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ALGEBRAS WHOSE EULER FORM IS NON-NEGATIVE

BY

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Introduction. Let A be a finite-dimensional algebra over an algebraically closed field k. We denote by mod_A the category of finite-dimensional left A-modules and by $\mathrm{D^b}(A)$ the derived category of mod_A. We say that two algebras, A and B, are *derived equivalent* if their derived categories, $\mathrm{D^b}(A)$ and $\mathrm{D^b}(B)$, are derived equivalent as triangulated categories. See [11] for definitions and basic concepts.

In recent years a considerable effort has been devoted to the characterizations of algebras which are derived equivalent to well understood classes of algebras (tame hereditary algebras, tubular algebras, some special biserial algebras) [1, 12, 3, 9]. An important invariant entering all these characterizations is the Euler form: if A has finite global dimension, the Grothendieck group $\mathrm{K}_\circ(A) \simeq \mathbb{Z}^n$ is equipped with a (non-symmetric) bilinear form $\langle -,- \rangle_A$ such that for two modules $X, Y \in \mathrm{mod}_A$ we have

$$\langle [X],[Y]\rangle_A = \sum_{i=0}^{\infty} \dim_k \mathrm{Ext}^i_A(X,Y),$$

where $[X]$ denotes the class of X in $\mathrm{K}_\circ(A)$. The associated quadratic form $\chi_A(v) = \langle v, v\rangle_A$ is the *Euler form* of A. For two derived equivalent algebras, A and B, the Euler forms χ_A and χ_B are equivalent. In particular, χ_A is non-negative if and only if so is χ_B. If χ_A is non-negative, $\mathrm{corank}\,\chi_A$ is the rank of the free abelian group $\mathrm{rad}\,\chi_A = \{v \in \mathrm{K}_\circ(A) \mid \chi_A(v) = 0\}$.

Algebras A whose form χ_A is non-negative are important. Examples include the algebras which are derived equivalent to tame hereditary and tubular algebras [11, 12], certain tree algebras which are derived tame [8, 16] and others. Recent results in [5] show that, if the Euler form of a connected algebra A is non-negative, then there exists an invertible linear transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\chi_A T(x_1, \ldots, x_n) = q_\Delta(x_1, \ldots, x_{n-s})$, where $s = \mathrm{corank}\,\chi_A$ and q_Δ is the quadratic form associated with a uniquely determined Dynkin graph Δ. The graph $\Delta = \mathrm{Dyn}(\chi_A)$ is called the *Dynkin type* of χ_A.

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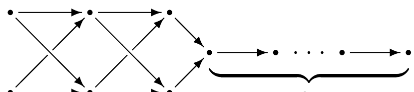
The main result of this work completes the description of the algebras A whose Euler form χ_A is non-negative with corank $\chi_A \le 2$ (at least for some classes of algebras).

THEOREM. *Let $A = k\mathrm{Q}_A/I$ be a connected finite-dimensional k-algebra such that χ_A is non-negative of corank* 2. *Assume that A is in one of the following classes*: (1) *tree algebras*; (2) *strongly simply connected poset algebras. Then A is derived equivalent to a tubular algebra or to a poset algebra* $\mathrm{P}(n)$ *of the form*



Moreover, *if A has more than* 6 *vertices*, *then A is derived equivalent to a tubular algebra* (*resp. to* $\mathrm{P}(n)$) *if and only if* $\mathrm{Dyn}(\chi_A) = \mathbb{E}_p$ $(p = 6, 7, 8)$ (*resp.* $\mathrm{Dyn}(\chi_A) = \mathbb{D}_{n-2}$).

The work is organized as follows. In Section 1, we recall some examples and properties of algebras whose Euler form is non-negative. In Section 2, we describe the Dynkin type of algebras derived equivalent to well-known classes of algebras. In particular, we show the following result.

PROPOSITION. *Let A be a strongly simply connected algebra whose Euler form is non-negative and of Dynkin type $\mathbb{A}_n$. Then A is derived equivalent to a hereditary algebra of type $\mathbb{A}_n$.*

In Section 3, we prove a useful lemma about the connectedness of the radical of a strongly simply connected algebra. In Sections 4 and 5, we give the proofs of the above theorem for tree algebras and strongly simply connected poset algebras respectively. Finally, in the last section, we treat the case where the associated Euler form is non-negative but has higher corank.

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1. Some algebras whose Euler form is non-negative

1.1. Let $A = k\mathrm{Q}_A/I$ be a finite-dimensional algebra. We shall *assume* that Q_A is connected and without oriented cycles (we say A is *connected* and *triangular*, respectively). In particular, A has finite global dimension.

A module $X \in \mathrm{mod}_A$ is also considered as a representation of Q_A. The dimension vector $\underline{\dim}\, X$ is identified with the class $[X]$ of X in the Grothendieck group $\mathrm{K}_\circ(A) \simeq \mathbb{Z}^n$.

For $x \in \mathrm{Q}_\circ$ we denote by S_x the simple module at x. By P_x (resp. I_x) we denote a projective cover (resp. injective envelope) of S_x. We also write e_x instead of $\underline{\dim}\, S_x$.

1.2. Given two derived equivalent algebras A and B with $F : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(B)$ a triangular equivalence, there is an induced isometry $f : \mathrm{K}_\circ(A) \to \mathrm{K}_\circ(B)$ satisfying $\langle x, y\rangle_A = \langle f(x), f(y)\rangle_B$.

Recall that $A[M]$ denotes the *one-point extension* of A by a module M (see [17]). The following result will be basic for our considerations.

THEOREM (see [2]). *Let A and B be two algebras and $M \in \mathrm{mod}_A$, $N \in \mathrm{mod}_B$ two modules. Suppose there is a triangular equivalence $F : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(B)$ which maps the stalk complex $M[0]$ to $N[0]$. Then there exists a triangular equivalence $\overline{F} : \mathrm{D}^{\mathrm{b}}(A[M]) \to \mathrm{D}^{\mathrm{b}}(B[N])$ extending F.*

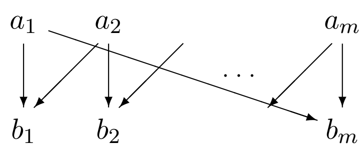
1.3. For a given subset J of the vertices of the quiver Q_A, the algebra $B = \mathrm{End}_A(\bigoplus_{x\in J} P_x)^{\mathrm{op}}$ is said to be *fully contained* in A. If J is path closed in Q_A, then B is said to be *convex* in A. If $\mathrm{Q}_\circ$ denotes the vertex set of Q_A and $J = \mathrm{Q}_\circ \setminus \{y\}$, we write $A \setminus \{y\} = B$.

LEMMA (see [3]). *Let B be fully contained in A and assume that χ_A is non-negative. Then χ_B is non-negative and* $\operatorname{corank} \chi_B \le \operatorname{corank} \chi_A$.

1.4. We recall that an algebra A is said to be *strongly simply connected* if for every algebra B convex in A, the first Hochschild cohomology $\mathrm{H}^1(B)$ vanishes [18]. Equivalently, A is strongly simply connected if and only if every algebra B convex in A is *separated*, that is, $B = k\mathrm{Q}_B/I'$ and for every vertex x in Q_B the following condition is satisfied: let $\mathrm{rad}\, P_x = \bigoplus_{i=1}^t M_i$ be a decomposition into indecomposable modules of the B-module $\mathrm{rad}\, P_x$; then for any $i \ne j$, the supports of M_i and M_j are contained in different connected components of $\mathrm{Q}_B \setminus \{y : \text{there is a path from } y \text{ to } x\}$. See also [6, 18].

EXAMPLES. (a) If $A = k\mathrm{Q}_A/I$ is a *tree algebra* (that is, the underlying graph of Q_A is a tree), then A is strongly simply connected.

(b) Let $A = k[S]$ be a *poset algebra* (that is, S is a poset and $A = k\mathrm{Q}_S/I_S$ where Q_S is the quiver of S, $k\mathrm{Q}_S$ the path algebra of Q_S and I_S the ideal in $k\mathrm{Q}_S$ generated by the differences of parallel paths in $k\mathrm{Q}_S$; see [10]). Then A is strongly simply connected if and only if A has no crowns (see [7]). We recall that a *crown* in A is an algebra C, fully contained in A, of the form



and such that the convex closure $\overline{\{a_i, b_i\}}$ of $\{a_i, b_i\}$ intersects $\overline{\{a_{i+1}, b_i\}}$ (resp. $\overline{\{a_i, b_{i-1}\}}$) in b_i (resp. in a_i) for $i = 1, \ldots, m$ and $a_{m+1} = a_1$, $b_0 = b_m$.

(c) Suppose we have the following setting: $A = k\mathrm{Q}_A/I$ is an algebra, χ_A is non-negative, x is a source in Q_A and there exists a vector $v \in \operatorname{rad}\chi_A$ with $v(x) \neq 0$. Then $\operatorname{corank} A_\circ = \operatorname{corank} A - 1$ where $A_\circ = A \setminus \{x\}$.

The following results are central in our considerations.

THEOREM. *Let A be a strongly simply connected algebra.*

(i) [1, 4] *A is derived equivalent to a tame hereditary algebra $k\Delta$ if and only if χ_A is non-negative with* $\operatorname{corank}\chi_A = 1$. *In this case*, *$\Delta$ is of type $\widetilde{\mathbb{D}}_n$ $(n \geq 4)$ or $\widetilde{\mathbb{E}}_p$ $(p = 6, 7, 8)$.*

(ii) [3] *If Q_A has more than* 6 *vertices*, *then A is derived equivalent to a tubular algebra $k\Delta$ if and only if χ_A is non-negative with* $\operatorname{corank}\chi_A = 2$ *and $\chi_A^{-1}(1) \cap \chi_A^{-1}(0)^{\perp} = \emptyset$ (where $V^{\perp} = \{w \in \mathrm{K}_\circ(A) : \langle v, w\rangle_A = 0$ for all $v \in V\}$).*

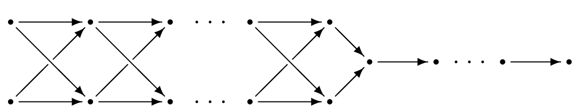
1.5. Following [16], we say that A is *derived tame* if A has finite global dimension and the repetitive category $\widehat{A}$ is tame. Examples of derived tame algebras are the following:

(a) By [11], hereditary tame algebras are derived tame. By [12], tubular algebras are derived tame.

(b) If A is derived tame and $\mathrm{D}^{\mathrm{b}}(A) \simeq \mathrm{D}^{\mathrm{b}}(B)$ is a triangular equivalence, then B is also derived tame (see [16]).

(c) Let C be a hereditary tame algebra of type $\widetilde{\mathbb{D}}_n$ and let M be an indecomposable regular C-module of regular length 2 lying in a tube of rank $n-2$ in the Auslander–Reiten quiver Γ_C. Then the one-point extension $C[M]$ is called a 2-*tubular* algebra (see [14]). In [16], it is shown that B is derived tame and derived equivalent to the poset algebra $\mathrm{P}(n+2)$ as defined in the introduction.

(d) Other examples of derived tame algebras are provided by the poset algebras associated with posets of the form



REMARKS. (1) All algebras in the above examples have a non-negative Euler form.

(2) Information on the structure of the module category of a derived tame algebra was recently obtained in [9].

2. The Dynkin type of non-negative Euler forms

2.1. Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be an integral quadratic form of the shape $q(v) = \sum_{i=1}^n q_i v(i)^2 + \sum_{i<j} q_{ij} v(i)v(j)$. We say that q is a *unit* (resp. *semi-unit*) form if $q_i = 1$ (resp. $q_i \in \{0,1\}$) for all i.

Associated with a semi-unit form we define a *bigraph* G_q with vertices $1,\dots,n$; two vertices $i \neq j$ are joined by $|q_{ij}|$ full edges if $q_{ij} < 0$ and by q_{ij} dotted edges if $q_{ij} \geq 0$; for every vertex i, there are $1 - q_i$ full loops at i. We say that q is *connected* if G_q is connected. The following are elementary facts.

(a) If $A = k\mathrm{Q}/I$ is a connected and triangular algebra, then χ_A is a connected unit form.

(b) Given a connected graph Δ formed by full edges and at most one loop at each vertex, there is a semi-unit form q_Δ such that $G_{q_\Delta} = \Delta$. Then q_Δ is positive (resp. non-negative) if and only if Δ is a Dynkin diagram (resp. an extended Dynkin diagram).

For Dynkin diagrams we consider the following partial order:

$$\mathbb{A}_m \leq \mathbb{A}_n \leq \mathbb{D}_n \leq \mathbb{D}_p \quad \text{for } m \leq n \leq p,$$
$$\mathbb{D}_p \leq \mathbb{E}_p \leq \mathbb{E}_q \quad \text{for } 6 \leq p \leq q \leq 8.$$

The following result is relevant to our discussion.

THEOREM (see [5]). *Let $q : \mathbb{Z}^n \to \mathbb{Z}$ be a connected, non-negative semi-unit form. Then there exists a $\mathbb{Z}$-invertible linear transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $qT(x_1,\dots,x_n) = q_\Delta(x_1,\dots,x_{n-c})$, where $c = \operatorname{corank} q$ and $\Delta = \operatorname{Dyn}(q)$ is a Dynkin diagram uniquely determined by q. Moreover, if q' is a connected restriction of q, then $\operatorname{Dyn}(q') \leq \operatorname{Dyn}(q)$.*

2.2. PROPOSITION. *Let A be a strongly simply connected algebra with a non-negative Euler form χ_A of type $\mathbb{A}_n$. Then A is derived equivalent to a hereditary algebra of type $\mathbb{A}_n$ and $\operatorname{corank} \chi_A = 0$.*

Proof. We show first that $\operatorname{corank} \chi_A = 0$, that is, χ_A is positive. Suppose that $\operatorname{corank} \chi_A > 0$. Then there exists an algebra B convex in A such that $\operatorname{corank} \chi_B = 1$. By 2.1, $\operatorname{Dyn}(\chi_B) \leq \operatorname{Dyn}(\chi_A) = \mathbb{A}_n$, thus $\operatorname{Dyn}(\chi_B) = \mathbb{A}_m$ for some $m \leq n$. By [1, 3], the algebra B is derived equivalent to a hereditary algebra of type $\widetilde{\mathbb{D}}_{m-1}$ or $\widetilde{\mathbb{E}}_{m-1}$ ($m = 7, 8, 9$), which implies $\operatorname{Dyn}(\chi_B) = \mathbb{D}_{m-1}$ or $\operatorname{Dyn}(\chi_B) = \mathbb{E}_{m-1}$, respectively—in any case a contradiction. Hence $\operatorname{corank} \chi_A = 0$.

By [1], A is derived equivalent to a hereditary algebra $k\Delta$, where Δ is a quiver of Dynkin type. Clearly, we have $\operatorname{Dyn}(\chi_A) = \Delta$. ∎

2.3. Let us restate the results in [1, 3] mentioned in 1.3. Let $A = k\mathrm{Q}/I$ be a connected and strongly simply connected algebra. Then we have:

(1) A is derived equivalent to a tame (but not representation-finite) hereditary algebra if and only if χ_A is non-negative and $\operatorname{corank}\chi_A = 1$. In this case, $\mathrm{Dyn}(\chi_A)$ is $\mathbb{D}_n$ ($n \geq 4$) or $\mathbb{E}_p$ ($p = 6, 7, 8$).

(2) If A is derived equivalent to a tubular algebra (resp. to a 2-tubular algebra), then χ_A is non-negative and $\operatorname{corank}\chi_A = 2$. If Q_A has more than 6 vertices, then $\mathrm{Dyn}(\chi_A) = \mathbb{E}_p$ ($p = 6, 7, 8$) (resp. $\mathrm{Dyn}(\chi_A) = \mathbb{D}_n$ ($n \geq 4$)), whereas if Q_A has 6 vertices in both cases we have $\mathrm{Dyn}(\chi_A) = \mathbb{D}_4$.

(3) Assume $A = B[M]$ is such that χ_A is non-negative, $\operatorname{corank}\chi_A = 2$, $\operatorname{corank}\chi_B = 1$ and M is indecomposable. Then A is derived equivalent to a tubular or a 2-tubular algebra.

We *conjecture* that the following hold for a strongly simply connected algebra A:

(4) If $\operatorname{corank}\chi_A = 2$, then

(4.1) if $\mathrm{Dyn}(\chi_A) = \mathbb{D}_n$ and $n \geq 5$, then A is derived equivalent to a 2-tubular algebra,

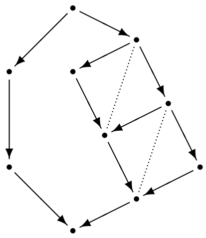
(4.2) if $\mathrm{Dyn}(\chi_A) = \mathbb{E}_p$ ($p = 6, 7, 8$), then A is derived equivalent to a tubular algebra.

(5) If $\operatorname{corank}\chi_A \geq 3$ then $\mathrm{Dyn}(\chi_A) = \mathbb{D}_n$.

The results we show in this work are special cases of conjecture (4). In [9], special cases of conjecture (5) are considered.

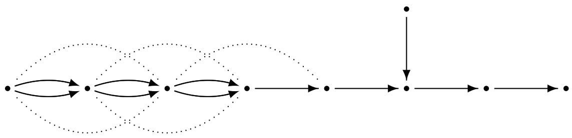
2.4. We recall from [4, 5] *examples* showing that the above conjectures may be expected only in the strongly simply connected case.

(a) Let A be the algebra given by the following quiver with commutativity relations as indicated by dotted lines.



Then χ_A is non-negative with $\operatorname{corank}\chi_A = 2$ and $\mathrm{Dyn}(\chi_A) = \mathbb{E}_8$. Moreover, A is wild and hence A cannot be derived tame, by 1.5.

(b) Let A be the algebra given by the following quiver with zero relations as indicated by dotted lines.



Then χ_A is non-negative with $\operatorname{corank} \chi_A = 3$ and $\operatorname{Dyn}(\chi_A) = \mathbb{E}_6$.

3. Connectivity of the radical. In the following we prove a general result about the convex closure of the support of the radical of a strongly simply connected algebra with non-negative Euler form. Although the proof is quite technical, it will be of great use in the forthcoming considerations.

PROPOSITION. *Let $A = k\mathrm{Q}/I$ be a strongly simply connected algebra with non-negative Euler form. Then the convex closure $\overline{\operatorname{rad}}\, \chi_A$ of the support of $\operatorname{rad} \chi_A$ is connected in A.*

Proof. Suppose that there exists a strongly simply connected algebra A such that $\overline{\operatorname{rad}}\, \chi_A$ is not connected in A. We assume that A is a minimal such example and let $\overline{\operatorname{rad}}\, \chi_A = \bigcup_{i=1}^{t} R_i$ $(t \geq 2)$ be a decomposition into connected algebras R_i which are convex in A.

The proof is done in several steps:

(i) *We first show that* $\operatorname{corank} \chi_A \geq 2$. Any vector $v \in \operatorname{rad} \chi_A$ decomposes as $v = \sum_{i=1}^{t} v_i$ with $v_i \in \mathrm{K}_\circ(R_i) \subset \mathrm{K}_\circ(A)$. Hence $0 = \chi_A(v) = \sum_{i=1}^{t} \chi_{R_i}(v_i)$ (since for $i \neq j$, $x \in \operatorname{supp} R_i$ and $y \in \operatorname{supp} R_j$ there are no directed paths between x and y implying that $\langle e_i, e_j \rangle_A = 0$). Since χ_A is non-negative, we have $v_i \in \operatorname{rad} \chi_{R_i}$ for $1 \leq i \leq t$, and therefore $\operatorname{corank} \chi_A \geq 2$.

(ii) *We show* $t = 2$, *that is*, $\overline{\operatorname{rad}}\, \chi_A = R_1 \cup R_2$ *where* R_1, R_2 *are connected and convex in* A. Choose $i \neq j$ such that there is a walk γ between R_i and R_j in Q_A of minimal length. Then the convex closure of R_i, R_j and γ in A is a strongly simply connected algebra $A_\circ$ with $\overline{\operatorname{rad}}\, \chi_{A_\circ} = R_i \cup R_j$. By the minimality of A we get $A = A_\circ$ and $t = 2$.

(iii) *Next we verify that for* $i = 1, 2$ *there is a source or a sink* y_i *such that* $A \setminus \{y_i\}$ *is connected and* $y_i \in R_i$. First observe that $A \setminus \{R_1 \cup R_2\}$ contains a vertex x_0 which is a source or a sink in Q_A, and that for any such point x_0, by minimality, $A \setminus \{x_0\}$ is not connected.

Choose such a point $x_0 \in A \setminus \{R_1 \cup R_2\}$, say a source, and set $A \setminus \{x_0\} = B_1 \cup B_2$ with $R_1 \subset B_1$ and $R_2 \subset B_2$. Now, choose a sink $x_1 \in B_1$. If $A \setminus \{x_1\}$ decomposes, say $A \setminus \{x_1\} = C_1 \cup C_2$ with $R_2 \subset C_2$, we choose a source $x_2 \in C_1$. Again, if $A \setminus \{x_2\}$ decomposes, say $A \setminus \{x_2\} = D_1 \cup D_2$ with $B_2 \subset D_2$, we choose a sink $x_3 \in D_1$. Observe that $|B_1| > |C_1| > |D_1| > \ldots$ This process may be continued until we find a source or a sink x_m not belonging to R_2 such that $A \setminus \{x_m\}$ is connected. By the above, we thus have $y_1 := x_m \in R_1$. Dually we find y_2.

(iv) *Now we prove that A is tubular or* 2*-tubular.* By 1.4(c), we have $\operatorname{corank} \chi_{A \setminus \{y_i\}} < \operatorname{corank} \chi_A$ for $i = 1, 2$, and hence by minimality $\operatorname{corank} \chi_{R_1} = 1 = \operatorname{corank} \chi_{R_2}$. Therefore $\operatorname{corank} \chi_A = 2$.

We may assume that y_1 is a source. Since A is strongly simply connected, $M = \operatorname{rad} P_{y_1}$ is indecomposable and $A' = A \setminus \{y_1\}$ is strongly simply connected with $\operatorname{corank}\chi_{A'} = 1$. By 2.3(3), A is either derived equivalent to a tubular or to a 2-tubular algebra.

(v) *Finally, we show that this leads to a contradiction.* In both cases we have $\operatorname{rad}\chi_A = kv_1 \oplus kv_2$ with $\langle v_1, v_2\rangle_A \neq 0$ (in the tubular case this follows from [17], in the 2-tubular case it may be easily verified for the poset algebra $\mathrm{P}(n)$). But this contradicts the fact that there are vectors $w_1, w_2 \in \operatorname{rad}\chi_A$ with $w_i \in \mathrm{K}_\circ(R_i) \subset \mathrm{K}_\circ(A)$ for $i = 1, 2$, which implies that $\langle w_1, w_2\rangle_A = 0$. This completes the proof of the proposition. ■

4. The tree case

4.1. The first result we state provides the *inductive step* dealing with tree algebras with non-negative Euler form.

PROPOSITION. *Let A be a tree algebra with non-negative Euler form and* $\operatorname{corank}\chi_A = c$. *Then there exists an algebra B with the following properties.*

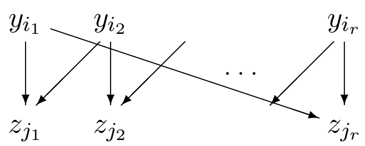
(i) *B is derived equivalent to a tree algebra and χ_B is non-negative with* $\operatorname{corank}\chi_B = c - 1$.

(ii) *A is derived equivalent to $B[M]$ for some indecomposable B-module M.*

We give the proof of the proposition in 4.4 after some preparation.

4.2. LEMMA. *Let $A = k\mathrm{Q}_A/I$ be a tree algebra. Consider the convex closure* $\overline{\operatorname{rad}}\,\chi_A$ *of the support of* $\operatorname{rad}\chi_A$ *and let x be a source or a sink in* $\overline{\operatorname{rad}}\,\chi_A$. *Then $A \setminus \{x\}$ is again a tree algebra.*

Proof. Suppose $A \setminus \{x\}$ is not a tree. Denote by $y_1, \ldots, y_t$ the vertices in Q_A such that there exists an arrow $\alpha_i : y_i \to x$ and denote by $z_1, \ldots, z_s$ those vertices with an arrow $\beta_i : x \to z_i$ (since χ_A is non-negative, we have $t + s \leq 4$). Since $A \setminus \{x\}$ is not a tree, it fully contains an algebra B of the form



with $r \geq 2$ and the arrows are compositions $\beta_j\alpha_i$ for some $1 \leq i \leq t$ and $1 \leq j \leq s$. Then there exists a vector $v \in \operatorname{rad}\chi_A$ with $v(y_{i_1}) \neq 0 \neq v(z_{j_1})$. This contradicts the fact that x was chosen to be a source or a sink in $\overline{\operatorname{rad}}\,\chi_A$. ■

4.3. Let $A = k\mathrm{Q}_A/I$ be a triangular algebra and x a source in Q_A. Let $A_\circ = A \setminus \{x\}$ and write $A = A_\circ[M]$ as a one-point extension with $M = \operatorname{rad} P_x$. Then $S_x^+ A = [M]A_\circ$ is the *source-reflection* of A at x. In

[11] it is shown that A and $S_x^+ A$ are derived equivalent. We denote the extension-vertex in $\mathrm{Q}_{S_x^+ A}$ by x^*, that is, $I_{x^*}/\mathrm{soc}\, I_{x^*} = M$.

For any vertex $x \in \mathrm{Q}_A$ we assume that

$$x_A^< := \{a \neq x : \text{there is a path from } a \text{ to } x\} = \{a_1, \ldots, a_t\}$$

is enumerated in such a way that the existence of a path from a_i to a_j implies that $i \leq j$. Define the algebra $A^x = S_{a_t}^+ \ldots S_{a_1}^+ A$, which is derived equivalent to A. Clearly, x is then a source in A^x. For any point $u \in x_A^<$ and $y = u^* \in \mathrm{Q}_{A^x}$ we also write $u = y^*$.

Lemma. *Let A be a tree algebra such that χ_A is non-negative, and let x be a source in $\overline{\mathrm{rad}}\, \chi_A$. Then for any arrow $\alpha : x \to y$ in Q_{A^x} we have $y \in \overline{\mathrm{rad}}\, \chi_A$.*

Proof. We assume that there exists $y \notin \overline{\mathrm{rad}}\, \chi_A$ such that there is an arrow $\alpha : x \to y$ in Q_{A^x} and proceed in several steps.

(i) *First we show that x is the only vertex in $\overline{\mathrm{rad}}\, \chi_A$ which is the starting point of a path in Q_{A^x} to y.* Assume there exists a vertex $x' \in \overline{\mathrm{rad}}\, \chi_A$, $x' \neq x$ and a path

$$x' \to z_0 \to z_1 \to \ldots \to z_t \to y$$

in Q_{A^x}. By Proposition 3, we know that it is possible to connect x and x' by a walk inside $\overline{\mathrm{rad}}\, \chi_A$, thus, since A is a tree algebra, we have $y \in (x_A^<)^*$. If there exists $i > 0$ such that z_{i-1} does not belong to $(x_A^<)^*$, we choose i maximal with this property. Then in A we have the following paths:

$$x' \to z_0 \to z_1 \to \ldots \to z_{i-1}$$

$$z_i^* \to z_{i+1}^* \to \ldots \to z_t^* \to y^* \xrightarrow{f} x$$

where f itself is a path. Together with a path from z_i^* to z_{i-1} and a walk inside $\overline{\mathrm{rad}}\, \chi_A$ between x' and x, we obtain a closed walk in Q_A, contrary to the assumption that A is a tree algebra. The case where $z_i^* \in x_A^<$ for all $i = 0, \ldots, t$ is similar.

(ii) *Now we show that the assumption leads to a contradiction.* Let $A' = A \setminus \{y_{A^x}^< \setminus \{x\}\}$. Clearly, A' is convex in A^x and $\overline{\mathrm{rad}}\, \chi_{A^x}$ is fully contained in A'. It is thus sufficient to show that for A' the assumption leads to a contradiction.

Consider a projective resolution of the simple module S_y in $\mathrm{mod}_{A'}$

$$0 \to P(n) \to P(n-1) \to \ldots \to P(0) \to S_y \to 0.$$

Then $\langle \underline{\dim}\, P(i), v \rangle_{A'} = 0$ for all $i = 0, \ldots, n$ and $v \in \mathrm{rad}\, \chi_{A'}$.

Let $v \in \mathrm{rad}\, \chi_{A'}$ be such that $v(x) \neq 0$. Then

$$\langle v, e_y \rangle_{A'} = \langle v, \underline{\dim}\, I_y \rangle_{A'} - \langle v, \underline{\dim}\, I_x \rangle_{A'} = -v(x) \neq 0$$

because $y \neq \overline{\mathrm{rad}}\,\chi_{A'}$ and x is the only predecessor of y in $\mathrm{Q}_{A'}$. On the other hand,

$$\langle e_y, v\rangle_{A'} = \sum_{i=0}^{n}(-1)^i \langle \underline{\dim}\, P(i), v\rangle_{A'} = 0.$$

Therefore $\chi_{A'}(2v+e_y) < 0$, contradicting the non-negativity of $\chi_{A'}$. ■

Obviously, the dual statement may be proved similarly.

4.4. *Proof of Proposition 4.1.* By Proposition 3, $\overline{\mathrm{rad}}\,\chi_A$ is connected. Choose a source or sink x in $\overline{\mathrm{rad}}\,\chi_A$ such that $\overline{\mathrm{rad}}\,\chi_A \setminus \{x\}$ is still connected. Say x is a source in $\overline{\mathrm{rad}}\,\chi_A$. Consider the algebra $A_\circ = A \setminus \{x\}$ which is fully contained in A. By 4.2, $A_\circ$ is again a tree algebra and by 1.4(c), $\mathrm{corank}\,\chi_{A_\circ} = c-1$.

As in the proof of Lemma 4.3, we have $\mathrm{rad}\,\chi_A = \mathrm{rad}\,\chi_{A^x}$ and in particular $\overline{\mathrm{rad}}\,\chi_A = \overline{\mathrm{rad}}\,\chi_{A^x}$. Observe that x is a source in A^x and define $B = A^x \setminus \{x\}$. Hence $B = S^+_{a_t}\dots S^+_{a_1} A_\circ$ (where $x^<_A = \{a_1,\dots,a_t\}$ is supposed to be "well-enumerated") and $\mathrm{corank}\,\chi_B = \mathrm{corank}\,\chi_{A_\circ} = c-1$.

It remains to show that the B-module $M = \mathrm{rad}\,P_x$ is indecomposable. First, observe that $M' = \mathrm{rad}\,P_x|_{\overline{\mathrm{rad}}\,\chi_{A^x}}$ is indecomposable (because $\overline{\mathrm{rad}}\,\chi_A = \overline{\mathrm{rad}}\,\chi_{A^x}$). By 4.3, any arrow $x\to y$ in Q_{A^x} belongs to $\overline{\mathrm{rad}}\,\chi_{A^x}$. Therefore a decomposition of M yields a decomposition of M', thus M is indecomposable. □

4.5. *Proof of the main theorem for tree algebras.* Let A be a tree algebra with non-negative Euler form of corank 2. By 4.1, there exists a triangular, connected algebra B which is derived equivalent to a tree algebra C and such that χ_B is non-negative of corank one and there exists an indecomposable B-module M such that A is derived equivalent to $B[M]$. In particular, χ_C is non-negative of corank one. The result follows from 2.3(3). ■

5. The poset case. A rereading of the proof of the main theorem in the tree case reveals that the assumption on A to be a tree algebra is only needed in the proof of Lemma 4.2 and in the step (i) of Lemma 4.3.

In the following we just give the arguments which establish the same assertions as 4.2 and 4.3 if A is a strongly simply connected poset algebra.

5.1. LEMMA. *Let A be a strongly simply connected poset algebra. Let x be a source or sink in $\overline{\mathrm{rad}}\,\chi_A$. Then $A\setminus\{x\}$ is again a strongly simply connected poset algebra.*

Proof. The algebra $B = A\setminus\{x\}$ is clearly a poset algebra. To show that B is strongly simply connected, it is enough to show that B admits no crown (see 1.4). This is shown exactly as in the proof of Lemma 4.2. ■

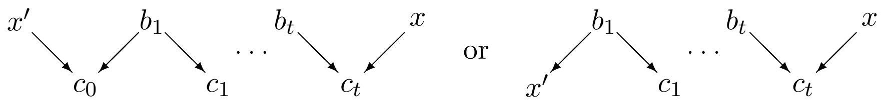
5.2. Lemma. *Let A be a strongly simply connected poset algebra such that χ_A is non-negative, and let x be a source in $\overline{\mathrm{rad}}\,\chi_A$. Then for any arrow $\alpha : x \to y$ in Q_{A^x} we have $y \in \overline{\mathrm{rad}}\,\chi_{A^x}$.*

Proof. Again, we assume that there exists an arrow $\alpha : x \to y$ such that $y \notin \overline{\mathrm{rad}}\,\chi_A$.

And again, we first show that then x is the only start point of a path from $\overline{\mathrm{rad}}\,\chi_{A^x}$ to y in Q_{A^x}. So assume that this is not so: let $x' \in \overline{\mathrm{rad}}\,\chi_{A^x}$ be different from x such that there exists a path

$$x' \to z_0 \to z_1 \to \ldots \to z_t \to y$$

in A^x. Since $\overline{\mathrm{rad}}\,A^x$ is connected there exists a fully contained algebra C in $\overline{\mathrm{rad}}\,A^x$ of the form

(∗)


First, suppose $y \notin (x_A^{<})^*$. Then we have $x' \not< x$ in A because there is an arrow $x \to y$ and A is a poset algebra. If there exists a j such that $c_j < y$ then we choose j maximal with that property. Observe that $j < t$. Thus $\{b_{j+1}, c_{j+1}, \ldots, b_t, c_t, x, y\}$ is a crown in A, contrary to the fact that A is strongly simply connected (see 1.4(b)). On the other hand, if there is no j with $c_j < y$ then (∗) together with y yields a crown in A.

Thus we have $y^* \in x_A^{<}$ and therefore $y^* < x < c_t$. On the other hand, since $x \to y$ is an arrow in Q_{A^x}, the vertex y^* cannot be smaller than c_t in A. This contradicts the fact that A is a poset algebra.

The rest of the proof goes as in 4.3. ■

6. Higher coranks

6.1. We shall prove the following result which is related to the conjecture about algebras A with $\mathrm{corank}\,\chi_A > 2$ (see 2.3(5)).

Proposition. *Let A be a tree algebra or a strongly simply connected poset algebra with non-negative Euler form. Then any properly contained, convex algebra B in A whose Euler form has corank 2 is derived equivalent to a poset algebra $\mathrm{P}(n)$.*

6.2. We shall need the following result.

Proposition. *Let A be an algebra which is derived equivalent to a tubular algebra and let M be an indecomposable A-module. Then*:

(i) $\chi_A(\underline{\dim}\,M) \in \{0, 1\}$.

(ii) *The Euler form of $A[M]$ is indefinite.*

Proof. (i) By [11], the inclusion $\mathrm{mod}_A \hookrightarrow \mathrm{D^b}(A)$, $X \mapsto X[0]$, induces an isometry $\mathrm{K_\circ}(A) \to \mathrm{K_\circ}(\mathrm{D^b}(A))$. Hence we shall prove that $\chi_{\mathrm{D^b}(A)}(M[0]) \in \{0,1\}$. By [12], for an indecomposable object $X \in \mathrm{D^b}(A)$, there is a tubular algebra B such that X lies in the image of the composition $\mathrm{mod}\, B \hookrightarrow \mathrm{D^b}(B) \to \mathrm{D^b}(A)$ of the inclusion with some triangular equivalence F, say $X = F(Y[0])$ for some indecomposable B-module Y. Hence $\chi_A(\underline{\dim}\, M) = \chi_{\mathrm{D^b}(A)}([M[0]]) = \chi_{\mathrm{D^b}(B)}([Y[0]]) = \chi_B(\underline{\dim}\, Y)$, and finally $\chi_B(\underline{\dim}\, Y) \in \{0,1\}$ by the results of [17].

(ii) Let M be an indecomposable A-module and $A' = A[M]$. Then $\chi_A(\underline{\dim}\, M) \in \{0,1\}$. Assume first $\chi_A(\underline{\dim}\, M) = 0$. As we have seen in the proof of Proposition 3, there exists a vector $v \in \mathrm{rad}\, \chi_A$ such that $\langle \underline{\dim}\, M, v\rangle_A \neq 0$. Let x be the extension vertex in $\mathrm{Q}_{A'}$ such that $\mathrm{rad}\, P_x = M$. Then $\langle v, e_x\rangle_{A'} = 0$ and $\langle e_x, v\rangle_{A'} = \langle \underline{\dim}\, P_x, v\rangle_{A'} - \langle \underline{\dim}\, M, v\rangle_{A'} = -\langle \underline{\dim}\, M, v\rangle_A \neq 0$, which implies that $\chi_{A'}$ is indeed indefinite.

Now assume $\chi_A(\underline{\dim}\, M) = 1$. Suppose that $\chi_{A'}$ is non-negative. We show that $\underline{\dim}\, M \in \chi_A^{-1}(1) \cap \chi_A^{-1}(0)^{\perp}$ contrary to 1.4. Indeed, if $v \in \chi_A^{-1}(0)$, then $\langle e_x, v\rangle_{A'} + \langle v, e_x\rangle_{A'} = 0$ (since otherwise $\chi_{A'}(2v \pm e_x) < 0$, a contradiction). Since $\langle v, e_x\rangle_{A'} = 0$, we have $0 = \langle e_x, v\rangle_{A'} = -\langle \underline{\dim}\, M, v\rangle_A$. ∎

6.3. *Proof of Proposition 6.1.* Let B be connected and convex in A with $B \neq A$ and such that $\mathrm{corank}\, \chi_B = 2$. By our main theorem, B is derived equivalent to a tubular algebra or to a 2-tubular algebra. Since $B \neq A$, there exists a B-module M such that $B[M]$ (or $[M]B$) is still convex in A. Since then $B[M]$ (resp. $[M]B$) is strongly simply connected, the module M has to be indecomposable and by 6.2, the algebra B cannot be tubular. ∎

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