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ISOMETRIC IMMERSIONS OF THE HYPERBOLIC SPACE $H^n(-1)$ INTO $H^{n+1}(-1)$

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We transform the problem of determining isometric immersions from $H^n(-1)$ into $H^{n+1}(-1)$ into that of solving equations of degenerate Monge–Ampère type on the unit ball $B^n(1)$. By presenting one family of special solutions to the equations, we obtain a great many noncongruent examples of such isometric immersions with or without umbilic set.

1. Introduction. Let $H^n(c)$ $(c < 0)$ be an n-dimensional hyperbolic space form with constant sectional curvature c. Its Cayley model is the hypersurface $F : \langle X, X\rangle_L = 1/c$, $x_{n+1} > 0$, in the Minkowski space $\mathbb{R}^{n+1}_1$, where $\langle\cdot,\cdot\rangle_L$ denotes the inner product in $\mathbb{R}^{n+1}_1$, i.e., $\langle X, Y\rangle_L = \sum_{i=1}^n x_i y_i - x_{n+1}y_{n+1}$ for $X = (x_1, \ldots, x_{n+1}), Y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$.

Denote by $M^n(c)$ the n-dimensional space form of constant sectional curvature c, i.e., $M^n(0) = E^n$, $M^n(c) = S^n(1/\sqrt{c})$ for $c > 0$ and $M^n(c) = H^n(c)$ for $c < 0$. For the problem of isometric immersions of $M^n(c)$ into $M^{n+1}(c)$, the following global results are well-known:

(i) $c=0$. Each isometrically immersed complete manifold M^n of E^n into E^{n+1} must be a cylinder over a plane curve, i.e., $M^n = E^{n-1} \times C$, where C is a curve in the plane orthogonal to E^{n-1}. This result is due to Hartman and Nirenberg [6] and Massey [10].

(ii) $c = 1$. An isometric immersion of $S^n(1)$ into $S^{n+1}(1)$ is rigid, i.e., it can only be a totally geodesic imbedding [1, 4, 13].

In the hyperbolic case, the situation looks quite different. Isometric immersions seem much more abundant. Indeed, Nomizu [11] constructed explicitly a one-parameter family of examples of isometric immersions of $H^2(-1)$ into $H^3(-1)$ with three different kinds of properties. At the same time, Ferus [5] showed that given a totally geodesic foliation of codimension 1 in $H^n(-1)$, there is a family of isometric immersions of $H^n(-1)$ into

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$H^{n+1}(-1)$ for which the relative nullity foliations coincide with the given foliation. These results completely characterize the space of nowhere umbilic isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$. By considering a broader class of isometric immersions, Abe and Haas [2] showed that given a differentiable lamination on $H^n(-1)$ there is a family of isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ so that the induced relative nullity foliations are completely determined by the lamination. While including the result of [5] and [11], Abe and Haas' approach provides also examples of isometric immersions with umbilic sets.

On the other hand, by making use of the fundamental theorem for hypersurfaces, Abe–Mori–Takahashi [3] completely parametrizes the space of isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ by a family of properly chosen (at most) countable n-tuples of real-valued functions defined on an open interval. This is an answer to the following open problem posed by Nomizu [11]: "*To determine all isometric immersions from* $H^n(-1)$ *into* $H^{n+1}(-1)$".

In this paper, we also consider that problem. By presenting a new approach we transform it into a problem of pure analysis. Precisely, we transform the problem of determining isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ into that of solving equations of degenerate Monge–Ampère type on the unit ball $B^n(1)$. Our main result is the following:

THEOREM. *Every smooth isometric immersion of* $H^n(-1)$ *into* $H^{n+1}(-1)$ *corresponds to a solution of the equations*

$$(1)\qquad \frac{\partial^2 u}{\partial\xi_i\partial\xi_j}\cdot\frac{\partial^2 u}{\partial\xi_k\partial\xi_l}-\frac{\partial^2 u}{\partial\xi_i\partial\xi_k}\cdot\frac{\partial^2 u}{\partial\xi_j\partial\xi_l}=0,\qquad \xi=(\xi_1,\dots,\xi_n)\in B^n(1),\quad i,j,k,l=1,\dots,n.$$

Conversely, for every smooth solution u *of* (1) *we define*

$$(2)\qquad \begin{cases} g_{ij}=\lambda^{-4}(\lambda^2\delta_{ij}+\xi_i\xi_j), & \lambda=\sqrt{1-\xi_1^2-\ldots-\xi_n^2},\\ h_{ij}=\lambda^{-1}\dfrac{\partial^2 u}{\partial\xi_i\partial\xi_j}, & i,j=1,\dots,n.\end{cases}$$

Then u *determines a smooth isometric immersion of* $H^n(-1)$ *into* $H^{n+1}(-1)$ *with* g *and* h *being its first and second fundamental forms, respectively.*

REMARK 1. Our theorem completely parametrizes the space of isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ by just one properly chosen real-valued function on $B^n(1)$. This can be compared with Theorem 1.1 of Abe–Mori–Takahashi [3].

REMARK 2. The special case $n=2$ has been considered by G. S. Zhao and the author in [8]. In another paper [7], we have studied the isometric immersions of $H^2(-1)$ into $H^3(c)$ $(c<-1)$, and basing on results obtained by Li [9], we partially classified those isometric immersions which are of bounded principal curvatures.

REMARK 3. Equations (1) imply that $\det(\partial^2 u/\partial\xi_i\partial\xi_j) = 0$ and $\mathrm{rank}(\partial^2 u/\partial\xi_i\partial\xi_j) \le 1$. For simplicity, we call (1) the *degenerate Monge–Ampère type equation* hereafter.

2. Proof of the theorem. Take the hyperplane Π: $x_{n+1} = 1$ in $\mathbb{R}^{n+1}_1$ and consider the central projection p of F: $x_1^2 + \ldots + x_n^2 - x_{n+1}^2 = -1$, $x_{n+1} > 0$, from the origin O of $\mathbb{R}^{n+1}$ into Π. Then F is mapped in a one-to-one fashion onto the open unit ball $B^n(1)$: $\xi_1^2 + \ldots + \xi_n^2 < 1$. The mapping p is given by

$$p : F \to B^n(1), \quad (x_1, \ldots, x_{n+1}) \to (\xi_1, \ldots, \xi_n),$$

where $x_{n+1} = \sqrt{1 + x_1^2 + \ldots + x_n^2}$ and $\xi_i = x_i/x_{n+1}$, $i = 1, \ldots, n$. The parametric representation of F with respect to $\{\xi_1, \ldots, \xi_n\}$ is given by

$$F : v(\xi_1, \ldots, \xi_n) = \left(\frac{\xi_1}{\lambda}, \ldots, \frac{\xi_n}{\lambda}, \frac{1}{\lambda}\right), \quad \lambda = \sqrt{1 - \xi_1^2 - \ldots - \xi_n^2}.$$

The metric of $H^n(-1)$ with respect to the global coordinates $\{\xi_1, \ldots, \xi_n\}$ on $B^n(1)$ is given by the tensor g with

$$g_{ij} = \lambda^{-4}(\lambda^2\delta_{ij} + \xi_i\xi_j), \quad i, j = 1, \ldots, n,$$

and $\det(g_{ij}) = \lambda^{-2n-2}$. The Christoffel symbols are $\Gamma_{ij}^k = \lambda^{-2}(\xi_i\delta_{jk} + \xi_j\delta_{ik})$.

Denote by h_{ij} the second fundamental form of the immersion $x : H^n(-1) \to H^{n+1}(-1)$ w.r.t. the global frame $\{\partial/\partial\xi_1, \ldots, \partial/\partial\xi_n\}$ on $H^n(-1)$. From the fundamental theorem of hypersurface theory, an isometric immersion is uniquely determined, in the sense of congruence, by its first and second fundamental forms. The Gauss equation and the Codazzi equation are the integrability conditions. It is well known that, for an isometric immersion of $H^n(-1)$ into $H^{n+1}(-1)$, the Codazzi equation is equivalent to the second fundamental form h being a Codazzi tensor on $H^n(-1)$. We will need the following

LEMMA (cf. Proposition 1.3.3 of [12]). *Let (M, g) be a Riemannian manifold of constant sectional curvature c (possibly zero) and T a Codazzi tensor of M. Then for every point of M, there exists a neighborhood U and a smooth function f: $M \to \mathbb{R}$ such that $T = \mathrm{Hess}\, f + cgf$ in U. In addition, if M is simply connected then such a representation is available on all of M.*

Conversely, on a manifold (M, g) of constant curvature c, any smooth function f generates a Codazzi tensor $T = \mathrm{Hess}\, f + cgf$.

By applying the lemma to $H^n(-1) = (B^n(1), g)$, we see that h being a Codazzi tensor on $H^n(-1)$ is equivalent to the existence of a globally defined smooth function f on $B^n(1)$ such that $h = \mathrm{Hess}\, f - gf$.

Define a function on $B^n(1)$ by $u = \lambda f$. Then the components of h are

$$h_{ij} = \nabla_{ij} f - g_{ij} f = \frac{\partial^2 f}{\partial \xi_i \partial \xi_j} - \Gamma_{ij}^k \frac{\partial f}{\partial \xi_k} - g_{ij} f = \lambda^{-1} \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}, \tag{3}$$

where ∇_{ij} are the operators of second covariant differentiation in the metric g.

For an isometric immersion of $H^n(-1)$ into $H^{n+1}(-1)$, the Gauss equation is expressed in the form

$$h_{ij} h_{kl} - h_{ik} h_{jl} = 0, \quad i, j, k, l = 1, \ldots, n. \tag{4}$$

Combining (3) with (4), we obtain the Monge–Ampère type equation (1) in $B^n(1)$.

Hence, every smooth isometric immersion x of $H^n(-1)$ into $H^{n+1}(-1)$ corresponds to a smooth solution u of (1) such that the first and second fundamental forms g and h of x are given by (2).

On the other hand, if u is a smooth solution of (1), then h defined by (2) is a Codazzi tensor on $H^n(-1)$ and g, h determine a smooth isometric immersion x of $H^n(-1)$ into $H^{n+1}(-1)$ with (1) as its Gauss equation.

This completes the proof of our theorem.

3. Further discussion. Assume that u is a smooth solution of (1), and that the first and second fundamental forms g and h of the immersion x which corresponds to u are given by (2). If we denote by σ the principal curvature of the immersion x, then

$$\det(h_{ij} - \sigma g_{ij}) = 0. \tag{5}$$

Now, we mention the following two obvious facts:

(i) Two solutions u and $\overline{u}$ of (1) determine two congruent immersions of $H^n(-1)$ into $H^{n+1}(-1)$ if $u - \overline{u}$ is a linear function in $\xi_1, \ldots, \xi_n$.

(ii) For any $0 \neq a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and $\xi = (\xi_1, \ldots, \xi_n) \in B^n(1)$, we have $-\sqrt{\sum_{i=1}^n a_i^2} < \sum_{i=1}^n a_i \xi_i < \sqrt{\sum_{i=1}^n a_i^2}$. Denote by I_a the interval $(-\sqrt{\sum_{i=1}^n a_i^2}, \sqrt{\sum_{i=1}^n a_i^2})$. Then for each $G \in C^\infty(I_a), u = G(a_1\xi_1 + \ldots + a_n\xi_n)$ is a smooth solution of (1).

Then, for any fixed $0 \neq a \in \mathbb{R}^n$ and $G \in C^\infty(I_a)$, the principal curvature σ which corresponds to the smooth solution $u_G = G(a_1\xi_1 + \ldots + a_n\xi_n)$ of (1) is determined by the equation

$$D_n :\equiv \det\left(\frac{\partial^2 u_G}{\partial \xi_i \partial \xi_j} - \lambda \sigma g_{ij}\right) = 0.$$

By using (2) and $\partial^2 u_G/\partial\xi_i\partial\xi_j = a_i a_j G''$, D_n equals

$$\det\begin{vmatrix} a_1^2\alpha-\lambda^{-1}\sigma-\lambda^{-3}\sigma\xi_1^2 & a_1a_2\alpha-\lambda^{-3}\sigma\xi_1\xi_2 & \dots & a_1a_n\alpha-\lambda^{-3}\sigma\xi_1\xi_n \\ a_2a_1\alpha-\lambda^{-3}\sigma\xi_2\xi_1 & a_2^2\alpha-\lambda^{-1}\sigma-\lambda^{-3}\sigma\xi_2^2 & \dots & a_2a_n\alpha-\lambda^{-3}\sigma\xi_2\xi_n \\ \vdots & \vdots & & \vdots \\ a_na_1\alpha-\lambda^{-3}\sigma\xi_n\xi_1 & a_na_2\alpha-\lambda^{-3}\sigma\xi_n\xi_2 & \dots & a_n^2\alpha-\lambda^{-1}\sigma-\lambda^{-3}\sigma\xi_n^2 \end{vmatrix}$$

$$\begin{aligned} &= -\lambda^{-1}\sigma D_{n-1} \\ &\quad + a_n\alpha\det\begin{vmatrix} -\lambda^{-1}\sigma-\lambda^{-3}\sigma\xi_1^2 & -\lambda^{-3}\sigma\xi_1\xi_2 & \dots & -\lambda^{-3}\sigma\xi_1\xi_n \\ -\lambda^{-3}\sigma\xi_2\xi_1 & -\lambda^{-1}\sigma-\lambda^{-3}\sigma\xi_2^2 & \dots & -\lambda^{-3}\sigma\xi_2\xi_n \\ \vdots & \vdots & & \vdots \\ -\lambda^{-3}\sigma\xi_{n-1}\xi_1 & -\lambda^{-3}\sigma\xi_{n-1}\xi_2 & \dots & -\lambda^{-3}\sigma\xi_{n-1}\xi_n \\ a_1 & a_2 & \dots & a_n \end{vmatrix} \\ &\quad - \lambda^{-3}\sigma\xi_n\det\begin{vmatrix} a_1^2\alpha-\lambda^{-1}\sigma & a_1a_2\alpha & \dots & a_1a_n\alpha \\ a_2a_1\alpha & a_2^2\alpha-\lambda^{-1}\sigma & \dots & a_2a_n\alpha \\ \vdots & \vdots & & \vdots \\ a_{n-1}a_1\alpha & a_{n-1}a_2\alpha & \dots & a_{n-1}a_n\alpha \\ \xi_1 & \xi_2 & \dots & \xi_n \end{vmatrix} \\ &\equiv: -\lambda^{-1}\sigma D_{n-1} + (A) + (B), \qquad \text{where } \alpha = G''(a_1\xi_1+\dots+a_n\xi_n). \end{aligned}$$

Direct computation gives

$$(A) = (-\lambda^{-1}\sigma)^{n-1}\alpha\Big[a_n^2 - \lambda^{-2}\sum_{i=0}^{n-1}(a_n\xi_n a_i\xi_i - a_n^2\xi_i^2)\Big],$$

$$(B) = \lambda^{-2}(-\lambda^{-1}\sigma)^n\xi_n^2 - \lambda^{-2}(-\lambda^{-1}\sigma)^{n-1}\alpha\sum_{i=1}^{n-1}(a_n\xi_n a_i\xi_i - a_i^2\xi_n^2).$$

Hence by induction

$$\begin{aligned} D_n &= (-\lambda^{-1}\sigma)D_{n-1} + (-\lambda^{-1}\sigma)^{n-1}\alpha a_n^2 + \lambda^{-2}(-\lambda^{-1}\sigma)^n\xi_n^2 \\ &\quad + (-\lambda^{-1}\sigma)^{n-1}\alpha\lambda^{-2}\sum_{i=1}^{n-1}(a_n\xi_i - a_i\xi_n)^2 \\ &= (-\lambda^{-1}\sigma)^{n-1}D_1 + (-\lambda^{-1}\sigma)^{n-1}\alpha(a_n^2+\dots+a_2^2) \\ &\quad + \lambda^{-2}(-\lambda^{-1}\sigma)^n(\xi_n^2+\dots+\xi_2^2) \\ &\quad + (-\lambda^{-1}\sigma)^{n-1}\alpha\lambda^{-2}\sum_{j=2}^{n}\sum_{i=1}^{j-1}(a_j\xi_i - a_i\xi_j)^2 \\ &= \lambda^{-2}(-\lambda^{-1}\sigma)^{n-1}\Big[-\lambda^{-1}\sigma + \lambda^2\alpha\sum_{i=1}^{n}a_i^2 + \alpha\sum_{j=2}^{n}\sum_{i=1}^{j-1}(a_j\xi_i - a_i\xi_j)^2\Big]. \end{aligned}$$

Thus, the only generally nonzero principal curvature of the immersion corresponding to u_G satisfies

$$\sigma = \lambda G''\Big(\sum_{i=1}^{n} a_i\xi_i\Big)\Big[\lambda^2\sum_{i=1}^{n} a_i^2 + \sum_{j=2}^{n}\sum_{i=1}^{j-1}(a_j\xi_i - a_i\xi_j)^2\Big].$$

Because $\lambda[\lambda^2\sum_{i=1}^{n} a_i^2 + \sum_{j=2}^{n}\sum_{i=1}^{j-1}(a_j\xi_i - a_i\xi_j)^2] > 0$ in $B^n(1)$, we see that, by choosing the function $G(t)$ properly, we can obtain a great many noncongruent isometric immersions of $H^n(-1)$ into $H^{n+1}(-1)$ with one of the following properties:

(1) The immersion has bounded principal curvature with no umbilics.

(2) The immersion has one-sided unbounded principal curvature with no umbilics.

(3) The immersion has bounded principal curvature with umbilic set consisting of the union of arbitrary finitely many $(n-1)$-dimensional submanifolds of $H^{n+1}(-1)$.

(4) The immersion has one-(or two-)sided unbounded principal curvatures with umbilic set consisting of the union of arbitrary finitely many $(n-1)$-dimensional submanifolds of $H^{n+1}(-1)$.

To study isometric immersions from $H^n(-1)$ into $H^{n+1}(-1)$, one may consider the following Dirichlet boundary problem for the degenerate Monge–Ampère type equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial\xi_i\partial\xi_j}\cdot\dfrac{\partial^2 u}{\partial\xi_k\partial\xi_l} - \dfrac{\partial^2 u}{\partial\xi_i\partial\xi_k}\cdot\dfrac{\partial^2 u}{\partial\xi_j\partial\xi_l} = 0 & \text{in } B^n(1), \\ & i,j,k,l = 1,\ldots,n, \\ u = \phi, & \text{on } \partial B^n(1). \end{cases} \tag{6}$$

However, the regularity properties of the solution to (6) are quite complicated. This is clearly seen even in the special case $n = 2$. Then $u = \sqrt{\xi_1^2 + \xi_2^2}$ satisfies (1) in $B^n(1)\setminus\{0\}$ and $u|_{\partial B^n(1)} \equiv 1$, whereas $\xi = 0$ is a singular point of u.

On the other hand, for $n = 2$, if $\phi \in C^0(\partial B^n(1))$, then (6) does have a unique convex generalized solution. One can consult Wachsmuth [14] for a more detailed discussion and references on the problem (6) for $n = 2$. We hope that a similar result can also be achieved in the general case.

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