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ON ARITHMETIC PROPERTIES OF THE CONVERGENTS OF EULER'S NUMBER

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1. Introduction and statement of results. The circle number π was already investigated by the ancients (particularly by Greeks like Archimedes), and everyone knows that the old problem of squaring the circle was answered in the negative by C. L. F. von Lindemann in 1882 [10], [15]. He proved that π is transcendental; 9 years before, Ch. Hermite [9] was the first to demonstrate the transcendence of e (see also [15]). Up till now we know much about e and π, but nevertheless there are still a lot of unsolved problems concerning these numbers. For instance, it follows from the famous theorem of A. O. Gelfond and T. Schneider [14] (Satz 14) that $e^\pi = (-1)^{-i}$ $(i^2 = -1)$ is transcendental, but nobody knows whether the numbers $\pi + e$ and πe are transcendental. Since both the roots of the polynomial $x^2 - (\pi + e)x + \pi e = (x - \pi)(x - e)$ are transcendental, it is clear that $\pi + e$ and πe cannot both be algebraic.

In the theory of diophantine approximations much work has been done to investigate rational approximations to π. Lower bounds of the form

$$\left|\pi - \frac{p}{q}\right| > \frac{1}{q^\alpha} \quad (p, q \in \mathbb{Z},\ q > 0 \text{ sufficiently large})$$

are known, where $\alpha = 42$ (K. Mahler) [11], [14] (p. 109), $\alpha = 13.398$ (M. Hata) [7], [8]. The crux is that the continued fraction expansion of π,

$$\pi = \langle 3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2, \ldots\rangle,$$

seems not to satisfy any algebraic rule. But it is a well-known fact from the theory of continued fractions (see e.g. [13]) that

$$e = \langle 2; 1, 2, 1, 1, 4, 1, 1, 6, 1, \ldots\rangle = \langle 2; \overline{1, 2k, 1}\rangle_{k\geq 1} =: \langle c_0; c_1, c_2, \ldots\rangle. \tag{1.1}$$

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The continued fraction expansions of e and of some related numbers have fascinated many mathematicians before. For instance, the expansion of $(m/n)e^{1/q}$ was investigated by K. R. Matthews and R. F. C. Walters [12], and H. Alzer [1] found sharp upper and lower bounds for $|e - p/q|$. Let

$$\frac{p_n}{q_n} := \langle c_0; c_1, c_2, \ldots, c_n \rangle \qquad (n = 0, 1, \ldots)$$

denote the convergents of e with positive coprime integers p_n and q_n. Then

$$\left| e - \frac{p_{3k+1}}{q_{3k+1}} \right| < \frac{1}{c_{3k+2} q_{3k+1}^2} = \frac{1}{2(k+1) q_{3k+1}^2} \qquad (k = 0, 1, \ldots), \tag{1.2}$$

$$\left| e - \frac{p_n}{q_n} \right| > \frac{1}{(2 + c_{n+1}) q_n^2} = \frac{1}{3 q_n^2} \qquad (n \geq 0,\ n \equiv 0, 2 \bmod 3). \tag{1.3}$$

Put

$$\begin{aligned} & P_k := p_{3k+1}, \quad Q_k := q_{3k+1} \quad (k = 0, 1, 2, \ldots); \\ & P_0 := 3,\ P_{-1} := 1,\ P_{-2} := 1,\ Q_0 := 1,\ Q_{-1} := 1,\ Q_{-2} := -1; \\ & P_{-k} := P_{k-3}, \quad Q_{-k} := -Q_{k-3} \quad (k = 3, 4, 5, \ldots). \end{aligned} \tag{1.4}$$

In this paper we investigate the sequences (P_n) and (Q_n) $(n \in \mathbb{Z})$ defined by (1.4). In view of (1.2) we may regard P_n/Q_n for $n \geq 0$ as the "best rational approximations" to Euler's number e. We have

$$\begin{aligned} (P_n)_{n \in \mathbb{Z}} &= (\ldots, 193, 19, 3, 1, 1, P_0 = 3, 19, 193, 2721, \ldots), \\ (Q_n)_{n \in \mathbb{Z}} &= (\ldots, -71, -7, -1, -1, 1, Q_0 = 1, 7, 71, 1001, \ldots). \end{aligned}$$

In the theorems and corollaries below we collect together some surprising arithmetical properties of these sequences of integers.

Theorem 1.1. *For every integer n we have*

$$P_{n+2} = 2(2n+5) P_{n+1} + P_n, \qquad Q_{n+2} = 2(2n+5) Q_{n+1} + Q_n.$$

Corollary 1.1. *For every integer n we have*

$$P_{n-1} Q_n - P_n Q_{n-1} = (-1)^{n+1} 2, \qquad P_{n-2} Q_n - P_n Q_{n-2} = (-1)^n 4(2n+1).$$

Corollary 1.2.

$$e = \cfrac{3}{1 + \cfrac{2}{19 + \cfrac{3}{10 + \cfrac{1}{14 + \cfrac{1}{18 + \ldots}}}}}$$

The partial quotients of this continued fraction are exactly the rationals P_n/Q_n ($n = 0, 1, \ldots$). This corollary is just a special instance of a result of O. Perron according to which the set of numbers such that the partial quotients are finite unions of arithmetic progressions is stable under

$$x \mapsto \frac{\alpha x + \beta}{\gamma x + \delta},$$

where $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$.

Moreover, it follows from Corollary 1.2 that

$$\frac{3-e}{7e-19} = \langle 10; 14, 18, 22, \ldots \rangle = \langle \overline{4k+2} \rangle_{k=2,3,\ldots},$$

or

$$\frac{e+1}{e-1} = \langle \overline{4k+2} \rangle_{k=0,1,\ldots}.$$

This expansion of $(e+1)/(e-1)$ was already known to L. Euler (see [5]).

Theorem 1.2. *For every integer $r \geq 0$ we have*

$$\sum_{t=0}^{r} \frac{P_t Q_{r-t}}{t!(r-t)!} = (r+1)(2r+3)4^r.$$

Corollary 1.3. *For every prime $p > 2$ we have*

$$\sum_{t=0}^{p} \frac{P_t Q_{p-t}}{t!(p-t)!} \equiv 12 \bmod p,$$

in particular the left-hand side of the congruence represents an integer.

We denote by $\|\alpha\|$ the distance of a real number α to the nearest integer. From (1.2) it can easily be seen that

$$\liminf_{q \geq 1} q\|qe\| = 0,$$

where q runs through all positive integers. But much more is true:

Theorem 1.3. *Let a and s be arbitrary positive integers. Then*

$$\liminf_{\substack{q \geq 1 \\ q \equiv a \bmod s}} q\|qe\| = 0.$$

A famous result of S. Uchiyama [16] states that for every real irrational α, any integers a, b and $s > 0$ there are infinitely many pairs u and $v \neq 0$ of integers such that

$$\left| \alpha - \frac{u}{v} \right| < \frac{s^2}{4v^2} \tag{1.5}$$

and

$$u \equiv a \bmod s, \qquad v \equiv b \bmod s,$$

provided that a and b are not both divisible by s. In 1949, S. Hartman [6] was the first to introduce such congruence conditions. He proved a corresponding result with $2s^2/v^2$ instead of $s^2/(4v^2)$ on the right-hand side of (1.5). Recently, the author [4] has shown that the factor $1/4$ on the right-hand side of (1.5) is best possible in general.

Let $k \geq 1$ and $(b_1, \ldots, b_k)$ be a k-tuple of integers (1). For some integer $s \geq 1$ we write

$$(b_1, \ldots, b_k) \bmod s$$

to denote the k-tuple $(\bar{b}_1, \ldots, \bar{b}_k)$ with $\bar{b}_i \equiv b_i \bmod s$ and $0 \leq \bar{b}_i < s$ for $i = 1, \ldots, k$.

THEOREM 1.4. *For every integer $s > 2$ the sequence $((P_n, Q_n) \bmod s)$ $(n \in \mathbb{Z})$ is periodic, and*

$$|\{(P_n, Q_n) \bmod s : n \in \mathbb{Z}\}| \leq \begin{cases} 1 + s & \text{if } s \not\equiv 0 \bmod 8, \\ s/2 & \text{if } s \equiv 0 \bmod 8, \end{cases}$$

and for every integer $s \geq 2$ one has $P_n \equiv Q_n \equiv 1 \bmod s$ for infinitely many n.

Some general results concerning approximation of real irrationals by rationals u/v with $u \equiv v \bmod s$ can be found in [2] and [3]. Theorem 1.4 may be regarded as the main theorem of this paper. Its proof yields the following

THEOREM 1.5. *For every integer $n > 0$ we have*

$$Q_n \equiv \begin{cases} (-1)^n & \bmod n, \\ (-1)^{n+1} & \bmod n+1, \\ (-1)^{n+1} & \bmod n+2, \\ (-1)^n & \bmod n+3, \end{cases} \qquad P_n \equiv \begin{cases} 3 & \bmod n, \\ 1 & \bmod n+1, \\ 1 & \bmod n+2, \\ 3 & \bmod n+3. \end{cases}$$

2. Proof of Theorems 1.1, 1.2 and of their corollaries

Proof of Theorem 1.1. It suffices to deduce the recurrence formula for the sequence (Q_n) $(n \in \mathbb{Z})$, since the arguments for (P_n) $(n \in \mathbb{Z})$ are the same. First one gets, using (1.1),

$$\begin{aligned} q_{3n-1} &= 2nq_{3n-2} + q_{3n-3} && (n \geq 1), \\ q_{3n} &= q_{3n-1} + q_{3n-2} && (n \geq 1), \\ q_{3n+1} &= q_{3n} + q_{3n-1} && (n \geq 0). \end{aligned} \tag{2.1}$$

(1) In this paper the usual notation for the greatest common divisor of some integers does not appear.

By induction, it can easily be seen that

$$\begin{aligned} q_{3n-1} &\le (2n)! \quad (n \ge 3),\\ q_{3n} &\le (2n)! \quad (n \ge 3),\\ q_{3n+1} &\le (2n)! \quad (n \ge 4). \end{aligned}$$

These inequalities imply the convergence of the generating functions

$$F(x) := \sum_{n=1}^{\infty} \frac{q_{3n-1}}{(2n)!} x^{2n}, \qquad G(x) := \sum_{n=0}^{\infty} \frac{q_{3n+1}}{(2n+1)!} x^{2n+1},$$
$$H(x) := \sum_{n=0}^{\infty} \frac{q_{3n}}{(2n+2)!} x^{2n+2}$$

at least for $|x| < 1$. Applying (2.1) again, we get a system of differential equations for the functions F, G and H:

$$F = xG + H, \qquad G' = F + H'', \qquad H''' = F' + G.$$

The first equation yields $H'' = F'' - 2G' - xG''$ and $H''' = F''' - 3G'' - xG'''$. Putting the expressions for H'' and H''' into the second and third equations, one gets

$$F'' + F = xG'' + 3G' \tag{2.2}$$

and

$$F''' - F' = xG''' + 3G'' + G. \tag{2.3}$$

From (2.3) we subtract the derivative of (2.2); this gives $F' = G''/2 - G/2$, $F''' = G^{(4)}/2 - G''/2$, and finally, by (2.3),

$$G^{(4)} - 2xG''' - 8G'' - G = 0. \tag{2.4}$$

Putting in the series of G and standard arguments yield

$$\frac{q_{3n+7}}{(2n+1)!} - \frac{2q_{3n+4}}{(2n)!} - \frac{8q_{3n+4}}{(2n+1)!} - \frac{q_{3n+1}}{(2n+1)!} = 0 \qquad (n \ge 0),$$

respectively

$$q_{3(n+2)+1} = (4n+10)q_{3(n+1)+1} + q_{3n+1} \qquad (n \ge 0).$$

Using the notation introduced in (1.4), we have

$$Q_{n+2} = 2(2n+5)Q_{n+1} + Q_n \qquad (n \ge 0).$$

The recurrence formula also holds for $n < 0$. This follows from the definition of the sequence (Q_n) $(n \in \mathbb{Z})$ in (1.4), and the assertion is proved.

The identities of Corollary 1.1 follow from Theorem 1.1 by induction.

Proof of Corollary 1.2. Let

$$\begin{aligned} a_0 &:= 0, & a_n &:= P_{n-1} & (n \geq 1),\\ b_0 &:= 2/3, & b_n &:= Q_{n-1} & (n \geq 1),\\ B_1 &:= 19/3, & B_n &:= 4n+2 & (n > 1). \end{aligned}$$

Then Theorem 1.1 implies for $n \geq 1$ the identities

$$a_{n+1} - B_n a_n - a_{n-1} = 0, \qquad b_{n+1} - B_n b_n - b_{n-1} = 0.$$

Hence,

$$\frac{a_{n+1}}{b_{n+1}} = \frac{B_n a_n + a_{n-1}}{B_n b_n + b_{n-1}} = \cfrac{a_1}{b_1 + \cfrac{2}{3}\cfrac{1}{B_1 + \cfrac{1}{B_2 + \cfrac{1}{\dots + \cfrac{1}{B_n}}}}} \qquad (n \geq 1).$$

The assertion of the corollary follows from

$$\frac{a_{n+1}}{b_{n+1}} = \frac{P_n}{Q_n} \to e \qquad (n \to \infty).$$

From the proof of Theorem 1.1 we know that $Q_n = q_{3n+1} \leq (2n)!$ for $n \geq 4$. Using the recurrence formula itself from Theorem 1.1, one gets the sharper bound

$$Q_n \leq 4^n (n+1)! \qquad (n \geq 0),$$

which is easily proved by induction. From (1.2) it follows at once that

$$P_n \leq 3 \cdot 4^n (n+1)! \qquad (n \geq 0).$$

Hence, the generating functions

$$J(x) := \sum_{n=0}^{\infty} \frac{Q_n}{n!} x^n \quad \text{and} \quad K(x) := \sum_{n=0}^{\infty} \frac{P_n}{n!} x^n \tag{2.5}$$

both exist at least for $|x| < 1/4$.

LEMMA 2.1. *The functions $J(x)$ and $K(x)$ are both solutions of the differential equation*

$$(4x-1)y'' + 10y' + y = 0 \qquad (|x| < 1/4).$$

Proof. Note that

$$J'' = Q_2 + \sum_{n=1}^{\infty} \frac{Q_{n+2}}{n!} x^n, \qquad J' = Q_1 + \sum_{n=1}^{\infty} \frac{Q_{n+1}}{n!} x^n,$$

$$xJ'' = \sum_{n=1}^{\infty} \frac{nQ_{n+1}}{n!} x^n$$

and $Q_{n+2} = 4nQ_{n+1} + 10Q_{n+1} + Q_n$ $(n \geq 1)$. One gets

$$J'' - Q_2 = 4xJ'' + 10(J' - Q_1) + (J - Q_0),$$

from which the assertion for $J(x)$ follows since $Q_2 - 10Q_1 - Q_0 = 0$.

LEMMA 2.2. *For* $|x| < 1/4$ *we have*

$$J(x) = \left(\frac{2}{\sqrt{(1-4x)}^3} - \frac{1}{1-4x}\right)e^{-1/2+\sqrt{1-4x}/2},$$
$$K(x) = \left(\frac{2}{\sqrt{(1-4x)}^3} + \frac{1}{1-4x}\right)e^{1/2-\sqrt{1-4x}/2}.$$

Proof. By straightforward computations it can easily be seen that both of the functions on the right-hand side satisfy the differential equation from the preceding lemma. Then the identities follow from

$$J(0) = Q_0 = 1, \qquad J'(0) = Q_1 = 7,$$
$$K(0) = P_0 = 3, \qquad K'(0) = P_1 = 19.$$

Proof of Theorem 1.2. Cauchy's product formula and Lemma 2.2 yield, for any real number $|x| < 1/4$,

$$\sum_{r=0}^{\infty}\left(\sum_{t=0}^{r}\frac{P_tQ_{r-t}}{t!(r-t)!}\right)x^r = J(x)K(x)$$
$$= \frac{4}{(1-4x)^3} - \frac{1}{(1-4x)^2} = \sum_{r=0}^{\infty}\left\{4\binom{2+r}{r} - \binom{1+r}{r}\right\}(4x)^r,$$

which gives the result.

3. Proof of the remaining results. First, Theorems 1.4 and 1.5 are proved. The assertion of Theorem 1.5 holds for $n = 1, 2, 3, 4$.

CASE 1: Let $n \geq 5$ be some odd integer. Put

$$N := \frac{n-3}{2}, \qquad a_m := 4(N + m - 2) + 10 \equiv 4m - 4 \bmod n \quad (m \in \mathbb{Z}). \tag{3.1}$$

The arguments of Case 1 are based on the important congruence

$$Q_{N+k} \equiv Q_{N-k} \bmod n \quad (k \geq 0). \tag{3.2}$$

There is nothing to prove for $k = 0$. Let $k \geq 1$ and assume (3.2) to be proved for $1, \ldots, k-1$. Applying k times the recurrence formula for Q_{n+2} from Theorem 1.1, one gets

$$Q_{N+k} = \sum_{\nu=0}^{k-1} a_{k-2\nu}Q_{N+k-(2\nu+1)} + Q_{N-k} =: S_k + Q_{N-k}. \tag{3.3}$$

Let k be even. Then

$$\begin{aligned} S_k &= \Big(\sum_{\nu=0}^{k/2-1} + \sum_{\nu=k/2}^{k-1}\Big) a_{k-2\nu} Q_{N+k-(2\nu+1)} \\ &= \sum_{\nu=0}^{k/2-1} (a_{k-2\nu} Q_{N+k-2\nu-1} + a_{-k+2\nu+2} Q_{N-k+2\nu+1}). \end{aligned}$$

In this formula, the indices satisfy $1 \le k-2\nu-1 \le k-1$; hence the induction hypothesis may be applied. It follows that

$$\begin{aligned} S_k &\equiv \sum_{\nu=0}^{k/2-1} (a_{k-2\nu} + a_{-k+2\nu+2}) Q_{N+k-2\nu-1} \\ &\overset{(3.1)}{\equiv} \sum_{\nu=0}^{k/2-1} (4(k-2\nu) - 4 + 4(-k+2\nu+2) - 4) Q_{N+k-2\nu-1} \\ &= 0 \bmod n, \end{aligned}$$

and we conclude $Q_{N+k} \equiv Q_{N-k} \bmod n$ from (3.3). It remains to consider the case where k is some odd integer. Clearly, $S_1 = a_1 Q_N \equiv 0 \bmod n$ by (3.1). Let $k \ge 3$. Then

$$\begin{aligned} S_k &= \Big(\sum_{\nu=0}^{(k-3)/2} + \sum_{\nu=(k+1)/2}^{k-1}\Big) a_{k-2\nu} Q_{N+k-2\nu-1} + a_1 Q_N \\ &\equiv \sum_{\nu=0}^{(k-3)/2} (a_{k-2\nu} + a_{-k+2\nu+2}) Q_{N+k-2\nu-1} \bmod n. \end{aligned}$$

Since $2 \le k-2\nu-1 \le k-1$, the last congruence follows from the induction hypothesis. Again, we get $Q_{N+k} \equiv Q_{N-k} \bmod n$, and (3.2) is proved.

In what follows we take special values for k in (3.2):

$$\begin{aligned} k = (n+3)/2: \quad & N+k = n,\ N-k = -3; \\ & Q_n \equiv Q_{-3} = -1 \bmod n\ (n \ge 5,\ \text{odd}). \\ k = (n+1)/2: \quad & N+k = n-1,\ N-k = -2; \\ & Q_{n-1} \equiv Q_{-2} = -1 \bmod n\ (n \ge 5,\ \text{odd}), \\ & \text{or } Q_n \equiv -1 \bmod n+1\ (n \ge 4,\ \text{even}). \\ k = (n-1)/2: \quad & N+k = n-2,\ N-k = -1; \\ & Q_{n-2} \equiv Q_{-1} = 1 \bmod n\ (n \ge 5,\ \text{odd}), \\ & \text{or } Q_n \equiv 1 \bmod n+2\ (n \ge 3,\ \text{odd}). \\ k = (n-3)/2: \quad & N+k = n-3,\ N-k = 0; \\ & Q_{n-3} \equiv Q_0 = 1 \bmod n\ (n \ge 5,\ \text{odd}), \\ & \text{or } Q_n \equiv 1 \bmod n+3\ (n \ge 2,\ \text{even}). \end{aligned}$$

From Theorem 1.1 one also gets

$$P_{N+k} \equiv P_{N-k} \bmod n \quad (k \geq 0), \tag{3.4}$$

which implies

$$\begin{aligned} P_n &\equiv 3 \mod n && (n \geq 5,\ \text{odd}),\\ P_n &\equiv 1 \mod n+1 && (n \geq 4,\ \text{even}),\\ P_n &\equiv 1 \mod n+2 && (n \geq 3,\ \text{odd}),\\ P_n &\equiv 3 \mod n+3 && (n \geq 2,\ \text{even}). \end{aligned}$$

Hence, half of the assertions of Theorem 1.5 are proved.

Now consider the sequence

$$W_m := (Q_m, Q_{m-1}, P_m, P_{m-1}) \bmod s \quad (m \in \mathbb{Z}), \tag{3.5}$$

where $s > 2$ denotes some odd integer. We already know that

$$W_s \equiv (-1, -1, 3, 1) \equiv (-Q_0, -Q_{-1}, P_0, P_{-1}) \bmod s \tag{3.6}$$

holds for odd integers $s \geq 3$. Hence, from the recurrence formulas in Theorem 1.1 it can easily be seen that

$$W_{2s} \equiv (Q_0, Q_{-1}, P_0, P_{-1}) \bmod s, \tag{3.7}$$

which implies

$$W_{m+2s} \equiv W_m \bmod s \tag{3.8}$$

for all integers m and every odd integer $s \geq 3$. We consider the following $2s$ successive elements of the sequence (W_m) $(m \in \mathbb{Z})$:

$$W_{n_i}, \quad \text{where} \quad \underbrace{n_1 := -\frac{s+1}{2}, n_2 := -\frac{s-1}{2}, \dots, n_s := \frac{s-3}{2}}_{\text{section A: } s \text{ elements}};$$

$$\underbrace{n_{s+1} := \frac{s-1}{2}, \dots, n_{2s} := \frac{3s-3}{2}}_{\text{section B: } s \text{ elements}}.$$

From (3.2) and (3.4) we conclude that the congruence

$$W_{n_s} = (Q_{n_s}, Q_{n_s-1}, P_{n_s}, P_{n_s-1}) \equiv (Q_{n_s}, Q_{n_s+1}, P_{n_s}, P_{n_s+1})$$

holds for W_{n_s} from section A, whereas we have

$$W_{n_s+1} = (Q_{n_s+1}, Q_{n_s}, P_{n_s+1}, P_{n_s})$$

for W_{n_s+1} from section B. Similarly, we have

$$W_{n_s-1} = (Q_{n_s-1}, Q_{n_s-2}, P_{n_s-1}, P_{n_s-2}) \equiv (Q_{n_s+1}, Q_{n_s+2}, P_{n_s+1}, P_{n_s+2})$$
$$\text{and} \quad W_{n_s+2} = (Q_{n_s+2}, Q_{n_s+1}, P_{n_s+2}, P_{n_s+1});$$
$$\dots$$
$$W_{n_1} = (Q_{n_1}, Q_{n_1-1}, P_{n_1}, P_{n_1-1}) \equiv (Q_{n_{2s}-1}, Q_{n_{2s}}, P_{n_{2s}-1}, P_{n_{2s}})$$
$$\text{and} \quad W_{n_{2s}} = (Q_{n_{2s}}, Q_{n_{2s}-1}, P_{n_{2s}}, P_{n_{2s}-1}),$$

where all the congruences hold modulo s. Collecting together pairs (P_j, Q_j) mod s from every quadruplet W_{n_i}, we get two such pairs from W_{n_1} at the beginning, namely (P_{n_1}, Q_{n_1}) mod s and (P_{n_1-1}, Q_{n_1-1}) mod s. Proceeding from W_{n_i} to W_{n_i+1}, we find at most one new pair (P_{n_i+1}, Q_{n_i+1}) mod s of residue classes modulo s. As was shown, there is a one-to-one correspondence between pairs from quadruplets belonging to section A to those from quadruplets of section B. By (3.8), it follows that for every odd integer $s \geq 3$,

$$|\{(P_m, Q_m) \bmod s : m \in \mathbb{Z}\}| \leq 2 + (s-1) = s+1. \tag{3.9}$$

Example ($s = 3$).

$$\{(P_m, Q_m) \bmod 3 : m \in \mathbb{Z}\} = \{(0,1), (0,2), (1,1), (1,2) \bmod 3\}.$$

Case 2: $n \geq 4$, even.

Case 2.1: $n \equiv 2 \bmod 4$; $n = 2u$, where $u \geq 3$ (odd). From (3.5)–(3.7) in the preceding case we conclude that

$$Q_{m+2u} \equiv Q_m \bmod u \quad \text{and} \quad P_{m+u} \equiv P_m \bmod u \tag{3.10}$$

hold for every integer m. Since $n = 2u$, one gets the congruences

$$\begin{aligned} Q_n &\equiv Q_0 = 1, & P_n &\equiv P_0 = 3, \\ Q_{n-1} &\equiv Q_{-1} = 1, & P_{n-1} &\equiv P_{-1} = 1, \\ Q_{n-2} &\equiv Q_{-2} = -1, & P_{n-2} &\equiv P_{-2} = 1, \\ Q_{n-3} &\equiv Q_{-3} = -1, & P_{n-3} &\equiv P_{-3} = 3 \end{aligned} \tag{3.11}$$

for the modulus u. It can easily be seen by (1.4) that all the integers P_m and Q_m are odd; hence the congruences in (3.11) also hold modulo 2. Since 2 and u are coprime, the modulus in (3.11) may be taken to be $n = 2u$. By the substitutions $n \to n+1, n+2, n+3$ (see the corresponding argument in Case 1) we have thus proved Theorem 1.5 in the following cases: the first congruences for $n \equiv 2$ and 6 mod 8, the second ones for $n \equiv 1$ and 5 mod 8, the third ones for $n \equiv 0$ and 4 mod 8, the fourth ones for $n \equiv 3$ and 7 mod 8. Moreover, the congruences (3.10) may be considered with respect to the modulus n, and writing s for n, we conclude that

$$W_{m+s} \equiv W_m \bmod s \tag{3.12}$$

holds for all integers m and every even integer $s \geq 2$, $s \equiv 2 \bmod 4$.

Case 2.2: $n \equiv 0 \bmod 4$; $n = 2^\alpha u$, where $\alpha \geq 2$, $u \geq 1$ (odd). Again we may assume the modulus u in (3.11) (see Case 2.1). Therefore it suffices to show that

$$
\begin{aligned}
Q_n &\equiv 1, & P_n &\equiv 3,\\
Q_{n-1} &\equiv 1, & P_{n-1} &\equiv 1,\\
Q_{n-2} &\equiv -1, & P_{n-2} &\equiv 1,\\
Q_{n-3} &\equiv -1, & P_{n-3} &\equiv 3
\end{aligned} \tag{3.13}
$$

hold modulo 2^α.

CASE 2.2.1: $\alpha = 2$. Theorem 1.1 implies $Q_{m+2} \equiv 2Q_{m+1} + Q_m \bmod 4$ and $P_{m+2} \equiv 2P_{m+1} + P_m \bmod 4$. It follows that

$$Q_{m+4} \equiv Q_m \bmod 4, \qquad P_{m+4} \equiv P_m \bmod 4 \qquad (m \in \mathbb{Z}),$$

and thus (3.13) holds modulo 4 (since $n \equiv 0 \bmod 4$). Hence Theorem 1.5 is true for: $n \equiv 4 \bmod 8$ (first congruences), $n \equiv 3 \bmod 8$ (second congruences), $n \equiv 2 \bmod 8$ (third congruences), $n \equiv 1 \bmod 8$ (fourth congruences). Moreover,

$$W_{m+s} \equiv W_m \bmod s \tag{3.14}$$

is true for all integers m and every positive integer $s \equiv 4 \bmod 8$.

CASE 2.2.2: $\alpha > 2$. The basic idea in this case arises from the congruences

$$
\begin{aligned}
Q_{2^{\alpha-1}-1} &\equiv 1 \bmod 2^\alpha, & Q_{2^{\alpha-1}} &\equiv 1 \bmod 2^\alpha,\\
P_{2^{\alpha-1}-1} &\equiv 1 \bmod 2^\alpha, & P_{2^{\alpha-1}} &\equiv 3 \bmod 2^\alpha.
\end{aligned} \tag{3.15}
$$

The proof by induction begins with the observation that

$$
\begin{aligned}
Q_3 &= 1001 \equiv 1 \bmod 8, & Q_4 &= 18089 \equiv 1 \bmod 8,\\
P_3 &= 2721 \equiv 1 \bmod 8, & P_4 &= 49171 \equiv 3 \bmod 8.
\end{aligned}
$$

Let $\alpha \geq 4$ and assume the assertion to be true for $\alpha - 1$. It suffices to explain the arguments for the Q's. By the induction hypothesis, there are integers g_1 and g_2 such that

$$Q_{2^{\alpha-2}-1} = 1 + g_1 2^{\alpha-1} \quad \text{and} \quad Q_{2^{\alpha-2}} = 1 + g_2 2^{\alpha-1}. \tag{3.16}$$

In what follows the $2^{\alpha-2}$ integers $Q_{2^{\alpha-2}+1}, Q_{2^{\alpha-2}+2}, \ldots, Q_{2^{\alpha-1}}$ are considered modulo 2^α. Theorem 1.1 yields

$$
\begin{aligned}
Q_{2^{\alpha-2}+1} &= (4(2^{\alpha-2} - 1) + 10)Q_{2^{\alpha-2}} + Q_{2^{\alpha-2}-1}\\
&\equiv 6Q_{2^{\alpha-2}} + Q_{2^{\alpha-2}-1} = 7 + 6g_2 2^{\alpha-1} + g_1 2^{\alpha-1}\\
&\equiv Q_1 + g_1 2^{\alpha-1} \bmod 2^\alpha,
\end{aligned}
$$

and similarly, mod 2^α,

$$
\begin{aligned}
Q_{2^{\alpha-2}+2} &\equiv Q_2 + g_2 2^{\alpha-1}, & Q_{2^{\alpha-1}-1} &\equiv Q_{2^{\alpha-2}-1} + g_1 2^{\alpha-1},\\
Q_{2^{\alpha-2}+3} &\equiv Q_3 + g_1 2^{\alpha-1}, & Q_{2^{\alpha-1}} &\equiv Q_{2^{\alpha-2}} + g_2 2^{\alpha-1}.\\
Q_{2^{\alpha-2}+4} &\equiv Q_4 + g_2 2^{\alpha-1}, \ldots
\end{aligned}
$$

Putting the representations of $Q_{2^{\alpha-2}-1}$ resp. $Q_{2^{\alpha-2}}$ from (3.16) into the last two congruences proves the assertion for the Q's in (3.15).

Repeating the arguments, it can be seen that both the sequences $((Q_m) \bmod 2^\alpha)$ and $((P_m) \bmod 2^\alpha)$ $(m \in \mathbb{Z})$ are periodic, and $2^{\alpha-1}$ successive numbers Q resp. P represent a period.

Since $n = 2^\alpha u$ is divisible by $2^{\alpha-1}$, we have, mod 2^α,

$$Q_n \equiv Q_0 = 1, \qquad Q_{n-2} \equiv Q_{-2} = -1,$$
$$Q_{n-1} \equiv Q_{-1} = 1, \qquad Q_{n-3} \equiv Q_{-3} = -1.$$

This proves (3.13), and similar arguments for the P's prove Theorem 1.5 upon verifying all the remaining cases: the first congruences for $n \equiv 0 \bmod 8$, the second ones for $n \equiv 7 \bmod 8$, the third ones for $n \equiv 6 \bmod 8$, the fourth ones for $n \equiv 5 \bmod 8$. Additionally, (3.11) and the arguments from Case 1 are used.

To complete the proof of Theorem 1.4, we finally investigate the sequence (W_m) $(m \in \mathbb{Z})$ for the modulus $s = 2^\alpha u$, where $\alpha \geq 3$ and $u \equiv 1 \bmod 2$.

We have

$$W_{m+s/2} \equiv W_m \bmod s \qquad (m \in \mathbb{Z}). \tag{3.17}$$

Proof of (3.17). For $u = 1$, the assertion follows from $s/2 = 2^{\alpha-1}$ and the result of Case 2.2.2.

If $u > 1$, then by (3.10), $2u$ successive elements of $(W_m) \bmod u$ represent a period. Moreover, $2^{\alpha-1}$ successive elements of $(W_m) \bmod 2^\alpha$ also represent a period. From $2u \mid (s/2)$ and $2^{\alpha-1} \mid (s/2)$ the assertion of (3.17) follows.

Collecting together (3.9), (3.12), (3.14) and (3.17), and counting pairs $(P_m, Q_m) \bmod s$ as demonstrated in the proof of (3.9), we finish the proof of Theorem 1.4.

Proof of Theorem 1.3. We know from Theorem 1.4 that there are infinitely many integers $m \geq 0$ such that

$$Q_m \equiv 1 \bmod s \tag{3.18}$$

for every integer $s \geq 2$. For $s = 1$ the assertion of Theorem 1.3 follows directly from (1.2). So we may assume that $s \geq 2$. Let $a > 0$ denote some integer. For every integer $m > 0$ satisfying (3.18) put

$$A_m := aP_m, \qquad B_m := aQ_m.$$

Then $B_m \equiv a \bmod s$, and also

$$\left| e - \frac{A_m}{B_m} \right| = \left| e - \frac{P_m}{Q_m} \right| \overset{(1.2)}{<} \frac{1}{2(m+1)Q_m^2} = \frac{a^2}{2(m+1)B_m^2}.$$

Hence for $m \to \infty$ the assertion follows from

$$0 < B_m |eB_m - A_m| < \frac{a^2}{2(m+1)} \to 0.$$

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