf_{t\to\infty}\|J^0(\widetilde{y}_t)h(\widetilde{H}_k(t))\widetilde{\varphi}_t\| = 0 \tag{4.12}$$

where we have set $\widetilde{\varphi}_t = \widetilde{U}_k(t,t_0)G_k(t_0)\varphi$.

Indeed, note first that (4.11) is the limit of the norms of $\varphi(t) = U_k(t,t_0)^* \times J^0(y_t^1-\widetilde{z}_k)U_k(t,t_0)\varphi$ and that $\varphi(t)$ converges in $L^2(\mathbb{R}^d)$, by a reasoning analogous to the proof of Proposition 3.7. Thus the limits (4.11) exist and we may replace them by lim inf.

However, taking $h_0 \in C_0^\infty(\mathbb{R})$ such that $h_0 = 1$ in a neighbourhood of 0, $0 \le h_0 \le 1$, we have $h_0(\widetilde{H}_k(T_0)/n)\psi \to \psi$ as $n \to \infty$ and by Lemma 4.2(b),

$$[((I - h_0(\widetilde{H}_k(t)/n))^2 \widetilde{\varphi}_t, \widetilde{\varphi}_t)]_{T_0}^T = \int_{T_0}^T dt\, (\mathbb{D}_{\widetilde{H}_k(t)}(I - h_0(\widetilde{H}_k(t)/n))^2 \widetilde{\varphi}_t, \widetilde{\varphi}_t)$$
$$\le \int_{T_0}^T dt\, C_\varphi t^{-1-2\min\{\mu,\mu_0\}}/n \le \widetilde{C}_\varphi/n,$$

which implies

$$\lim_{n\to\infty} \sup_{t \ge T_0} \|(I - h_0(\widetilde{H}_k(t)/n))\widetilde{\varphi}_t\| = 0.$$

Step 3. Instead of (4.12) it suffices to show that $M(t) \in \mathcal{G}(\widetilde{H}_k(t))$ with

$$M(t) = \frac{1}{t} h(\widetilde{H}_k(t)) J^0(\widetilde{y}_t)^2 h(\widetilde{H}_k(t)). \tag{4.13}$$

Indeed, $(M(t)\widetilde{\varphi}_t, \widetilde{\varphi}_t) \in L^1([t_0;\infty[, dt)$ implies

$$0 = \liminf_{t\to\infty} t(M(t)\widetilde{\varphi}_t, \widetilde{\varphi}_t) = \liminf_{t\to\infty} \|J^0(\widetilde{y}_t) h(\widetilde{H}_k(t))\widetilde{\varphi}_t\|^2.$$

Step 4. To complete the proof of Proposition 4.1 it suffices to prove

Lemma 4.4. *Let* $g \in C_0^\infty\big(\big]-\frac{3}{4}|\widetilde{z}_k|; \frac{3}{4}|\widetilde{z}_k|\big[\big)$ *be such that* $g = 1$ *on* $\big[-\frac{2}{3}|\widetilde{z}_k|; \frac{2}{3}|\widetilde{z}_k|\big]$. *Then*

$$(1 - g)(\widetilde{w}_t) J^0(\widetilde{y}_t) h(\widetilde{H}_k(t)) = O(t^{-1}). \tag{4.14}$$

Indeed, if g, M, $\widetilde{M}_h$ are as before, then Lemma 4.4 and Proposition 4.3 give

$$M(t) = \frac{1}{t} h(\widetilde{H}_k(t)) g(\widetilde{w}_t) J^0(\widetilde{y}_t)^2 g(\widetilde{w}_t) h(\widetilde{H}_k(t)) + O(t^{-2})$$
$$\le \widetilde{M}_h(t) + Ct^{-2} \in \mathcal{G}(\widetilde{H}_k(t)). \ \blacksquare$$

Proof (of Lemma 4.4). We set $J = J^0$ and $\overline{g} = 1 - g$. Then (4.14) follows if we show

$$(-i + \widetilde{H}_k)^{-1} J(\widetilde{y}_t) \overline{g}(\widetilde{w}_t)^2 J(\widetilde{y}_t)(i + \widetilde{H}_k)^{-1} \le Ct^{-2}. \tag{4.14$'$}$$

Writing $\overline{g}(\lambda)^2 = \widetilde{g}(\lambda)\big(\lambda^2 - \frac{1}{4}\widetilde{z}_k^2\big)\widetilde{g}(\lambda)$ we have $\widetilde{g} \in S_1^{-1}(\mathbb{R}) \Rightarrow [J(\widetilde{y}_t), \widetilde{g}(\widetilde{w}_t)] \times (1 + |\widetilde{w}_t|) = O(t^{-3})$ (cf. Appendix), and $J(\widetilde{y}_t) \ne 0 \Rightarrow |\widetilde{y}_t| \le 4\tau \le \frac{1}{4}|\widetilde{z}_k| \Rightarrow -\widetilde{z}_k\widetilde{y}_t \le \frac{1}{4}\widetilde{z}_k^2$ allows us to estimate

$$J(\widetilde{y}_t)\widetilde{g}(\widetilde{w}_t)\big(\widetilde{w}_t^2 - \tfrac{1}{4}\widetilde{z}_k^2\big)\widetilde{g}(\widetilde{w}_t)J(\widetilde{y}_t)$$
$$= \widetilde{g}(\widetilde{w}_t)J(\widetilde{y}_t)\big(\widetilde{w}_t^2 - \tfrac{1}{4}\widetilde{z}_k^2\big)J(\widetilde{y}_t)\widetilde{g}(\widetilde{w}_t) + O(t^{-3})$$
$$\le \widetilde{g}(\widetilde{w}_t)J(\widetilde{y}_t)(\widetilde{w}_t^2 + \widetilde{z}_k\widetilde{y}_t)J(\widetilde{y}_t)\widetilde{g}(\widetilde{w}_t) + Ct^{-3}$$
$$\le \widetilde{g}(\widetilde{w}_t)J(\widetilde{y}_t)2t^{-2}H_{0k}J(\widetilde{y}_t)\widetilde{g}(\widetilde{w}_t) + Ct^{-3}.$$

Since $[\widetilde{y}_t, \widetilde{g}(\widetilde{w}_t)]$ and $[\widetilde{w}_t^2, J(\widetilde{y}_t)]\widetilde{g}(\widetilde{w}_t)$ are $O(t^{-3})$, we obtain (4.14′) noting that $H_{0k}J(\widetilde{y}_t)\widetilde{g}(\widetilde{w}_t)(i+\widetilde{H}_k)^{-1} = J(\widetilde{y}_t)\widetilde{g}(\widetilde{w}_t)H_{0k}(i+\widetilde{H}_k)^{-1} + O(t^{-1}) = O(1)$. ■

5. Interaction potentials with singularities. Let $\widehat{C}$ be as in (1.11b) and $\theta \in C_0^\infty(\mathbb{R})$ be such that $\theta(x_1) = 1$ for $|x_1| \le \widehat{C}/|E|$. Then

(5.1) $$\|V_k(x)\theta(x_1)\varphi\| \le \tfrac{1}{5}\|p^2\theta(x_1)\varphi\| + C\|\varphi\| \le \tfrac{1}{2}\|H_0\varphi\| + C'\|\varphi\|$$

and $V_k(x)(1-\theta)(x_1)$ is bounded. Therefore $H_0 + V_k(x)$ is well defined as a self-adjoint operator on the domain of H_0 and the operators $H_0(H_0+V_k(x)+i)^{-1}$, $(H_0+V_k(x))(H_0+i)^{-1}$ are bounded. The analogous assertion clearly holds if $V_k(x)$ is replaced by $V_k(x-\chi_k(t))$ or by $V(t,x)$ (using constants locally bounded with respect to t).

Further on $\beta > 0$ is fixed small enough. Following [7] or [9] we may state

LEMMA 5.1. *There exist functions $u_t^j \in C_0^\infty(\mathbb{R})$, $j = 1,\dots,d$, such that for $t \ge 1$ one has*

(5.2a) $$u_t^1(\lambda) = \dot{\chi}_k^1(t)/t - z_0^1 \quad \textit{for } \lambda \in [\widetilde{z}_k - t^{-\beta}; \widetilde{z}_k + t^{-\beta}],$$

(5.2b) $$u_t^1(\lambda) = \lambda \quad \textit{for } \lambda \notin \bigcup_{1\le k\le N} [\widetilde{z}_k - 2t^{-\beta}; \widetilde{z}_k + 2t^{-\beta}],$$

(5.2c) $$u_t^j(\lambda) = \dot{\chi}_k^j(t) \quad \textit{for } \lambda \in [\dot{\chi}_k^j(t) - t^{-\beta}; \dot{\chi}_k^j(t) + t^{-\beta}], \qquad j \ge 2,$$

(5.2d) $$u_t^j(\lambda) = \lambda \quad \textit{for } \lambda \in [-\overline{C} + t^{-\beta}; \overline{C} - t^{-\beta}] \setminus \bigcup_{1\le k\le N} [\dot{\chi}_k^j(t) - 2t^{-\beta}; \dot{\chi}_k^j(t) + 2t^{-\beta}], \qquad j \ge 2,$$

(5.2e) $$(u_t^j)'(\lambda) = \frac{d}{d\lambda}u_t^j(\lambda) = 0 \quad \textit{for } |\lambda| \ge \overline{C},\ j \ge 2,$$

(5.2f) $$\left|\frac{d}{dt}u_t^j(\lambda)\right| \le Ct^{-1-\beta}, \quad (u_t^j)'(\lambda) \ge 0,$$
$$|(u_t^j)^{(n)}(\lambda)| = \left|\frac{d^n}{d^n\lambda}u_t^j(\lambda)\right| \le C_n t^{(n-1)\beta} \quad \textit{for } \lambda \in \mathbb{R},\ n \ge 1,$$

where $\widetilde{\chi}_k'(t) = (\dot{\chi}_k^1(t),\dots,\dot{\chi}_k^d(t))$ and $\overline{C}$ is fixed large enough.

We write $a_t = O(b_t)$ if $b_t \ge I$ for $t \ge T_0$ and $b_t^{-1/2}a_t b_t^{-1/2} = O(1)$. Note that $a_t = O(b_t)$ holds if we have $a_t b_t^{-1} = O(1)$ and $b_t^{-1}a_t = O(1)$. Further, we denote $x_\perp = (x_2,\dots,x_d)$, $\widetilde{y}_t^\perp = x_\perp/t$, $u_t^\perp(\widetilde{y}_t^\perp) = (u_t^2(x_2/t),\dots,u_t^d(x_d/t))$, $u_t^{\perp\prime}(\widetilde{y}_t^\perp) = ((u_t^2)'(x_2/t),\dots,(u_t^d)'(x_d/t))$,

(5.3) $$\eta_t^\perp = \tfrac{1}{2}|w_t^\perp|^2 - u_t^\perp(\widetilde{y}_t^\perp)\cdot w_t^\perp/t + hc + C_\perp I$$

with $C_\perp > 0$ large enough and

(5.3′) $$\eta_t^0 = \tfrac{1}{2}(w_t^1 - u_t^1(y_t^1))^2 + \tfrac{1}{4}(y_t^1)^2.$$

PROPOSITION 5.2. *Let* $\eta_t = \eta_t^0 + \eta_t^\perp + V(t)/t^2$. *If* $\varepsilon > 0$ *is small enough, then*

$$(5.4a)\quad (w_t)^{2\theta} \le C(\eta_t^0 + \eta_t^\perp)^\theta \quad \text{for } 0 \le \theta \le 1,$$

$$(5.4b)\quad \eta_t - (\eta_t^0 + \eta_t^\perp) = t^{-2}V(t) = O(t^{-2\varepsilon}(\eta_t^0 + \eta_t^\perp)),$$

$$(5.4c)\quad \mathbb{D}_{H(t)}\eta_t = \mathbb{D}_{H_0}(\eta_t^0 + \eta_t^\perp) + O(t^{-1-\varepsilon}\eta_t^{1-\varepsilon}),$$

$$(5.4d)\quad \mathbb{D}_{H_0}\eta_t^\perp = -\frac{1}{t}\sum_{2\le j\le d} w_t^j(1 + (u_t^j)'(x_j/t))w_t^j + O(t^{-1-\varepsilon}\eta_t^{1/2}),$$

$$(5.4e)\quad \mathbb{D}_{H_0}\eta_t^0 = -\frac{1}{t}(w_t^1 - y_t^1)(1 + 2(u_t^1)'(y_t^1))(w_t^1 - y_t^1) + O(t^{-1-\varepsilon}\eta_t^{1/2}).$$

Proof. By interpolation it suffices to prove (5.4a) for $\theta = 1$. As $u_t^\perp(\widetilde{y}_t^\perp)$ is bounded, we have $|w_t^\perp|^2 \le C\eta_t^\perp$. Then using $u_t(y_t^1)^2 = (y_t^1)^2 + O(t^{-\beta})$ we may estimate

$$(5.5)\quad \begin{aligned}(w_t^1)^2 &= (w_t^1 - u_t^1(y_t^1))^2 + u_t^1(y_t^1)^2 + 2(w_t^1 - u_t^1(y_t^1))u_t(y_t^1) + hc\\ &\le 2(w_t^1 - u_t^1(y_t^1))^2 + 2u_t^1(y_t^1)^2 + 1 \le 12\eta_t^0 + Ct^{-\beta}.\end{aligned}$$

Thus (5.4b) follows from (5.4a) by the estimate

$$t^{-2}e^{-i\chi_k(t)\cdot p}V_k(x)e^{i\chi_k(t)\cdot p} \le Ct^{-2}(1+p^2)^{1-\varepsilon} \le C't^{-2\varepsilon}(1+|w_t|^2)^{1-\varepsilon}.$$

Next we note that

$$\begin{aligned} u_t^1(y_t^1) + z_0^1 - \dot{\chi}_k^1(t)/t \ne 0 &\Rightarrow |x_1 - \chi_k^1(t)| \ge \tfrac{1}{2}t^{2-\beta}\\ &\Rightarrow \nabla V_k(x - \chi_k(t)) = O(t^{-\mu(2-\beta)}),\\ u_t^\perp(\widetilde{y}_t^\perp) - \dot{\chi}_k^\perp(t) \ne 0 &\Rightarrow |x_\perp - \chi_k^\perp(t)| \ge \tfrac{1}{2}t^{1-\beta}\\ &\Rightarrow \nabla V_k(x - \chi_k(t)) = O(t^{-\mu(1-\beta)}),\end{aligned}$$

hence using the fact that $\dot{\chi}_k^1(t)/t$, $\dot{\chi}_k^\perp(t)$, $u_t^1(y_t^1)\eta_t^{-1/2}$, $u_t^\perp(\widetilde{y}_t^\perp)$ are $O(1)$, we obtain

$$(5.6)\quad \partial_{x_1}V_k(x - \chi_k(t))(u_t^1(y_t^1) + z_0^1 - \dot{\chi}_t^1(t)/t) = O(t^{-\varepsilon}\eta_t^{1/2}),$$

$$(5.6')\quad \partial_{x_\perp}V_k(x - \chi_k(t))(u_t^\perp(\widetilde{y}_t^\perp) - \dot{\chi}_k^\perp(t)) = O(t^{-\varepsilon}).$$

Then reasoning as in the proof of Lemma 2.4 we can see that (5.6)–(5.6′) imply (5.4c).

Finally, we obtain (5.4d, e) calculating

$$\begin{aligned} \mathbb{D}_{H_0}u_t^\perp(\widetilde{y}_t^\perp) &= \frac{1}{t}u_t^{\perp\prime}(\widetilde{y}_t^\perp)(p_\perp - \widetilde{y}_t^\perp) + O(t^{-2+\beta}),\\ -t\mathbb{D}_{H_0}\eta_t^0 + O(t^{-\varepsilon}) &= (w_t^1)^2 - 2(y_t^1 - w_t^1)u_t^{1\prime}(y_t^1)w_t^1 - u_t^1(y_t^1)w_t^1\\ &\quad + 2(y_t^1 - w_t^1)(u_t^{1\prime}u_t^1)(y_t^1) + (y_t^1 - w_t^1)y_t^1\\ &= (w_t^1 - y_t^1)^2\\ &\quad + 2(w_t^1 - y_t^1)(1 + 2u_t^{1\prime})(y_t^1)(w_t^1 - y_t^1) + O(t^{-\varepsilon}). \ \blacksquare\end{aligned}$$

Now it is clear that Corollary 3.1 holds. However, $\widetilde{\eta}_{n,t}(\eta_t)^{1-\varepsilon} = O(n^{1-\varepsilon})$ and (3.6) holds if $O(t^{-2})$ is replaced by $O(n^{1-\varepsilon}t^{-1-\varepsilon})$. Thus the proof of Proposition 3.2 is valid if C/n is replaced by $Cn^{-\varepsilon}$. All the remaining proofs of Section 3 are valid if $O(t^{-2})$ is replaced by $O(t^{-1-\varepsilon})$. In Section 4 we use (5.1) with $V_k(x)$, H_0 replaced by $V_k(x-\widetilde{\chi}_k(t))$, H_{0k} to conclude that $\widetilde{H}_k$, $\widetilde{H}_k(t)$ are self-adjoint on the domain of H_{0k} and that

$$(5.7)\qquad H_{0k}(\widetilde{H}_k+i)^{-1}, H_{0k}(\widetilde{H}_k(t)+i)^{-1},\\ \widetilde{H}_k(H_{0k}+i)^{-1}, \widetilde{H}_k(t)(H_{0k}+i)^{-1} \in B(L^2(\mathbb{R}^d)).$$

The second inequality of (5.1) with H_0 and φ replaced by H_{0k} and $(H_{0k}+i)^{-1}\varphi$ gives $\theta(x_1)p^2(H_{0k}+i)^{-1} \in B(L^2(\mathbb{R}^d))$, hence

$$(5.8)\qquad (H_{0k}+i)^{-1}[ip, \theta(x_1)V_k^s(x-\widetilde{\chi}_k(t))](H_{0k}+i)^{-1} = O(1).$$

Since $\nabla V(x) = [ip, V(x)]$ we obtain the following version of Lemma 4.2(a):

$$(5.9)\qquad \frac{d}{dt}V_k^s(x-\widetilde{\chi}_k(t)) = O(t^{-1-2\mu_0}(I+|H_{0k}|^2)),$$

$$(5.9')\qquad V_k^s(x-\widetilde{\chi}_k(t)) = V_k^s(x-\omega_k) + O(t^{-2\mu_0}(I+|H_{0k}|^2))$$

and by (5.7) we may always replace H_{0k} by $\widetilde{H}_k$ or $\widetilde{H}_k(t)$. It is checked in the Appendix that the assertions of Lemma 4.2(b), (c) still hold and moreover one has

$$(5.10)\qquad (\widetilde{H}_k+i)[\widetilde{h}(\widetilde{H}_k), J^0(\widetilde{y}_t)] = O(t^{-1}).$$

We also note that

$$(5.11)\qquad B = (\widetilde{H}_k+i)^{-1}[ip, \theta(x_1)V_k^s(x-\omega_k)](\widetilde{H}_k+i)^{-1}\\ \text{is compact on } L^2(\mathbb{R}^d).$$

In order to show that the assertion of Proposition 4.3 still holds it suffices to fix $\lambda \in [-n; n]$ and to find $\delta > 0$ such that for $h \in C_0^\infty(]\lambda-\delta; \lambda+\delta[)$, $|h| \le 1$, one has

$$(5.12)\qquad \pm h(\widetilde{H}_k)[i\theta(x_1)V_k^s(x-\omega_k), g_1(\widetilde{w}_t)]h(\widetilde{H}_k) \le \tfrac{1}{8}|\widetilde{z}_k|\widetilde{M}_h(t) + Ct^{-2},$$

where we assume that $g_1(\lambda) = -\lambda$ for $|\lambda| \le \frac{2}{3}|\widetilde{z}_k|$, i.e. $g(\lambda) = 1$ for $|\lambda| \le \frac{2}{3}|\widetilde{z}_k|$.

First of all we introduce $\widetilde{g}(\lambda) = g_1(\lambda) + \lambda$ and we check that

$$(5.13)\qquad \theta(x_1)\widetilde{g}(\widetilde{w}_t)h(\widetilde{H}_k) = O(t^{-2}).$$

Indeed, if $\theta_1 \in C_0^\infty(\mathbb{R})$ is such that $\theta_1 = 1$ on $\operatorname{supp}\theta$, then the standard pseudo-differential expansion [cf. (A.1) of Appendix] gives $\theta(x_1)\widetilde{g}(\widetilde{w}_t) \times (1-\theta_1)(x_1) = O(t^{-N})$ for every $N \in \mathbb{N}$. To obtain (5.13) we note that $(1+|p_1|^2)\theta_1(x_1)h(\widetilde{H}_k) = O(1)$ and $\widetilde{g}(\lambda) = 0$ for $|\lambda| \le \frac{2}{3}|\widetilde{z}_k|$ implies $|\widetilde{g}(\lambda)| \le C\lambda^2$, hence $|\widetilde{g}(\widetilde{w}_t)|(1+|p_1|^2)^{-1} \le C|p_1|^2t^{-2}(1+|p_1|^2)^{-1} \le Ct^{-2}$.

From (5.13) it is clear that modulo $O(t^{-2})$ we may replace $g_1(\widetilde{w}_t)$ by $-\widetilde{w}_t$ in (5.12). Next we note that $|x_1| \le \tau t \Rightarrow J^0(x_1/t) = 1$ and there is $T_0 > 0$ such that $\theta(x_1) = \theta(x_1)J^0(\widetilde{y}_t)$ for $t \ge T_0$. Writing $h = h\widetilde{h}$ with $\widetilde{h} \in C_0^\infty(]\lambda - 2\delta; \lambda + 2\delta[)$ and using (5.10) we have

$$\begin{aligned}(5.14)\qquad & \pm t^{-1}h(\widetilde{H}_k)J^0(\widetilde{y}_t)[ip_1, \theta(x_1)V_k^s(x-\omega_k)]J^0(\widetilde{y}_t)h(\widetilde{H}_k)\\ &= \pm t^{-1}h(\widetilde{H}_k)J^0(\widetilde{y}_t)\widetilde{h}_1(\widetilde{H}_k)B\widetilde{h}_1(\widetilde{H}_k)J^0(\widetilde{y}_t)h(\widetilde{H}_k) + O(t^{-2}),\end{aligned}$$

where $\widetilde{h}_1(\lambda) = \widetilde{h}(\lambda)(\lambda + i)$ and B is the compact operator given by (5.11). Thus for δ small enough we may estimate (5.14) by

$$\begin{aligned}&\frac{1}{8t}|\widetilde{z}_k|h(\widetilde{H}_k)J^0(\widetilde{y}_t)^2h(\widetilde{H}_k) + O(t^{-2})\\ &\qquad = \frac{1}{8t}|\widetilde{z}_k|h(\widetilde{H}_k)g(\widetilde{w}_t)J^0(\widetilde{y}_t)^2g(\widetilde{w}_t)h(\widetilde{H}_k) + O(t^{-2})\\ &\qquad \le \tfrac{1}{8}|\widetilde{z}_k|\widetilde{M}_h(t) + Ct^{-2},\end{aligned}$$

where the cut-off $g(\widetilde{w}_t)$ was introduced in view of (4.14).

Thus Propositions 4.3 and 4.1 still hold under the general hypotheses of Section 1.

Appendix. Let $J, \eta \in C(\mathbb{R}^d)$ and $n \in \mathbb{N}$ be such that $J^{(\alpha)} \in L^\infty(\mathbb{R}^d)$ for $|\alpha| = n$ and $\eta^{(\alpha)} \in L^1(\mathbb{R}^d)$ for $|\alpha| \ge n$. Then

$$\begin{aligned}(\mathrm{A}.1)\qquad J(x)\eta(D) = &\sum_{|\alpha|\le n-1} \eta^{(\alpha)}(D)J^{(\alpha)}(x)i^{-|\alpha|}/\alpha!\\ &+ O\big(\max_{|\alpha|=n\le|\alpha'|\le n+d+1} \|J^{(\alpha)}\|_{L^\infty(\mathbb{R}^d)}\|\eta^{(\alpha')}\|_{L^1(\mathbb{R}^d)}\big).\end{aligned}$$

In particular, we may apply (A.1) with $n > m + d$ if $J, \eta \in S_1^m(\mathbb{R}^d)$, where the notation $f \in S_1^m(\mathbb{R}^d)$ means that for any $\alpha \in \mathbb{N}^d$ one has the estimate $|f^{(\alpha)}(x)| \le C_\alpha(1 + |x|)^{m-|\alpha|}$.

It is easy to check that applying formula (A.1) we obtain the commutator estimates needed in the proof of Lemma 4.4.

Proof of Lemma 3.6. Let $\varphi, \psi \in D(H_0)$, $\varphi_t = U(t, t_0)\varphi$ and $\widetilde{\psi}_t = \widetilde{U}(t, t_0)\psi$. Then

$$\|\Omega_{t''}\varphi - \Omega_{t'}\varphi\| = \sup_{\substack{\|\psi\|\le 1\\ \psi\in D(H_0)}} |(\Omega_{t''}\varphi - \Omega_{t'}\varphi, \psi)| \le \sup_{\substack{\|\psi\|\le 1\\ \psi\in D(H_0)}} \int_{t'}^{t''} dt\, \left|\frac{d}{dt}(\Omega_t\varphi, \psi)\right|,$$

$$\begin{aligned}\left|\frac{d}{dt}(\Omega_t\varphi, \psi)\right| &= |((\mathbb{D}_{H_0}M(t) + O(t^{-1-\varepsilon}))\varphi_t, \widetilde{\psi}_t)|\\ &\le 4(\widetilde{M}_0(t)\varphi_t, \varphi_t)^{1/2}(\widetilde{M}_0(t)\widetilde{\psi}_t, \widetilde{\psi}_t)^{1/2} + Ct^{-1-\varepsilon}\|\varphi\| \cdot \|\psi\|,\end{aligned}$$

and we obtain $\|\Omega_{t''}\varphi - \Omega_{t'}\varphi\| \to 0$ as $t', t'' \to \infty$ estimating $\int_{t'}^{t''} dt\, \big|\frac{d}{dt}(\Omega_t\varphi, \psi)\big|$ by

$$\Big[\int_{t'}^{t''} (\widetilde{M}_0(t)\varphi_t, \varphi_t)\, dt\Big]^{1/2} \Big[\int_{t'}^{t''} (\widetilde{M}_0(t)\widetilde{\psi}_t, \widetilde{\psi}_t)\, dt\Big]^{1/2} + Ct'^{-\varepsilon}\|\varphi\| \cdot \|\psi\|. \ \blacksquare$$

Proof of Lemma 4.2. By (5.9)–(5.9′), for $\zeta \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\begin{aligned}(\zeta - \widetilde{H}_k(t))^{-1} - (\zeta - \widetilde{H}_k)^{-1} &= (\zeta - \widetilde{H}_k(t))^{-1}(\widetilde{H}_k(t) - \widetilde{H}_k)(\zeta - \widetilde{H}_k)^{-1} \\ &= O\Big(t^{-\varepsilon}\frac{1 + |\zeta|^2}{|\mathrm{Im}\,\zeta|^2}\Big), \\ \frac{d}{dt}(\zeta - \widetilde{H}_k(t))^{-1} &= (\zeta - \widetilde{H}_k(t))^{-1}\Big(\frac{d}{dt}\widetilde{H}_k(t)\Big)(\zeta - \widetilde{H}_k(t))^{-1} \\ &= O\Big(t^{-1-\varepsilon}\frac{1 + |\zeta|^2}{|\mathrm{Im}\,\zeta|^2}\Big)\end{aligned}$$

with $\varepsilon > 0$. We complete the proof of part (b) by using $a = \widetilde{H}_k(t)$ or $a = \widetilde{H}_k$ in the formula

$$h(a) = i\int \partial_{\overline{\zeta}}\widetilde{h}(\zeta)(\zeta - a)^{-1}\, d\zeta \wedge d\overline{\zeta}/(2\pi), \tag{A.2}$$

where $\widetilde{h} \in C_0^\infty(\mathbb{C})$ is an almost analytic extension satisfying $|\partial_{\overline{\zeta}}\widetilde{h}(\zeta)| \le C_k|\mathrm{Im}\,\zeta|^k$ for every $k \in \mathbb{N}$ and $\widetilde{h} = h$ on $\mathbb{R}$ (cf. [11]). To prove (c) we note that

$$(\zeta - \widetilde{H}_k)^{-1}[\widetilde{H}_k, \widetilde{w}_t](\zeta - \widetilde{H}_k)^{-1} = O\Big(t^{-1}\frac{1 + |\zeta|^2}{|\mathrm{Im}\,\zeta|^2}\Big)$$

and (A.2) with $a = \widetilde{H}_k$ implies $[h(\widetilde{H}_k), \widetilde{w}_t] = O(t^{-1})$.

We complete the proof using an almost analytic extension of g, allowing one to express $g(\widetilde{w}_t)$ similarly to (A.2) and obtain the estimate

$$\|[h(\widetilde{H}_k), g(\widetilde{w}_t)]\| \le C\|[h(\widetilde{H}_k), \widetilde{w}_t]\|. \ \blacksquare$$

Proof of (5.10). Let $J \in C_0^\infty(\mathbb{R})$. Then

$$\begin{aligned}2J(\widetilde{y}_t)\widetilde{w}_t^2 J(\widetilde{y}_t) &= J(\widetilde{y}_t)^2\widetilde{w}_t^2 + \widetilde{w}_t^2 J(\widetilde{y}_t)^2 + [[\widetilde{w}_t^2, J(\widetilde{y}_t)], J(\widetilde{y}_t)] \\ &= J(\widetilde{y}_t)^2\widetilde{w}_t^2 + \widetilde{w}_t^2 J(\widetilde{y}_t)^2 + O(t^{-6})\end{aligned}$$

and for $\zeta \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\begin{aligned}&(\overline{\zeta} - \widetilde{H}_k)^{-1}J(\widetilde{y}_t)\widetilde{w}_t^2 J(\widetilde{y}_t)(\zeta - \widetilde{H}_k)^{-1} \\ &\quad\le (\overline{\zeta} - \widetilde{H}_k)^{-1}J(\widetilde{y}_t)(2t^{-2}H_{0k} - \widetilde{z}_k\widetilde{y}_t)J(\widetilde{y}_t)(\zeta - \widetilde{H}_k)^{-1} \\ &\quad= (\overline{\zeta} - \widetilde{H}_k)^{-1}(t^{-2}J(\widetilde{y}_t)^2 H_{0k} + t^{-2}H_{0k}J(\widetilde{y}_t)^2 + O(1))(\zeta - \widetilde{H}_k)^{-1} \\ &\quad= O\Big(\frac{1 + |\zeta|^2}{|\mathrm{Im}\,\zeta|^2}\Big).\end{aligned}$$

Hence $\widetilde{w}_t J^{0\prime}(\widetilde{y}_t)(\zeta - \widetilde{H}_k)^{-1} = O\big(\frac{1+|\zeta|}{|\mathrm{Im}\,\zeta|}\big)$ and it remains to use (A.2) as before noting that

$$\begin{aligned}(i+\widetilde{H}_k)&[(\zeta-\widetilde{H}_k)^{-1}, J^0(\widetilde{y}_t)]\\ &= (i+\widetilde{H}_k)(\zeta-\widetilde{H}_k)^{-1}(2t^{-1}\widetilde{w}_t J^{0\prime}(\widetilde{y}_t) + O(t^{-2}))(\zeta-\widetilde{H}_k)^{-1}\\ &= O\bigg(t^{-1}\frac{1+|\zeta|^2}{|\mathrm{Im}\,\zeta|^2}\bigg). \ \blacksquare\end{aligned}$$

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