ON FULL SUSLIN TREES

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0. Introduction. In the present paper we answer a combinatorial question of Kunen listed in Arnie Miller's Problem List. We force, e.g. for the first strongly inaccessible Mahlo cardinal λ, a full (see 1.1(2)) λ-Suslin tree and we remark that the existence of such trees follows from $\mathbf{V} = \mathbf{L}$ (if λ is Mahlo strongly inaccessible). This answers [Mi91, Problem 15.13].

Our notation is rather standard and compatible with those of classical textbooks on Set Theory. However, in forcing considerations, we keep the older tradition that

a stronger condition is the larger one.

We will keep the following conventions concerning use of symbols.

NOTATION 0.1. (1) λ, μ will denote cardinal numbers and $\alpha, \beta, \gamma, \delta, \xi, \zeta$ will be used to denote ordinals.

(2) Sequences (not necessarily finite) of ordinals are denoted by ν, η, ϱ (with possible indices).

(3) The length of a sequence η is $\lg(\eta)$.

(4) For a sequence η and an ordinal $\alpha \leq \lg(\eta)$, $\eta{\restriction}\alpha$ is the restriction of the sequence η to α (so $\lg(\eta{\restriction}\alpha) = \alpha$). If a sequence ν is a proper initial segment of a sequence η then we write $\nu \vartriangleleft \eta$ (and $\nu \trianglelefteq \eta$ has the obvious meaning).

(5) A tilde indicates that we are dealing with a name for an object in forcing extension (like $\underset{\sim}{x}$).

1. Full λ-Suslin trees. A subset T of $^{\alpha>}2$ is an *α-tree* whenever (α is a limit ordinal and) the following three conditions are satisfied:

- $\langle\rangle \in T$, if $\nu \vartriangleleft \eta \in T$ then $\nu \in T$,
- $\eta \in T$ implies $\eta^\frown\langle 0\rangle, \eta^\frown\langle 1\rangle \in T$, and

• for every $\eta \in T$ and $\beta < \alpha$ such that $\lg(\eta) \leq \beta$ there is $\nu \in T$ such that $\eta \trianglelefteq \nu$ and $\lg(\eta) = \beta$.

A λ-*Suslin tree* is a λ-tree $T \subseteq {}^{\lambda>}2$ in which every antichain is of size less than λ.

DEFINITION 1.1. (1) For a tree $T \subseteq {}^{\alpha>}2$ and an ordinal $\beta \leq \alpha$ we let
$$T_{[\beta]} := T \cap {}^{\beta}2 \quad \text{and} \quad T_{[<\beta]} := T \cap {}^{\beta>}2.$$
If $\delta \leq \alpha$ is limit then we define
$$\lim_\delta T_{[<\delta]} := \{\eta \in {}^{\delta}2 : (\forall \beta < \delta)(\eta{\upharpoonright}\beta \in T)\}.$$

(2) An α-tree T is *full* if for every limit ordinal $\delta < \alpha$ the set $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ has at most one element.

(3) An α-tree $T \subseteq {}^{\alpha>}2$ has *true height* α if for every $\eta \in T$ there is $\nu \in {}^{\alpha}2$ such that
$$\eta \vartriangleleft \nu \quad \text{and} \quad (\forall \beta < \alpha)(\nu{\upharpoonright}\beta \in T).$$

We will show that the existence of full λ-Suslin trees is consistent assuming the cardinal λ satisfies the following hypothesis.

HYPOTHESIS 1.2. (a) λ is a strongly inaccessible (Mahlo) cardinal,

(b) $S \subseteq \{\mu < \lambda : \mu$ is a strongly inaccessible cardinal$\}$ is a stationary set,

(c) $S_0 \subseteq \lambda$ is a set of limit ordinals,

(d) for every cardinal $\mu \in S$, $\diamondsuit_{S_0 \cap \mu}$ holds true.

Further in this section we will assume that λ, S_0 and S are as above and we may forget to repeat these assumptions.

Let us recall that the *diamond principle* $\diamondsuit_{S_0 \cap \mu}$ postulates the existence of a sequence $\bar{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$ (called a $\diamondsuit_{S_0 \cap \mu}$-*sequence*) such that $\nu_\delta \in {}^{\delta}2$ (for $\delta \in S_0 \cap \mu$) and
$$(\forall \nu \in {}^{\mu}2)[\text{the set } \{\delta \in S_0 \cap \mu : \nu{\upharpoonright}\delta = \nu_\delta\} \text{ is stationary in } \mu].$$

Now we introduce a forcing notion $\mathbb{Q}$ and its relative $\mathbb{Q}^*$ which will be used in our proof.

DEFINITION 1.3. (1) A condition in $\mathbb{Q}$ is a tree $T \subseteq {}^{\alpha>}2$ of a true height $\alpha = \alpha(T) < \lambda$ (see 1.1(3); so α is a limit ordinal) such that $\|\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}\| \leq 1$ for every limit ordinal $\delta < \alpha$; the order on $\mathbb{Q}$ is defined by $T_1 \leq T_2$ if and only if $T_1 = T_2 \cap {}^{\alpha(T_1)>}2$ (so it is the end-extension order).

(2) For a condition $T \in \mathbb{Q}$ and a limit ordinal $\delta < \alpha(T)$, let $\eta_\delta(T)$ be the unique member of $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ if there is one, otherwise $\eta_\delta(T)$ is not defined.

(3) Let $T \in \mathbb{Q}$. A function $f : T \to \lim_{\alpha(T)}(T)$ is called a *witness* for T if $(\forall \eta \in T)(\eta \vartriangleleft f(\eta))$.

(4) A condition in $\mathbb{Q}^*$ is a pair (T, f) such that $T \in \mathbb{Q}$ and $f : T \to \lim_{\alpha(T)}(T)$ is a witness for T; the order on $\mathbb{Q}^*$ is defined by $(T_1, f_1) \leq (T_2, f_2)$ if and only if $T_1 \leq_{\mathbb{Q}} T_2$ and $(\forall \eta \in T_1)(f_1(\eta) \trianglelefteq f_2(\eta))$.

PROPOSITION 1.4. (1) *If* $(T_1, f_1) \in \mathbb{Q}^*$, $T_1 \leq_{\mathbb{Q}} T_2$ *and*

$(*)$ *either* $\eta_{\alpha(T_1)}(T_2)$ *is not defined or it does not belong to* $\mathrm{rang}(f_1)$

then there is $f_2 : T_2 \to \lim_{\alpha(T_2)}(T_2)$ *such that* $(T_1, f_1) \leq (T_2, f_2) \in \mathbb{Q}^*$.

(2) *For every* $T \in \mathbb{Q}$ *there is a witness* f *for* T.

P r o o f. Should be clear. ∎

PROPOSITION 1.5. (1) *The forcing notion* $\mathbb{Q}^*$ *is* $(< \lambda)$-*complete, in fact any increasing chain of length* $< \lambda$ *has the least upper bound in* $\mathbb{Q}^*$.

(2) *The forcing notion* $\mathbb{Q}$ *is strategically* γ-*complete for each* $\gamma < \lambda$.

(3) *Forcing with* $\mathbb{Q}$ *adds no new sequences of length* $< \lambda$. *Since* $\|\mathbb{Q}\| = \lambda$, *forcing with* $\mathbb{Q}$ *preserves cardinal numbers, cofinalities and cardinal arithmetic.*

P r o o f. (1) It is straightforward: suppose that $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is an increasing sequence of elements of $\mathbb{Q}^*$. Clearly we may assume that $\xi < \lambda$ is a limit ordinal and $\zeta_1 < \zeta_2 < \xi \Rightarrow \alpha(T_{\zeta_1}) < \alpha(T_{\zeta_2})$. Let $T_\xi = \bigcup_{\zeta < \xi} T_\zeta$ and $\alpha = \sup_{\zeta < \xi} \alpha(T_\zeta)$. Clearly, the union is increasing and T_ξ is a full α-tree. For $\eta \in T_\xi$ let $\zeta_0(\eta)$ be the first $\zeta < \xi$ such that $\eta \in T_\zeta$ and let $f_\xi(\eta) = \bigcup\{f_\zeta(\eta) : \zeta_0(\eta) \leq \zeta < \xi\}$. By the definition of the order on $\mathbb{Q}^*$ we see that the sequence $\langle f_\zeta(\eta) : \zeta_0(\eta) \leq \zeta < \xi \rangle$ is $\triangleleft$-increasing and hence $f_\xi(\eta) \in \lim_\alpha(T_\xi)$. Plainly, the function f_ξ witnesses that T_ξ has true height α, and thus $(T_\xi, f_\xi) \in \mathbb{Q}^*$. It should be clear that (T_ξ, f_ξ) is the least upper bound of the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$.

(2) For our purpose it is enough to show that for each ordinal $\gamma < \lambda$ and a condition $T \in \mathbb{Q}$ the second player has a winning strategy in the following game $\mathcal{G}_\gamma(T, \mathbb{Q})$. (Also we can let Player I choose T_ξ for ξ odd.)

The game lasts γ moves and during a play the players, called I and II, choose successively open dense subsets $\mathcal{D}_\xi$ of $\mathbb{Q}$ and conditions $T_\xi \in \mathbb{Q}$. At stage $\xi < \gamma$ of the game, Player I chooses an open dense subset $\mathcal{D}_\xi$ of $\mathbb{Q}$ and Player II answers playing a condition $T_\xi \in \mathbb{Q}$ such that

$$T \leq_{\mathbb{Q}} T_\xi, \quad (\forall \zeta < \xi)(T_\zeta \leq_{\mathbb{Q}} T_\xi), \quad \text{and} \quad T_\xi \in \mathcal{D}_\xi.$$

The second player wins if he always has legal moves during the play.

Let us describe the winning strategy for Player II. At each stage $\xi < \gamma$ of the game he plays a condition T_ξ and writes down a function f_ξ such that $(T_\xi, f_\xi) \in \mathbb{Q}^*$. Moreover, he keeps an extra obligation that $(T_\zeta, f_\zeta) \leq_{\mathbb{Q}^*} (T_\xi, f_\xi)$ for each $\zeta < \xi < \gamma$.

So arriving at a non-limit stage of the game he takes the condition (T_ζ, f_ζ) he constructed before (or just (T, f), where f is a witness for T,

if this is the first move; by 1.4(2) we can always find a witness). Then
he chooses $T_\zeta^* \geq_{\mathbb{Q}} T_\zeta$ such that $\alpha(T_\zeta^*) = \alpha(T_\zeta) + \omega$ and $(T_\zeta^*)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta)$. Thus $\eta_{\alpha(T_\zeta)}(T_\zeta^*)$ is not defined. Now Player II takes $T_{\zeta+1} \geq_{\mathbb{Q}} T_\zeta^*$ from the open dense set $\mathcal{D}_{\zeta+1}$ played by his opponent at this stage.
Clearly $\eta_{\alpha(T_\zeta)}(T_{\zeta+1})$ is not defined, so Player II may use 1.4(1) to choose
$f_{\zeta+1}$ such that $(T_\zeta, f_\zeta) \leq_{\mathbb{Q}^*} (T_{\zeta+1}, f_{\zeta+1}) \in \mathbb{Q}^*$.

At a limit stage ξ of the game, the second player may take the least
upper bound $(T_\xi', f_\xi') \in \mathbb{Q}^*$ of the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ (exists by (1))
and then apply the procedure described above.

(3) Follows from (2) above. ∎

DEFINITION 1.6. Let $\mathbf{T}$ be the canonical $\mathbb{Q}$-name for a generic tree added
by forcing with $\mathbb{Q}$:

$$\Vdash_{\mathbb{Q}} \mathbf{T} = \bigcup \{ T : T \in G_{\mathbb{Q}} \}.$$

It should be clear that $\mathbf{T}$ is (forced to be) a full λ-tree. The main point
is to show that it is λ-Suslin and this is done in the following theorem.

THEOREM 1.7. $\Vdash_{\mathbb{Q}}$ "$\mathbf{T}$ is a λ-Suslin tree".

P r o o f. Suppose that A is a $\mathbb{Q}$-name such that

$$\Vdash_{\mathbb{Q}} \text{ "}A \subseteq \mathbf{T} \text{ is an antichain"},$$

and let T_0 be a condition in $\mathbb{Q}$. We will show that there are $\mu < \lambda$ and a
condition $T^* \in \mathbb{Q}$ stronger than T_0 such that $T^* \Vdash_{\mathbb{Q}}$ "$A \subseteq \mathbf{T}_{[<\mu]}$" (and thus
it forces that the size of A is less than λ).

Let $\mathbf{A}$ be a $\mathbb{Q}$-name such that

$$\Vdash_{\mathbb{Q}} \text{ "}\mathbf{A} = \{ \eta \in \mathbf{T} : (\exists \nu \in A)(\nu \trianglelefteq \eta) \text{ or } \neg(\exists \nu \in A)(\eta \trianglelefteq \nu) \}\text{"}.$$

Clearly, $\Vdash_{\mathbb{Q}}$ "$\mathbf{A} \subseteq \mathbf{T}$ is dense open".

Let χ be a sufficiently large regular cardinal ($\beth_7(\lambda^+)^+$ is enough).

CLAIM 1.7.1. There are $\mu \in S$ and $\mathfrak{B} \prec (\mathcal{H}(\chi), \in, <_\chi^*)$ such that:

(a) $A, \mathbf{A}, S, S_0, \mathbb{Q}, \mathbb{Q}^*, T_0 \in \mathfrak{B}$,
(b) $\|\mathfrak{B}\| = \mu$ and $^{\mu>}\mathfrak{B} \subseteq \mathfrak{B}$,
(c) $\mathfrak{B} \cap \lambda = \mu$.

P r o o f. First construct inductively an increasing continuous sequence
$\langle \mathfrak{B}_\xi : \xi < \lambda \rangle$ of elementary submodels of $(\mathcal{H}(\chi), \in, <_\chi^*)$ such that $A, \mathbf{A}, S,$
$S_0, \mathbb{Q}, \mathbb{Q}^*, T_0 \in \mathfrak{B}_0$ and for every $\xi < \lambda$,

$$\|\mathfrak{B}_\xi\| = \mu_\xi < \lambda, \quad \mathfrak{B}_\xi \cap \lambda \in \lambda, \quad \text{and} \quad ^{\mu_\xi \geq}\mathfrak{B}_\xi \subseteq \mathfrak{B}_{\xi+1}.$$

Note that for a club E of λ, for every $\mu \in S \cap E$ we have

$$\|\mathfrak{B}_\mu\| = \mu, \quad ^{\mu>}\mathfrak{B}_\mu \subseteq \mathfrak{B}_\mu, \quad \text{and} \quad \mathfrak{B}_\mu \cap \lambda = \mu.$$

Choose $\mu \in S \cap E$ and let $\mathfrak{B} = \mathfrak{B}_\mu$.

Let $\mu \in S$ and $\mathfrak{B} \prec \big(\mathcal{H}(\chi), \in, <_\chi^*\big)$ be given by 1.7.1. We know that $\Diamond_{S_0 \cap \mu}$ holds, so fix a $\Diamond_{S_0 \cap \mu}$-sequence $\overline{\nu} = \langle \nu_\delta : \delta \in S_0 \cap \mu \rangle$.

Let

$$\underset{\sim}{\mathcal{I}} := \{ T \in \mathbb{Q} : T \text{ is incompatible (in } \mathbb{Q}) \text{ with } T_0 \text{ or:}$$

$$T \geq T_0 \text{ and } T \text{ decides the value of } \underset{\sim}{\mathbf{A}} \cap {}^{\alpha(T)>}2 \text{ and}$$

$$(\forall \eta \in T)(\exists \varrho \in T)(\eta \trianglelefteq \varrho \ \& \ T \Vdash_{\mathbb{Q}} \varrho \in \underset{\sim}{\mathbf{A}}) \}.$$

CLAIM 1.7.2. $\underset{\sim}{\mathcal{I}}$ *is a dense subset of* $\mathbb{Q}$.

P r o o f. Should be clear (remember 1.5(2)).

Now we choose by induction on $\xi < \mu$ a continuous increasing sequence $\langle (T_\xi, f_\xi) : \xi < \mu \rangle \subseteq \mathbb{Q}^* \cap \mathfrak{B}$.

STEP: $i = 0$. T_0 is already chosen and it belongs to $\mathbb{Q} \cap \mathfrak{B}$. We take any f_0 such that $(T_0, f_0) \in \mathbb{Q}^* \cap \mathfrak{B}$ (exists by 1.4(2)).

STEP: limit ξ. Since ${}^{\mu>}\mathfrak{B} \subseteq \mathfrak{B}$, the sequence $\langle (T_\zeta, f_\zeta) : \zeta < \xi \rangle$ is in $\mathfrak{B}$. By 1.5(1) it has the least upper bound (T_ξ, f_ξ) (which belongs to $\mathfrak{B}$).

STEP: $\xi = \zeta + 1$. First we take the (unique) tree T_ξ^* of true height $\alpha(T_\xi^*) = \alpha(T_\zeta) + \omega$ such that $T_\xi^* \cap {}^{\alpha(T_\zeta)>}2 = T_\zeta$ and: if $\alpha(T_\zeta) \in S_0$ and $\nu_{\alpha(T_\zeta)} \notin \mathrm{rang}(f_\zeta)$ then $(T_\xi^*)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta) \setminus \{\nu_{\alpha(T_\zeta)}\}$, otherwise $(T_\xi^*)_{[\alpha(T_\zeta)]} = \lim_{\alpha(T_\zeta)}(T_\zeta)$.

Let $T_\xi \in \mathbb{Q} \cap \underset{\sim}{\mathcal{I}}$ be strictly above T_ξ^* (exists by 1.7.2). Clearly we may choose such T_ξ in $\mathfrak{B}$. Now we have to define f_ξ. We do it by 1.4, but additionally we require that

$$\text{if} \quad \eta \in T_\xi \quad \text{then} \quad (\exists \varrho \in T_\xi)(\varrho \vartriangleleft f_\xi(\eta) \ \& \ T \Vdash_{\mathbb{Q}} \text{``} \varrho \in \underset{\sim}{\mathbf{A}} \text{''}).$$

Plainly the additional requirement causes no problems (remember the definition of $\underset{\sim}{\mathcal{I}}$ and the choice of T_ξ) and the choice can be done in $\mathfrak{B}$.

There are no difficulties in carrying out the induction. Finally we let

$$T_\mu := \bigcup_{\xi < \mu} T_\xi \quad \text{and} \quad f_\mu = \bigcup_{\xi < \mu} f_\xi.$$

By the choice of $\mathfrak{B}$ and μ we are sure that T_μ is a μ-tree. It follows from 1.5(1) that $(T_\mu, f_\mu) \in \mathbb{Q}^*$, so in particular the tree T_μ has enough μ branches (and belongs to $\mathbb{Q}$).

CLAIM 1.7.3. *For every* $\varrho \in \lim_\mu(T_\mu)$ *there is* $\xi < \mu$ *such that*

$$(\exists \beta < \alpha(T_{\xi+1}))(T_{\xi+1} \Vdash_{\mathbb{Q}} \text{``} \varrho{\restriction}\beta \in \underset{\sim}{\mathbf{A}} \text{''}).$$

P r o o f. Fix $\varrho \in \lim_\mu(T_\mu)$ and let

$$S_\nu^* := \{ \delta \in S_0 \cap \mu : \alpha(T_\delta) = \delta \text{ and } \nu_\delta = \varrho{\restriction}\delta \}.$$

Plainly, the set S_ν^* is stationary in μ (remember the choice of $\overline{\nu}$). By the

definition of the T_ξ's (and by $\varrho \in \lim_\mu(T_\mu)$) we conclude that for every $\delta \in S_\nu^*$,

$$\text{if} \quad \eta_\delta(T_{\delta+1}) \text{ is defined} \quad \text{then} \quad \varrho{\restriction}\delta \neq \eta_\delta(T_\mu) = \eta_\delta(T_{\delta+1}).$$

But $\varrho{\restriction}\delta = \nu_\delta$ (as $\delta \in S_\nu^*$). So look at the inductive definition: necessarily for some $\varrho_\delta^* \in T_\delta$ we have $\nu_\delta = f_\delta(\varrho_\delta^*)$, i.e. $\varrho{\restriction}\delta = f_\delta(\varrho_\delta^*)$. Now, $\varrho_\delta^* \in T_\delta = \bigcup_{\xi<\delta} T_\xi$ and hence for some $\xi(\delta) < \delta$, we have $\varrho_\delta^* \in T_{\xi(\delta)}$. By Fodor's lemma we find $\xi^* < \mu$ such that the set

$$S_\nu' := \{\delta \in S_\nu^* : \xi(\delta) = \xi^*\}$$

is stationary in μ. Consequently, we find ϱ^* such that the set

$$S_\nu^+ := \{\delta \in S_\nu' : \varrho^* = \varrho_\delta^*\}$$

is stationary (in μ). But the sequence $\langle f_\xi(\varrho^*) : \xi^* \leq \xi < \mu\rangle$ is $\trianglelefteq$-increasing, and hence the sequence ϱ is its limit. Now we easily obtain the claim using the inductive definition of the (T_ξ, f_ξ)'s.

It follows from the definition of $\underset{\sim}{\mathbf{A}}$ and 1.7.3 that

$$T_\mu \Vdash_\mathbb{Q} \text{ "}\underset{\sim}{A} \subseteq T_\mu\text{"}$$

(remember that $\underset{\sim}{A}$ is a name for an antichain of $\mathbf{T}$), and hence

$$T_\mu \Vdash_\mathbb{Q} \text{ "}\|\underset{\sim}{A}\| < \lambda\text{",}$$

finishing the proof of the theorem. ∎

DEFINITION 1.8. A λ-tree T is S_0-*full*, where $S_0 \subseteq \lambda$, if for every limit $\delta < \lambda$,

• if $\delta \in \lambda \setminus S_0$ then $T_{[\delta]} = \lim_\delta(T)$,
• if $\delta \in S_0$ then $\|T_{[\delta]} \setminus \lim_\delta(T)\| \leq 1$.

COROLLARY 1.9. *Assuming Hypothesis 1.2:*

(1) *The forcing notion* $\mathbb{Q}$ *preserves cardinal numbers, cofinalities and cardinal arithmetic.*

(2) $\Vdash_\mathbb{Q}$ "$\mathbf{T} \subseteq {}^{\lambda>}2$ *is a* λ-*Suslin tree which is full and even* S_0-*full". [So, in* $\mathbf{V}^\mathbb{Q}$, *in particular we have: for every* $\alpha < \beta < \mu$, *for all* $\eta \in T \cap {}^\alpha 2$ *there is* $\nu \in T \cap {}^\beta 2$ *such that* $\eta \lhd \nu$, *and for a limit ordinal* $\delta < \lambda$, $\lim_\delta(T_{[<\delta]}) \setminus T_{[\delta]}$ *is either empty or has a unique element (and then* $\delta \in S_0$).]

P r o o f. By 1.5 and 1.7. ∎

Of course, we do not need to force.

DEFINITION 1.10. Let $S_0, S \subseteq \lambda$. A sequence $\langle (C_\alpha, \nu_\alpha) : \alpha < \lambda \text{ limit}\rangle$ is called a *squared diamond sequence* for (S, S_0) if for each limit ordinal $\alpha < \lambda$,

(i) C_α is a club of α disjoint from S,
(ii) $\nu_\alpha \in {}^\alpha 2$,

(iii) if $\beta \in \mathrm{acc}(C_\alpha)$ then $C_\beta = C_\alpha \cap \beta$ and $\nu_\beta \lhd \nu_\alpha$,

(iv) if $\mu \in S$ then $\langle \nu_\alpha : \alpha \in C_\mu \cap S_0 \rangle$ is a diamond sequence.

PROPOSITION 1.11. *Assume* (*in addition to* 1.2)

(e) *there exists a squared diamond sequence for* (S, S_0).

Then there is a λ-Suslin tree $T \subseteq {}^{\lambda >}2$ which is S_0-full.

P r o o f. Look carefully at the proof of 1.7. ∎

COROLLARY 1.12. *Assume that* $\mathbf{V} = \mathbf{L}$ *and λ is Mahlo strongly inaccessible. Then there is a full λ-Suslin tree.*

P r o o f. Let $S \subseteq \{\mu < \lambda : \mu \text{ is strongly inaccessible}\}$ be a stationary non-reflecting set. By Beller and Litman [BeLi80], there is a square $\langle C_\delta : \delta < \lambda \text{ limit} \rangle$ such that $C_\delta \cap S = \emptyset$ for each limit $\delta < \lambda$. As in Abraham, Shelah and Solovay [AShS 221, §1] we can also have the squared diamond sequence. ∎

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