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ON RESIDUALLY FINITE GROUPS AND THEIR GENERALIZATIONS

BY

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The paper is concerned with the class of groups satisfying the finite embedding (FE) property. This is a generalization of residually finite groups. In [2] it was asked whether there exist FE-groups which are not residually finite. Here we present such examples. To do this, we construct a family of three-generator soluble FE-groups with torsion-free abelian factors. We study necessary and sufficient conditions for groups from this class to be residually finite. This answers the questions asked in [1] and [2].

1. The construction of the group $G(\phi)$**.** Let ϕ be a map from $\mathbb{Z}$ into $\mathbb{Z}\setminus\{0\}$. We define $G(\phi)$ to be the group generated by elements $\{x_i\}_{i\in\mathbb{Z}}\cup\{y_j\}_{j\in\mathbb{Z}}\cup\{z\}$ with the following relations:

$$[x_i,x_j]=[y_i,y_j]=1,\ z^{-1}x_iz=x_{i-1},\ z^{-1}y_jz=y_{j-1},\ y_j^{-1}x_iy_j=x_i^{\phi(i-j)}.$$

It is obvious that the group $G(\phi)$ is generated by three elements $x=x_0$, $y=y_0$ and z.

Let us start with a lemma describing the abelian subgroups of $G(\phi)$.

Lemma 1.1. *Let H be a normal subgroup of a group G and let $h\in H$ be an element of infinite order. Assume we are given a set S consisting of integers s such that h is conjugate to $h^s\in G$. For each $s\in S$ we choose an element $y_s\in G$ such that $y_s^{-1}hy_s=h^s$. Let Y denote the subgroup of G generated by the set $\{y_s\}_{s\in S}$ and let C be the multiplicative semigroup generated by S. Then*:

(i) *There exists a subgroup A of H such that $h\in A$ and A is isomorphic to the additive group of $\mathbb{Z}C^{-1}$.*

(ii) *For any y in Y there exist a and b in C such that $y^{-1}h^ay=h^b$.*

(iii) *For any a and b in C there exists y in Y such that $y^{-1}h^ay=h^b$.*

(iv) *If Y is abelian then the subgroup A of H generated by $\{y^{-1}hy : y\in Y\}$ is isomorphic to the additive group of $\mathbb{Z}C^{-1}$.*

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Proof. (i) Let $c_1, c_2, c_3, \ldots$ be the list of all elements of C. By induction we can construct a sequence $h_0, h_1, h_2, \ldots$ of elements of H such that $h_0 = h$, $h_n^{c_n} = h_{n-1}$ and each h_n is conjugate in G to h. The subgroup $A = \langle h_0, h_1, h_2, \ldots\rangle$ of H is clearly isomorphic to the additive group of $\mathbb{Z}C^{-1}$.

(ii) We proceed by induction on the length of the word y written in the letters y_s.

If $y = y_s$ we set $a = s$ and $b = s^2$, if $y = y_s^{-1}$ we set $a = s^2$ and $b = s$.

Let $y = y_s z$ or $y = y_s^{-1} z$ where z is an element of Y of smaller length. By induction, there exist a and b in C such that $z^{-1}h^a z = h^b$. Now we have $y^{-1}h^a y = h^{bs}$ or $y^{-1}h^{as} y = h^b$.

(iii) There exist g and x in Y such that $g^{-1}hg = h^a$ and $x^{-1}hx = h^b$. Therefore, $(g^{-1}x)^{-1}h^a g^{-1}x = h^b$.

(iv) By (i), it is sufficient to prove that for all $y \in Y$ the element $y^{-1}hy$ belongs to the subgroup $A = \langle h_0, h_1, h_2, \ldots\rangle$. Take some $y \in Y$. By (ii), there exist c_n and c_t in C such that $y^{-1}h^{c_n} y = h^{c_t}$. By (i), there exists $z \in Y$ such that $z^{-1}hz = h_n$. Since Y is abelian and $h = h_n^{c_n c_{n-1}\ldots c_1}$, we get

$$y^{-1}h_n^{c_n} y = \left(y^{-1}z^{-1}hzy\right)^{c_n} = z^{-1}y^{-1}h^{c_n} yz = z^{-1}h^{c_t} z = h_n^{c_t}.$$

Hence,

$$y^{-1}hy = (y^{-1}h_n^{c_n} y)^{c_{n-1}\ldots c_2 c_1} = h_n^{c_t c_{n-1}\ldots c_1} \in A.$$

NOTATION. Similarly to Lemma 1.1, for the group $G(\phi)$ we will denote by C the subsemigroup of $\mathbb{Z}$ generated by $\operatorname{im}\phi$.

PROPOSITION 1.2. *Every element of the group $G(\phi)$ can be uniquely written as a finite product*

$$\prod_{i\in\mathbb{Z}} x_i^{\alpha(i)} \cdot \prod_{j\in\mathbb{Z}} y_j^{\beta(j)} \cdot z^t,$$

where $t \in \mathbb{Z}$ and $\alpha(i) \in \mathbb{Z}C^{-1}$, $\beta(j) \in \mathbb{Z}$ for all integers i, j.

Proof. By Lemma 1.1(iv), the subgroup $X_i = \langle y_j x_i y_j^{-1} : j \in \mathbb{Z}\rangle$ is isomorphic to the additive group of $\mathbb{Z}C^{-1}$. Now it is sufficient to use the fact that $z^{-1}X_i z = X_{i-1}$.

Let X be the normal subgroup of $G(\phi)$ generated by $x = x_0$ and let Y be the normal subgroup of $G(\phi)$ generated by $x = x_0$ and $y = y_0$. These definitions yield:

COROLLARY 1.3. *There exist normal subgroups X and Y of $G(\phi)$ such that X is isomorphic to the infinite product of the additive group $\mathbb{Z}C^{-1}$, and Y/X, $G(\phi)/Y$ are free abelian groups.*

2. Residually finite groups. In this section we describe some conditions for the group $G(\phi)$ to be residually finite.

DEFINITION. We will say that a group G is *approximated by finite p-groups* if for every $1 \neq g \in G$ there exists a normal subgroup H of G such that $g \notin H$ and the index of H in G is p^n for some n.

Clearly, if G is approximated by finite p-groups then G is approximated by finite groups and so G is a residually finite group.

Consider the following two simple examples.

EXAMPLE 2.1. Let $\phi(i) = 1$ for all i. Then $G(\phi)$ is approximated by finite p-groups for any prime p. This is clear since $G(\phi)$ is a wreath product of the free abelian group generated by x and y by the infinite cyclic group generated by z.

EXAMPLE 2.2. Let ϕ be a map onto the set of all primes. Then $G(\phi)$ contains subgroups isomorphic to the additive group of rational numbers so it is not residually finite.

This example was described by P. Hall in [5], Theorem 2. He proved that this is a minimal example (in the sense of minimal soluble rank) of a soluble group which is not residually finite. Moreover, this group contains a maximal subgroup of infinite index. See also [9], Theorem 9.58.

LEMMA 2.3. *Let H be the normal subgroup of $G(\phi)$ generated by z^n and y^m. Then H consists of finite products*

$$\prod_{i\in\mathbb{Z}} x_i^{\alpha(i)} \cdot \prod_{j\in\mathbb{Z}} y_j^{\beta(j)} \cdot z^{nt},$$

where $\alpha(i) \in \mathbb{Z}C^{-1}$ and $\sum_{i\in\mathbb{Z}} \alpha(in+k)$ belongs to the ideal $J(n,m)$ of $\mathbb{Z}C^{-1}$ generated by the integers $\phi(j) - \phi(j-n)$ and $\phi(j)^m - 1$ for all j. Moreover, $\beta(j) \in \mathbb{Z}$ and $\sum_{j\in\mathbb{Z}} \beta(jn+k) \in m\mathbb{Z}$ for all integers k.

Proof. We have $z^n \in H$ so H contains also

$$x_i x_{i+n}^{-1} = x_i z^n x_i^{-1} z^{-n} \quad \text{and} \quad y_j y_{j+n}^{-1} = y_j z^n y_j^{-1} z^{-n},$$

for all integers i and j. Consequently, H contains

$$x_i^{\phi(j)-\phi(j-n)} = x_i^{\phi(j)} y_{i-j} y_{i-j+n}^{-1} x_i^{-\phi(j)} y_{i-j+n} y_{i-j}^{-1}$$

and

$$x_i^{\phi(j)^m-1} = x_i^{-1} y_{i-j}^{-m} x_i y_{i-j}^m.$$

Let $k \in J(n,m)$. Then there exists an integer $c \in C$ such that ck is a sum of integers of the form $\phi(j) - \phi(j-n)$ or $\phi(j)^m - 1$. Then x_i^{ck} is a product of $x_i^{\phi(j)-\phi(j-n)}$, $x_i^{\phi(j)^m-1}$ and their inverses. By Lemma 1.1, there exists $y \in Y$ such that $y^{-1} x_i y = x_i^c$. This yields $x_i^k \in H$. Using elements of the form $x_i x_{i+n}^{-1}$, we can prove that a finite product $\prod_{i\in\mathbb{Z}} x_i^{\alpha(i)}$ belongs to

H, where $\alpha(i) \in \mathbb{Z}C^{-1}$ and $\sum_{i\in\mathbb{Z}} \alpha(in+k)$ belongs to the ideal $J(n,m)$ of $\mathbb{Z}C^{-1}$. Similarly we can prove that for all integers k, the product $\prod_{j\in\mathbb{Z}} y_j^{\beta(j)}$ belongs to H, where $\prod_{j\in\mathbb{Z}} \beta(jn+k) \in m\mathbb{Z}$. To end the proof, one can easily check that the subgroup defined above is stable under conjugations by x, y and z.

THEOREM 2.4. *Let ϕ be a map from $\mathbb{Z}$ into $\mathbb{Z}\setminus\{0\}$. Let C be the multiplicative semigroup generated by the image of ϕ. Then the group $G(\phi)$ is residually finite if and only if for any positive integer N there exist integers $m > N$, $n > N$ and $t > N$ such that $t \notin C$ and the ideal $J(n,m)$ of $\mathbb{Z}C^{-1}$ generated by the set $\{\phi(j) - \phi(j-n),\ \phi(j)^m - 1 : j \in \mathbb{Z}\}$ is contained in $t\mathbb{Z}C^{-1}$.*

Proof. $\Rightarrow$ Suppose $G(\phi)$ is residually finite. Take an integer $N > 0$. Then there exists a normal subgroup H of $G(\phi)$ such that z^i, y^i and x^i do not belong to H for $i \le N$. Let n, m and t be the smallest positive integers such that H contains z^n, y^m and x^t. By Lemma 2.3, H contains x^j for all $j \in J(n,m)$. Hence $J(n,m) \subset t\mathbb{Z}C^{-1}$.

$\Leftarrow$ Fix a positive integer N. Let $m,n,t > N$ be integers such that $t \notin C$ and $J(n,m) \subset t\mathbb{Z}C^{-1}$. Let H_N be the normal subgroup generated by z^n, y^m and x^t. Then by Lemma 2.3, the subgroup H_N consists of finite products

$$\prod_{i\in\mathbb{Z}} x_i^{\alpha(i)} \cdot \prod_{j\in\mathbb{Z}} y_j^{\beta(j)} \cdot z^{ns},$$

where $\alpha(i) \in \mathbb{Z}C^{-1}$, $\sum_{i\in\mathbb{Z}} \alpha(in+k) \in t\mathbb{Z}C^{-1}$, $\beta(j) \in \mathbb{Z}$ and $\sum_{j\in\mathbb{Z}} \beta(jn+k) \in m\mathbb{Z}$ for all integers k. This subgroup has a finite index equal to $nm^n t^n$. It is clear that the intersection of the subgroups H_N over all positive integers N is trivial. Hence $G(\phi)$ is residually finite.

THEOREM 2.5. *Let $\phi : \mathbb{Z} \to \mathbb{Z}\setminus\{0\}$ be periodic with period n (that is, $\phi(n+i) = \phi(i)$ for all $i \in \mathbb{Z}$). Then $G(\phi)$ is residually finite.*

Proof. Suppose p is a prime with does not divide any of $\phi(1),\ldots,\phi(n)$ where n is the period of ϕ. Let Gp be the normal subgroup of $G(\phi)$ generated by z^{np}, y^{p-1} and x^p. Since p divides $\phi(i)^{p-1} - 1$ for all i, by Lemma 2.3 the group Gp consists of elements of the form

$$\prod_{i\in\mathbb{Z}} x_i^{\alpha(i)} \cdot \prod_{j\in\mathbb{Z}} y_j^{\beta(j)} \cdot z^{pn},$$

where $\sum_{k\in\mathbb{Z}} \alpha(i+pk) \in p\mathbb{Z}C^{-1}$ for all i and $\sum_{k\in\mathbb{Z}} \beta(j+pk) \in (p-1)\mathbb{Z}$ for all j. One can easily check that the index of Gp in $G(\phi)$ is $np(p-1)^{np}p^{np}$. It is clear that the intersection of all subgroups Gp, for p prime not dividing any of $\phi(1),\ldots,\phi(n)$, is trivial.

THEOREM 2.6. *Let p be a prime. Then $G(\phi)$ is approximated by finite p-groups if and only if $p \notin C$ and for any positive integer N there exist integers $m > N$, $n > N$ and $t > N$ such that the ideal $J(p^n, p^m)$ is contained in $p^t \mathbb{Z}C^{-1}$.*

Proof. $\Rightarrow$ Suppose $G(\phi)$ is approximated by finite p-groups. Let $N > 0$ be an integer. Then there exists a normal subgroup H of $G(\phi)$ such that $G(\phi)/H$ is a finite p-group and z^i, y^i and x^i do not belong to H for $i \le p^N$. Let n, m and t be the smallest positive integers such that H contains z^n, y^m and x^t. By Lemma 2.3, H contains x^j for all $j \in J(n,m)$. Hence $J(n,m) \subset t\mathbb{Z}C^{-1}$. Furthermore, n, m and t are some powers of p since the index of H is a power of p.

$\Leftarrow$ Let H_N be a normal subgroup of $G(\phi)$ defined in the following way: Let $m, n, t > N$ be integers such that $J(p^n, p^m) \subset p^t \mathbb{Z}C^{-1}$. Let H_N be the normal subgroup generated by z^{p^n}, y^{p^m} and x^{p^t}. Then by Lemma 2.3, H_N consists of finite products

$$\prod_{i\in\mathbb{Z}} x_i^{\alpha(i)} \cdot \prod_{j\in\mathbb{Z}} y_j^{\beta(j)} \cdot z^{sp^n},$$

where $\alpha(i) \in \mathbb{Z}C^{-1}$, $\sum_{i\in\mathbb{Z}} \alpha(in+k) \in p\mathbb{Z}C^{-1}$, $\beta(j) \in \mathbb{Z}$ and $\sum_{j\in\mathbb{Z}} \beta(jn+k) \in p^m\mathbb{Z}$ for all integers k. This subgroup has a finite index equal to $p^{n+mp^n+tp^n}$. It is clear that the intersection of all subgroups H_N over all positive integers N is trivial. Hence $G(\phi)$ is residually finite.

THEOREM 2.7. *Let $m > 1$. Let $\phi : \mathbb{Z} \to \mathbb{Z}$ be defined by $\phi(i) = im + 1$. Then $G(\phi)$ is approximated by finite p-groups if and only if the prime p divides m.*

Proof. $\Rightarrow$ Suppose that p does not divide m. Then there exists an integer i such that p divides $im + 1$. Hence $p \in C$ and consequently $G(\phi)$ is not approximated by finite p-groups.

$\Leftarrow$ Suppose p divides m. Let n be a positive integer. Then the ideal $J(p^n, p^n)$ is generated by

$$\phi(j) - \phi(j - p^n) = jm + 1 - (j - p^n)m - 1 = p^n m$$

and by

$$\phi(j)^{p^n} - 1 = (jm+1)^{p^n} - 1.$$

One can easily show by induction on n that all these elements belong to $p^n\mathbb{Z}$. This yields $J(p^n, p^n) \subset p^n\mathbb{Z}C^{-1}$. By Theorem 2.6, $G(\phi)$ is approximated by finite p-groups.

Now we show that the residual finiteness of $G(\phi)$ does not depend on the semigroup C.

EXAMPLE 2.8. Let $m > 0$ be an integer and let $\phi(i) = m$ for all i. Then $G(\phi)$ is approximated by finite p-groups for all primes p relatively prime to m.

EXAMPLE 2.9. Let $\phi(i) = 1$ for $i \neq 0$ and $\phi(0) = m$, where $m > 1$ is an integer. Then $G(\phi)$ is not residually finite.

Proof. Suppose H is a normal subgroup of $G(\phi)$ of a finite index. Then H contains z^n for some n. This yields

$$y_0 y_n^{-1} = y_0 z^n y_0^{-1} z^{-n} \in H.$$

Consequently, H contains

$$x_0^{m-1} = x_0^{\phi(0)-\phi(n)} = x_0^{\phi(0)} y_0 y_n^{-1} x_0^{-\phi(0)} y_n^{-1} y_0.$$

Hence $G(\phi)$ is not residually finite.

3. Groups with the finite embedding property

DEFINITION. Following [3], we will say that a group G is a *Finite Embedding group* (*FE-group*) if for every finite subset X of G there exists an injection Ψ of X into a finite group H such that if x, y and xy are in X then

$$\Psi(xy) = \Psi(x)\Psi(y).$$

THEOREM 3.1 ([3], Proposition 1.2). *All residually finite groups are FE-groups.*

THEOREM 3.2. *Every finitely related FE-group G is residually finite.*

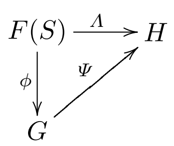
Proof. Let G be a FE-group generated by a set S with relations $r_1, \ldots, r_n$. Then $G = F(S)/R$ where $F(S)$ is the free group generated by S and R is the normal subgroup of $F(S)$ generated by the set of relations. Let $\phi : F(S) \to G$ be the canonical projection. Let $v \neq 1$ be an element of G and $w \in F(S)$ be such that $\phi(w) = v$. Let X be the set of all subwords of $w, r_1, \ldots, r_n$ including the empty word. By definition, there exists an injection Ψ of $\phi(X)$ into a finite group H such that if x, y and xy are in $\phi(X)$ then

$$\Psi(xy) = \Psi(x)\Psi(y).$$

Let $\Lambda : F(S) \to H$ be the group homomorphism given by

$$\Lambda(s) = \begin{cases} \Psi(\phi(s)) & \text{if } s \in X \cap S, \\ 1 & \text{if } s \in S \setminus X. \end{cases}$$

We arrive at a commutative diagram of group morphisms:



By the properties of Ψ, the set $\{r_1, \ldots, r_n\}$ of relations is contained in $\ker \Lambda$. Hence we can extend Ψ to a group homomorphism $\lambda : G \to H$. Since Ψ is an injection, $\lambda(v) \neq 1$. Furthermore, $\ker \lambda$ is a subgroup of G of finite index.

PROPOSITION 3.3. *Let G be a group such that for every finite subset X of G there exists an injection Ψ of X into a residually finite group Γ such that if x, y and xy are in X then $\Psi(xy) = \Psi(x)\Psi(y)$. Then G is a FE-group.*

Proof. Let X, Ψ and Γ be as in the assumptions. Since $\Psi(X)$ is a finite subset of Γ, there exists an injection τ of $\Psi(X)$ into a finite group H such that if x, y and xy are in X then $\tau(\Psi(xy)) = \tau(\Psi(x))\tau(\Psi(y))$. Now $\tau \circ \Psi : X \to H$ is the required injection.

The aim of this section is to prove that $G(\phi)$ is a FE-group for every ϕ. This gives us a series of not residually finite FE-groups.

THEOREM 3.4. *The group $G(\phi)$ satisfies the FE condition for all functions ϕ.*

Proof. Let $\phi : \mathbb{Z} \to \mathbb{Z} \setminus \{0\}$ and let X be a finite subset of $G(\phi)$. Then there exists a positive integer n such that all elements of X can be written as products

$$\prod_{i=-n}^{n} x_i^{\alpha(i)} \cdot \prod_{j=-n}^{n} y_j^{\beta(j)} \cdot z^t,$$

where for all i and j we have $\alpha(i) \in \mathbb{Z}C^{-1}$, $\beta(j) \in \mathbb{Z}$ and $-n \leq t \leq n$. The multiplication in X looks as follows:

$$\begin{aligned}
&\prod_{i=-n}^{n} x_i^{\alpha(i)} \cdot \prod_{j=-n}^{n} y_j^{\beta(j)} \cdot z^t \cdot \prod_{i=-n}^{n} x_i^{\delta(i)} \cdot \prod_{j=-n}^{n} y_j^{\gamma(j)} \cdot z^k \\
&\quad = \prod_{i=-n}^{n} x_i^{\alpha(i)} \cdot \prod_{j=-n}^{n} y_j^{\beta(j)} \cdot \prod_{i=-n}^{n} x_{i+t}^{\delta(i)} \cdot \prod_{j=-n}^{n} y_{j+t}^{\gamma(j)} \cdot z^{t+k} \\
&\quad = \prod_{i=-n}^{n+t} x_i^{\alpha(i)+\delta(i-t)\prod_{j=-n}^{n} \phi(j+t-i)^{\beta(j)}} \cdot \prod_{j=-n}^{n+t} y_j^{\beta(j)+\gamma(j-t)} \cdot z^{t+k}.
\end{aligned}$$

Let $\psi : \mathbb{Z} \to \mathbb{Z} \setminus \{0\}$ be a periodic function with period $6n+2$ defined by

$$\psi(i) = \begin{cases} \phi(i) & \text{for } -3n \leq i \leq 3n, \\ M & \text{for } i = 3n+1, \end{cases}$$

where M is an integer so large that every element of X can be considered as an element of $G(\phi)$. Let $\lambda : X \to G(\psi)$ be the injection given by

$$\lambda\Big(\prod_{i=-n}^{n} x_i^{\alpha(i)} \cdot \prod_{j=-n}^{n} y_j^{\beta(j)} \cdot z^t\Big) = \prod_{i=-n}^{n} x_i^{\alpha(i)} \cdot \prod_{j=-n}^{n} y_j^{\beta(j)} \cdot z^t$$

It is clear that $\lambda(ab) = \lambda(a)\lambda(b)$ for $a, b \in X$. Since by Theorem 2.5, $G(\psi)$ is residually finite, it is a FE-group by Proposition 3.3.

COROLLARY 3.5. *There exists a finitely generated FE-group which is not locally residually finite.*

Proof. Let ϕ be a function from $\mathbb{Z}$ onto the set of all primes. Then $G(\phi)$ is generated by 3 elements, it is not residually finite since it contains subgroups isomorphic to the additive group of $\mathbb{Q}$ and by Theorem 3.3, it is a FE-group.

4. Idempotents. One of the famous open problems in group theory is the following one formulated by Kaplansky [6]:

CONJECTURE. *The group algebra $k[G]$ of a torsion free group G over a field has no nontrivial idempotents.*

Formanek [4] gave a partial answer to this conjecture in the case when K is a field of characteristic 0 and for groups satisfying the following non-divisibility condition:

(∗) For each $1 \neq g \in G$ there are infinitely many primes p such that g is not conjugate to any of $g^p, g^{p^2}, g^{p^3}, \ldots$

Zalesskiĭ and Mikhalev [8] studied idempotents in group algebras of positive characteristic p and formulated the following condition:

(D_p) For any $g \in G$, if g is conjugate to g^{p^N} for some integer $N > 0$ then g has finite order.

In [1] Bass reformulated the condition (∗) follows:

(D) Suppose H is a finitely generated subgroup of G, $g \in G$, N is an integer > 0 and for all but finitely many primes p, g is conjugate in H to g^{p^N}. Then g has finite order.

He proved that linear groups satisfy condition (D) and the torsion free linear groups satisfy Kaplansky's Conjecture. He also proved that the (D)-groups satisfy the following conjecture:

BASS' STRONG CONJECTURE [1]. *Let P be a finitely generated projective module over the integral group ring $\mathbb{Z}[G]$. Then $r_p(g)=0$ for $g\neq 1$, where r_p is the trace map.*

Strojnowski [10] proved Bass' Strong Conjecture for groups satisfying the following condition:

(WD) Suppose H is a finitely generated subgroup of G, $g \in H$, N is an integer > 0 and for all primes p, g is conjugate to g^{p^N}. Then $g = 1$.

In this paper we give a series of examples to show how these conditions differ.

THEOREM 4.1. (i) *$G(\phi)$ satisfies condition* (D_p) *if and only if the group CC^{-1} does not contain any power of the prime p.*

(ii) *$G(\phi)$ satisfies condition* (D) *if and only if for all integers $N > 0$ the group CC^{-1} does not contain infinitely many elements of the set $\{p^N\colon\ p$ is a prime$\}$.*

(iii) *$G(\phi)$ satisfies condition* (WD) *if and only if for any integer $N > 0$ there exists a prime number p such that p^N does not belong to the group CC^{-1}.*

Proof. Since the proofs of all parts are similar we only show (i). Let $p^n \in CC^{-1}$. Then by Lemma 1.1(iii), there exists an element g of the subgroup generated by all y_s such that $g^{-1}xg = x^{p^N}$. Hence $G(\phi)$ does not satisfy condition (D_p).

Conversely, if $G(\phi)$ does not satisfy (D_p) then there exists $h \in G(\phi)$ of infinite order such that h is conjugate to its p^nth power. Since the groups $G(\phi)/Y$ and Y/X are free abelian, they do not contain the additive group $\mathbb{Z}[1/p]$. Hence by Lemma 1.1(i), $h \in X$. Let $h = \prod_{i=a}^{b} x_i^{\alpha(i)}$ and let $g = \prod_{j=c}^{d} y_j^{\beta(j)} \cdot z^t \in G(\phi)$ be such that $g^{-1}hg = h^{pN}$. Then

$$h^{pN} = \prod_{i=a}^{b} x_i^{\alpha(i)p^n} = z^{-t}\Big(\prod_{i=a}^{b} x_i^{\alpha(i)\prod_{j=c}^{d}\phi(i-j)^{\beta(j)}}\Big)z^t.$$

Hence $t = 0$ and for all i, if $\alpha(i) \neq 0$ then $\prod_{j=c}^{d} \phi(i-j)^{\beta(j)} = p^n$. Thus, $p^n \in CC^{-1}$.

EXAMPLE 4.2. Let ϕ be a map from the integers onto the set $\{p^p : p$ is a prime$\}$. Then $G(\phi)$ satisfies conditions (D) and (WD) but does not satisfy $(*)$ or (D_p) for any prime p.

EXAMPLE 4.3. Let ϕ be a map from the integers onto $\{2p : p$ is an odd prime$\}$. Then $G(\phi)$ satisfies (WD), (D), $(*)$, and (D_p) for all primes p but is not residually finite since it contains a subgroup isomorphic to the additive group of all rational numbers.

Now we show that nondivisibility conditions are not stable under infinite extensions by cyclic groups.

EXAMPLE 4.4. Let $H = G(\phi) \rtimes \langle g\rangle$ be the semidirect product of the group $G(\phi)$ from Example 4.3 and the infinite cyclic group generated by g such that $gz = zg$, $gy = yg$ and $g^{-1}xg = x^2$. Then x is conjugate in H to x^p for all primes p. Hence the group H does not satisfy any of the conditions (WD), (D), $(*)$ or (D_p).

PROPOSITION 4.5. *The following classes of groups are closed under subdirect products*:

(i) (D_p)*-groups having at most p-torsion.*
(ii) *Torsion free* (D)*-groups.*
(iii) (WD)*-groups.*

P r o o f. Since proofs of all parts are similar we only show (i). Let $G \subseteq \prod_{j\in J} G_j$ be a subdirect product of (D_p)-groups with p-torsion only. Let $g = (g_j), h = (h_j) \in G$ be such that $h^{-1}gh = g^{p^N}$. Then for each j, $h_j^{-1}g_jh_j = g_j^{p^N}$ so $g_j = 1$. Hence $g = 1$.

In [1] Bass wrote: "We do not know whether all residually finite groups satisfy condition (D)". The negative answer was given by Wilson [11]. Now we present a new construction of such "bad" groups.

THEOREM 4.6. *Let* $m > 1$. *Let* $\phi : \mathbb{Z} \to \mathbb{Z}$ *be defined by* $\phi(i) = im + 1$. *Then* $G(\phi)$ *satisfies the condition* (WD) *but does not satisfy* (D) *or* $(*)$. *Moreover*, *for each prime p the following conditions are equivalent*:

(i) *p divides m.*
(ii) $G(\phi)$ *is approximated by finite p-groups.*
(iii) $G(\phi)$ *satisfies condition* (D_p).

P r o o f. The implication (i)⇒(ii) follows from Theorem 2.7.

(ii)⇒(iii) follows from Proposition 4.5.

(iii)⇒(i). Take a prime q such that q does not divide m. Since at least two of the integers $1, q, q^2, \ldots, q^m$ are congruent modulo m, m divides $q^{m!} - 1$ so $q^{m!}$ has the form $im + 1$. Hence $G(\phi)$ does not satisfy (D_q).

Furthermore, by Theorem 4.1(ii), $q^{m!} - 1 \in C$ for primes $q > m$ implies that $G(\phi)$ satisfies neither (D) nor $(*)$.

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