ON THE METRIC THEORY OF CONTINUED FRACTIONS

BY

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Introduction. For any positive integer \( n \) we denote by \( P(n) \) the Lebesgue measure of the set of irrational numbers \( x \in (0,1) \) whose closest rational approximation with denominator \( \leq n \) is a convergent of the continued fraction expansion of \( x \).

The question of the behaviour of \( P(n) \) was asked by M. Deléglise to A. Schinzel. Recently I. Aliev, S. Kanemitsu and A. Schinzel [1] proved that

\[
P(n) = \frac{1}{2} + \frac{6}{\pi^2}(\log 2)^2 + O\left(\frac{1}{n}\right).
\]

In this article we shall improve this result to the following

**Theorem.** There exists \( c > 0 \) such that

\[
P(n) = \frac{1}{2} + \frac{6}{\pi^2}(\log 2)^2 + O\left(\frac{1}{n}\exp\left(-c\frac{(\log n)^{3/5}}{\log\log n^{1/5}}\right)\right).
\]

Under the Riemann hypothesis we have

\[
P(n) = \frac{1}{2} + \frac{6}{\pi^2}(\log 2)^2 + O(n^{-4/3+\varepsilon}).
\]

**Remark.** I. Aliev, S. Kanemitsu and A. Schinzel [1] also note that the main term, but not the error term, can be derived from Theorem 1.3 of P. Kargaev and A. Zhigljavsky [2].

Classical results. We denote by \( \lfloor x \rfloor \) the greatest integer not exceeding \( x \) and write \( \psi(x) = x - \lfloor x \rfloor - \frac{1}{2} \).

**Lemma 1.** Let \( f \) be a function with a continuous derivative in the interval \([a, b] \). Then

\[
\sum_{a < n \leq b} f(n) = \int_a^b f(x) \, dx + \psi(a)f(a) - \psi(b)f(b) + \int_a^b \psi(x)f'(x) \, dx.
\]

**Proof.** See for example Titchmarsh [4], formula 2.1.2, page 13.

\[\square\]

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[9]
Applying this lemma and writing $\phi(x) = \psi(x)/x$, $\log^+ x = \max(\log x, 1)$ we obtain

**Lemma 2.** For arbitrary positive numbers $a < b$ we have

$$
\sum_{a < n \leq b} \frac{1}{n} = \log b - \log a + \phi(a) - \phi(b) + O\left(\frac{1}{a^2}\right),
$$

$$
\sum_{a < n \leq b} \frac{\log n}{n} = \frac{(\log b)^2 - (\log a)^2}{2} + \phi(a) \log a - \phi(b) \log b + O\left(\frac{\log^+ a}{a^2}\right),
$$

$$
\sum_{a < n \leq b} \frac{1}{n^2} = \frac{1}{a} - \frac{1}{b} + O\left(\frac{1}{a^2}\right).
$$

**Corollary 1.** For any positive number $x$, we have

$$
\sum_{x/2 < k \leq x} \frac{1}{k} = \log 2 + \phi\left(\frac{x}{2}\right) - \phi(x) + O\left(\frac{1}{x^2}\right),
$$

$$
\sum_{x/2 < k \leq x} \frac{\log k}{k} = \log \left(\frac{x}{2}\right) \log 2 + \frac{(\log 2)^2}{2}
$$

$$
+ \phi\left(\frac{x}{2}\right) \log \left(\frac{x}{2}\right) - \phi(x) \log x + O\left(\frac{\log^+ x}{x^2}\right),
$$

$$
\sum_{x/2 < k \leq x} \frac{1}{k^2} = \frac{1}{x} + O\left(\frac{1}{x^3}\right).
$$

**Lemma 3.** There exists $c > 0$ such that for any $x \geq 1$ we have

$$
\sum_{1 \leq d \leq x} \frac{\mu(d)}{d} = O\left(\exp\left(-c \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)\right),
$$

$$
\sum_{1 \leq d \leq x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O\left(\frac{1}{x} \exp\left(-c \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right)\right).
$$

Under the Riemann hypothesis, for any $x \geq 1$ we have

$$
\sum_{1 \leq d \leq x} \frac{\mu(d)}{d} = O(x^{-1/2+\varepsilon}), \quad \sum_{1 \leq d \leq x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O(x^{-3/2+\varepsilon}).
$$

**Proof.** By partial summation, for any $1 \leq x \leq y$ we have

$$
\sum_{x < d \leq y} \frac{\mu(d)}{d} = \frac{1}{y} \sum_{x < d \leq y} \mu(d) + \int_{x}^{y} \left( \sum_{x < d \leq t} \mu(d) \right) dt \frac{dt}{t^2},
$$

$$
\sum_{x < d \leq y} \frac{\mu(d)}{d^2} = \frac{1}{y^2} \sum_{x < d \leq y} \mu(d) + 2 \int_{x}^{y} \left( \sum_{x < d \leq t} \mu(d) \right) dt \frac{dt}{t^3}.
$$
By Satz 3 of A. Walfisz [5], page 191, there exists $c' > 0$ such that

$$\sum_{1 \leq d \leq x} \mu(d) = O\left(x \exp\left(-c' \frac{(\log x)^{3/5}}{\log \log x}^{1/5}\right)\right).$$

Writing

$$\delta(t) = \exp\left(-c' \frac{(\log t)^{3/5}}{\log \log t}^{1/5}\right),$$

we have $\sum_{x < d \leq y} \mu(d) \ll t \delta(t)$, hence

$$\sum_{x < d \leq y} \frac{\mu(d)}{d} \ll \delta(y) + \int \frac{\delta(t) \, dt}{t} \ll \delta(x) \log x,$$

$$\sum_{x < d \leq y} \frac{\mu(d)}{d^2} \ll \frac{\delta(y)}{y} + \int \frac{\delta(t) \, dt}{t^2} \ll \frac{\delta(y)}{y} + \delta(x) \int \frac{\delta(t) \, dt}{t^2} \ll \frac{\delta(x)}{x}.$$ 

In 1909, Landau [3] proved that $\sum_{d=1}^{\infty} \mu(d)/d = 0$. Hence for any $c$ with $0 < c < c'$ we have

$$\sum_{1 \leq d \leq x} \frac{\mu(d)}{d} = -\lim_{y \to \infty} \sum_{x < d \leq y} \frac{\mu(d)}{d} \ll \delta(x) \log x \ll \exp\left(-c \frac{(\log x)^{3/5}}{(\log \log x)^{1/5}}\right),$$

which proves the first estimate of Lemma 3.

Furthermore,

$$\sum_{1 \leq d \leq x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} - \sum_{d > x} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O\left(\frac{\delta(x)}{x}\right),$$

which proves the second estimate of Lemma 3.

The proof of the estimates under the Riemann hypothesis is similar.

**Proof of the theorem.** In [1] I. Aliev, S. Kanemitsu and A. Schinzel reduce the problem to the evaluation of an elementary sum by proving

**Lemma 4 (Aliev, Kanemitsu, Schinzel).** For $n > 1$ we have

$$P(n) = \frac{1}{2} + 2 \sum_{b,c} \frac{1}{bc}$$

where the sum is taken over all integers $b, c$ such that $1 \leq b \leq n < c < 2b$ and $(b, c) = 1$. 
Applying this lemma we have

\[ P(n) = \frac{1}{2} + 2 \sum_{b=1}^{n} \sum_{n < c < 2b} \frac{1}{b} \sum_{d \mid (b,c)} \mu(d) \]

\[ = \frac{1}{2} + 2 \sum_{d=1}^{n} \frac{\mu(d)}{d^2} \sum_{1 \leq k \leq n/d} \frac{1}{k} \sum_{m \leq n < 2k} \frac{1}{m} = \frac{1}{2} + 2 \sum_{d=1}^{n} \frac{\mu(d)}{d^2} S\left(\frac{n}{d}\right) \]

where

\[ S(x) = \sum_{x/2 < k \leq x} \frac{1}{k} \sum_{x < m < 2k} \frac{1}{m}. \]

**Lemma 5.** For any positive number \( x \) we have

\[ S(x) = \frac{(\log 2)^2}{2} - \frac{1}{4x} + O\left(\frac{\log x}{x^2}\right). \]

**Proof.** The argument is similar to those used in [1]. We use Corollary 1, which is a simple application of the Euler–Maclaurin summation:

\[ S(x) = \sum_{x/2 < k \leq x} \frac{1}{k} - \sum_{x/2 < k \leq x} \frac{1}{2k^2} \]

\[ = \sum_{x/2 < k \leq x} \frac{1}{k} \left( \log 2k - \log x + \phi(x) - \phi(2k) + O(x^{-2}) \right) - \sum_{x/2 < k \leq x} \frac{1}{2k^2} \]

\[ = \sum_{x/2 < k \leq x} \frac{\log k}{k} - \sum_{x/2 < k \leq x} \frac{1}{4k^2} - \left( \log(x/2) - \phi(x) + O(x^{-2}) \right) \sum_{x/2 < k \leq x} \frac{1}{k} \]

\[ = \frac{(\log 2)^2}{2} - \frac{1}{4x} + O\left(\frac{\log^2 x}{x^2}\right). \]

If we replace \( S(n/d) \) by the asymptotic formula above and do a straightforward summation over \( d \) we obtain the result of I. Aliev, S. Kanemitsu and A. Schinzel in [1].

However, we observe that if \( d \) is large, then \( n/d \) is small and therefore the error term in the asymptotic formula above is bad. Hence we need a different argument when \( d \) is large.

Let \( R \) be an integer such that \( R \approx n^{1/3} \) (this choice will be explained later). We then have

\[ P(n) = \frac{1}{2} + 2 \sum_{1 \leq d \leq n/R} \frac{\mu(d)}{d^2} S\left(\frac{n}{d}\right) + 2 \sum_{1 \leq r < R} \sum_{n/(r+1) < d \leq n/r} \frac{\mu(d)}{d^2} S\left(\frac{n}{d}\right). \]

We observe that for any real number \( x > 0 \) we have \( S(x) = S(\lfloor x \rfloor) \). Indeed,
if \( k \) and \( m \) are integers we have

\[
x/2 < k \leq x \iff \lfloor x/2 \rfloor < k \leq \lfloor x \rfloor, \quad x < m \leq 2k \iff \lfloor x \rfloor < m \leq 2k.
\]

Now for \( n/(r+1) < d \leq n/r \) we have \( \lfloor n/d \rfloor = r \). Hence

\[
(3) \quad P(n) = \frac{1}{2} + 2 \sum_{1 \leq d \leq n/R} \frac{\mu(d)}{d^2} S\left(\frac{n}{d}\right) + 2 \sum_{1 \leq r < R} S(r) \sum_{n/(r+1) < d \leq n/r} \frac{\mu(d)}{d^2}.
\]

We will use Lemma 5 to replace \( S(n/d) \) and \( S(r) \) by the corresponding asymptotic formula. We recall that \( R \approx n^{1/3} \), which implies that \( \log(n/R) \approx \log n \). We deduce from Lemma 3 that there exists \( c > 0 \) such that for \( 1 \leq r \leq R \),

\[
\sum_{d > n/r} \frac{\mu(d)}{d^2} \ll \frac{r}{n} \exp\left(-c \frac{(\log n)^{3/5}}{(\log \log n)^{1/5}}\right).
\]

The term \((\log 2)^2/2\) from Lemma 5 for \( S(n/d) \) and \( S(r) \) contributes to \( P(n) \) (in (3)) the amount

\[
(\log 2)^2 \left( \sum_{1 \leq d \leq n/R} \frac{\mu(d)}{d^2} + \sum_{1 \leq r < R} \sum_{n/(r+1) < d \leq n/r} \frac{\mu(d)}{d^2} \right)
\]

\[
= (\log 2)^2 \sum_{1 \leq d \leq n} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} (\log 2)^2 - (\log 2)^2 \sum_{d > n} \frac{\mu(d)}{d^2},
\]

which gives the constant term \( \frac{6}{\pi^2} (\log 2)^2 \) and an admissible error term by Lemma 3.

The term \(-1/(4(n/d))\) from Lemma 5 for \( S(n/d) \) contributes to \( P(n) \) (in (3)) the amount

\[
-\frac{1}{2n} \sum_{1 \leq d \leq n/R} \frac{\mu(d)}{d},
\]

which by Lemma 3 is an error term of order

\[
O\left(\frac{1}{n} \exp\left(-c \frac{(\log n)^{3/5}}{(\log \log n)^{1/5}}\right)\right)
\]

and under the Riemann hypothesis

\[
O\left(\frac{1}{n} \left(\frac{n}{R}\right)^{-1/2+\epsilon} R^{1/2} n^{-3/2+\epsilon}\right).
\]

The term \(-1/(4r)\) from Lemma 5 for \( S(r) \) contributes to \( P(n) \) (in (3)) the amount
\[-\frac{1}{2} \sum_{1 \leq r < R} \frac{1}{r} \sum_{n/(r+1) < d \leq n/r} \frac{\mu(d)}{d^2} \]
\[= -\frac{1}{2} \sum_{1 \leq r < R} \frac{1}{r} \left( \sum_{d > n/(r+1)} \frac{\mu(d)}{d^2} - \sum_{d > n/r} \frac{\mu(d)}{d^2} \right) \]
\[= -\frac{1}{2} \left( \frac{1}{R} \sum_{d > n/R} \frac{\mu(d)}{d^2} - \sum_{d > n} \frac{\mu(d)}{d^2} + \sum_{2 \leq r \leq R} \frac{1}{r(r-1)} \sum_{d > n/r} \frac{\mu(d)}{d^2} \right), \]

which is of order
\[O\left( \frac{1}{R} n + \frac{1}{n} + \frac{1}{n} \sum_{2 \leq r \leq R} \frac{1}{r-1} \exp\left( -c \frac{(\log n)^{3/5}}{\log \log n} \right) \right), \]

which in turn is
\[O\left( \frac{1}{n} \exp\left( -c' \frac{(\log n)^{3/5}}{\log \log n} \right) \right) \quad \text{for} \quad 0 < c' < c. \]

Under the Riemann hypothesis this error term becomes
\[O\left( \frac{n}{R} \right)^{-3/2+\varepsilon} + n^{-3/2+\varepsilon} + \sum_{1 \leq r \leq R} \frac{1}{r^2} \left( \frac{n}{r} \right)^{-3/2+\varepsilon} = O(R^{1/2} n^{-3/2+\varepsilon}). \]

The error term \(O((\log^+(n/d))/(n/d)^2)\) from Lemma 5 for \(S(n/d)\) contributes to \(P(n)\) (in (3)) the amount
\[O\left( \sum_{1 \leq d \leq n/R} \frac{\log^+(n/d)}{n^2} \right) = O\left( \frac{\log n}{nR} \right). \]

The error term \(O((\log^+(r))/r^2)\) from Lemma 5 for \(S(r)\) contributes to \(P(n)\) (in (3)) the amount
\[O\left( \sum_{1 \leq r < R} \frac{\log^+(r)}{r^2} \left| \sum_{n/(r+1) < d \leq n/r} \frac{\mu(d)}{d^2} \right| \right), \]

which is
\[O\left( \log n \sum_{1 \leq r < R} \frac{1}{r^2} \frac{r+1}{n} \exp\left( -c \frac{(\log n)^{3/5}}{\log \log n} \right) \right), \]

which in turn is
\[O\left( \frac{1}{n} \exp\left( -c' \frac{(\log n)^{3/5}}{\log \log n} \right) \right) \quad \text{for} \quad 0 < c' < c. \]

Under the Riemann hypothesis this error term becomes
\[O\left( \log n \sum_{1 \leq r < R} \frac{1}{r^2} \left( \frac{n}{r+1} \right)^{-3/2+\varepsilon} \right) = O((\log n) R^{1/2-\varepsilon} n^{-3/2+\varepsilon}). \]
We now see that the choice $R \approx n^{1/3}$ permits us to optimize the sum $R^{1/2}n^{-3/2} + 1/(nR)$ and completes the proof of the theorem.

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