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ON THE ISOPERIMETRY OF GRAPHS WITH MANY ENDS

BY

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Let X be a connected graph with uniformly bounded degree. We show that if there is a radius r such that, by removing from X any ball of radius r, we get at least three unbounded connected components, then X satisfies a strong isoperimetric inequality. In particular, the non-reduced l^2-cohomology of X coincides with the reduced l^2-cohomology of X and is of uncountable dimension. (Those facts are well known when X is the Cayley graph of a finitely generated group with infinitely many ends.)

1. Introduction. We consider graphs X with $\deg(X) < \infty$ (that is, there is a constant D such that for any vertex v of X we have $\deg(v) < D$).

THEOREM 1.1. *Let X be a connected graph with* $\deg(X) < \infty$. *If there is a constant $r > 0$ such that for each $x \in X$ the complement of the ball of radius r with center x has at least three unbounded connected components, then there is an $\varepsilon > 0$ such that for all finite sets Ω of vertices of X,*

$$|\partial\Omega|/|\Omega| \geq \varepsilon.$$

(See §2 below for the definition of the metric on X and for the definition of the boundary $\partial\Omega$.) Notice that a Cayley graph X of a finitely generated group with infinitely many ends satisfies the hypothesis of the theorem. Before giving the proof, we mention that the idea of the proof comes from differential geometry. If x_0 is a base point in a complete Riemannian manifold X and if

$$Z(x) = -\operatorname{grad}(d(x_0, x))$$

is well defined (this is the case if for example X is simply connected and of non-positive sectional curvature), and if this vector field satisfies

$$\operatorname{div}(Z(x)) \leq -\delta^2 < 0$$

for all $x \in X$, then the divergence version of the Stokes formula shows that X

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satisfies a strong isoperimetric inequality (see [Gr], 05C, for a more general situation; see also [Av]).

If x_0 is a base point of a graph X, we define (on a quasi-isometric subset of X) a map Z which moves points in the direction of x_0 by a uniformly bounded distance (see the proof of Lemma 5.1). The key point is then to show that a hypothesis on the number of ends of X implies that many points of X have at least two preimages (that is, if x is such a point then, by definition of the discrete divergence, $\operatorname{div}(Z(x)) \leq -1$). See the proof of Lemma 5.1. The existence on X of such a map implies a strong isoperimetric inequality (see Proposition 2.1). The reverse implication also holds and is an application of the marriage lemma of P. Hall (we give a self-contained proof of this implication in the last section).

The application to l^2-cohomology mentioned in the abstract is proved as follows. One shows that the space of locally constant functions (modulo the constants) on the (totally disconnected and compact) space of ends of X injects into the l^2-cohomology group $H^1_{(2)}(X)$ (see [ABCKT], Ch. 4, §2.4, p. 50). Then, using the strong isoperimetric inequality of Theorem 1.1, one shows that the (bounded) coboundary operator

$$d : C^0_{(2)}(X) \to C^1_{(2)}(X)$$

satisfies, for some $\varepsilon > 0$,

$$\|df\|_2 \geq \varepsilon \|f\|_2$$

for all functions f with compact support (see [Ge] or [Gr], 8.C, "Examples and applications", p. 248) and as d is continuous, the inequality also holds on the whole Hilbert space $C^0_{(2)}(X)$. This implies that the image of d is closed, i.e. that the reduced and non-reduced cohomologies coincide. The above argument provides a direct way (that is, without using Stalling's structure theorem on groups with many ends) to prove that the l^2-cohomology of groups with infinitely many ends is non-zero in dimension 1. This fact is used in the proof of the theorem of Gromov saying that the fundamental group of a closed Kähler manifold has at most one end (see [Gr89] and [ABCKT], Ch. 4).

When the graph X has homogeneity properties (for example if X is a Cayley graph), see [SW] and [Du], [St]. The motivation for writing this paper came from a discussion with A. Valette. I'm very grateful to him. Thanks are also due to R. Brooks for mentioning the marriage lemma.

2. A criterion for strong isoperimetry. Let X be a connected graph with $\deg(X) < \infty$. We consider the path metric d on X in which each edge has length one. We denote by V the set of vertices of X. If $\Omega \subset V$ is a subset, its *boundary* in X is the set

$$\partial\Omega = \{u \in \Omega : \exists v \in V \setminus \Omega : d(u,v) = 1\}.$$

We say that X has the property of *strong isoperimetry* if there exists $\varepsilon > 0$ such that for all finite non-empty subsets Ω in V,

$$|\partial\Omega|/|\Omega| \geq \varepsilon.$$

PROPOSITION 2.1 (cf. [GLP], 6.17). *With the notations as above, if there is a map $Z : V \to V$ and a constant $C > 0$ such that*

$$d(Z(v),v) \leq C, \qquad |Z^{-1}(v)| \geq 2, \qquad \forall v \in V,$$

then X has the property of strong isoperimetry.

Proof. We first assume that $C=1$. Let $\Omega \subset V$. We define $\Omega^0 = \Omega \setminus \partial\Omega$. By hypothesis the distance between a point and its image under the map Z is at most one. Hence

$$Z^{-1}(\Omega^0) \subset \Omega.$$

According to the second condition imposed on Z, we have

$$2|\Omega^0| \leq |Z^{-1}(\Omega^0)|.$$

Hence $|\Omega^0| \leq \frac{1}{2}|\Omega|$ so that $|\partial\Omega| \geq \frac{1}{2}|\Omega|$.

To handle the cases when $C \geq 1$, let X' be the graph obtained from X by adding edges between vertices at mutual distance less than or equal to C. We consider on X' the path metric in which each edge has length one. On this new graph X', the map Z satisfies the hypothesis of the proposition with $C = 1$. Moreover, there is a constant $K > 0$ (depending on C and on the degree of X) such that, for any finite subset Ω of V,

$$K|\partial\Omega| \geq |\partial'\Omega|.$$

Hence $\varepsilon = 1/(2K)$ is as required. ∎

3. Example. The following example illustrates the above proposition and shows directly that a finitely generated group containing a free subgroup on two generators has the property of strong isoperimetry.

Let F be the free group on two letters a and b. Let w be a non-empty reduced word on the alphabet $S = \{a, a^{-1}, b, b^{-1}\}$ (that is, a standard representative for an element of F). We denote by $f(w)$ the word obtained by forgetting the last letter of w. With respect to the word metric on F induced by S, the map $Z : F \to F$ defined by $Z(e) = e$ and $Z(w) = f(w)$ for $w \neq e$ moves points by a distance of one at the most. The set of preimages of any point contains at least three points.

More generally, let Γ be a finitely generated group containing an isomorphic copy of F. Let S be a generating set of Γ containing a and b. Let T be a left-transversal of F in Γ. That is,

$$\Gamma = \bigcup_{t\in T} tF$$

and the union is disjoint. We can extend the map Z to Γ as follows:

$$Z(t) = t, \qquad Z(tw) = tf(w).$$

With respect to the word metric on Γ induced by S, the map Z moves points by a distance of one at the most. The set of preimages of any point contains at least three points. Hence the above proposition shows that Γ has the property of strong isoperimetry.

4. The quasi-isometry relation

DEFINITION 4.1. Two metric spaces X, Y are *quasi-isometric* if there exist a constant $\lambda > 1$ and maps $f : X \to Y$ and $g : Y \to X$ such that:

1. $d(f(x), f(x')) \le \lambda d(x, x') + \lambda,\ \forall x, x' \in X.$
2. $d(g(y), g(y')) \le \lambda d(y, y') + \lambda,\ \forall y, y' \in Y.$
3. $d(g \circ f(x), x) \le \lambda,\ \forall x \in X.$
4. $d(f \circ g(y), y) \le \lambda,\ \forall y \in Y.$

The map f is a *quasi-isometry*, g (which is also a quasi-isometry) is a *quasi-inverse* of f, and λ is a *quasi-isometry constant* for f.

Let X be a metric space and let $R > 0$. Let $\Omega \subset X$. Let

$$N_R(\Omega) = \{x \in X : \exists x' \in \Omega : d(x, x') \le R\}$$

be the R-neighborhood of Ω. If $x \in X$ we denote by $d(x, \Omega) = \inf_{y\in\Omega} d(x, y)$ the distance between x and Ω.

PROPOSITION 4.1 (cf. [Ka], Lemma 4.2). *Let X and Y be two connected graphs with $\deg(X) < \infty$ and $\deg(Y) < \infty$. If X is quasi-isometric to Y then X has the property of strong isoperimetry if and only if Y does.*

Proof. Let V be the set of vertices of X with metric induced by inclusion. Let W be the set of vertices of Y with metric induced by inclusion. Let $f : V \to W$ be a quasi-isometry. The proposition is an immediate consequence of the following statement (extracted from [Pi]).

There are constants $C > 1$ and $R > 0$ such that for all finite subsets $\Omega \subset V$,

$$|\partial N_R(f(\Omega))| \le C|\partial\Omega|, \qquad |f(\Omega)| \le |\Omega| \le C|f(\Omega)|.$$

We prove the first inequality. Let g be a quasi-inverse of f and let λ be a constant of quasi-isometry. We can assume $\lambda \in \mathbb{N}$. We choose $R = \lambda + 1$. We want to define a map

$$h : \partial N_R(f(\Omega)) \to \partial\Omega$$

which is "almost injective". First, we notice that if $y \in \partial N_R(f(\Omega))$ then $g(y) \notin \Omega$. This is because if $g(y) \in \Omega$ then

$$d(y, f(\Omega)) \le d(y, f \circ g(y)) \le \lambda < R$$

and this contradicts $y \in \partial N_R(f(\Omega))$. We choose $x \in \Omega$ such that

$$d(g(y), x) = d(g(y), \Omega).$$

As $g(y) \notin \Omega$ it follows that $x \in \partial\Omega$. We put $h(y) = x$.

Now we check that there is a constant $C > 1$ such that, if $x \in \partial\Omega$ then $|h^{-1}(x)| \le C$. Let $y \in h^{-1}(x)$. Then

$$\begin{aligned} d(g(y), x) = d(g(y), \Omega) &\le \lambda d(f \circ g(y), f(\Omega)) + \lambda \\ &\le \lambda(\lambda + d(y, f(\Omega))) + \lambda \le \lambda^2 + \lambda R + \lambda = M. \end{aligned}$$

Hence

$$d(y, f(x)) \le d(f \circ g(y), f(x)) + \lambda \le \lambda M + 2\lambda.$$

We choose

$$C = \deg(Y)^{\lambda M + 2\lambda}.$$

This proves the first inequality of the above statement. The others are obvious. ■

5. Proof of the theorem. Let X be a metric space. Let $m > 0$. We consider subsets U of X with the following property (cf. [Gr], 1A): if $u, v \in U$ and if $u \ne v$ then

$$d(u, v) \ge m.$$

If V is such a subset which is maximal with respect to inclusion, we also get

$$\forall x \in X,\ \exists v \in V: \quad d(x, v) < m.$$

Such a subset is called *maximal m-separated*.

LEMMA 5.1. *Let X be a connected graph with path metric in which each edge has length one. Assume that there is an $r > 0$ such that for every x in X the set $X \setminus B_r(x)$ has at least three unbounded connected components. Let $m \in \mathbb{N}$ be such that $m > 4r + 2$. Let V be a subset of the vertices of X which is maximal m-separated. Then there exists a map $Z : V \to V$ with the following properties*:

$$d(Z(v), v) \le 2m + 1, \quad |Z^{-1}(v)| \ge 2, \quad \forall v \in V.$$

Proof. (The following definition of Z was suggested by G. Levitt.) We choose a base point $v_0 \in V$. Let $v \in V$. If $d(v_0, v) \le m$, we put $Z(v) = v_0$. If $d(v_0, v) > m$, we choose a geodesic segment g_v of X between v and v_0. Let $x \in g_v$ be the point of g_v at a distance of $m + 1$ from v. Let $w \in V$ be such that

$$d(x, w) = d(x, V).$$

We define $Z(v) = w$. If $v \in V$ and if $d(v_0, v) > m$, we have

$$d(v, Z(v)) \leq d(v, x) + d(x, w) < m + 1 + m$$

by maximality of V. We want to show that

$$|Z^{-1}(u)| \geq 2, \quad \forall u \in V.$$

Claim. *Let $u \in V$. Let C be an unbounded connected component of $X \setminus B_r(u)$ such that $v_0 \notin C$. Let $v \in C \cap V$ be such that*

$$d(v_0, v) = d(v_0, C \cap V).$$

(*Such a v exists because C is unbounded and V is maximal.*) *Then $Z(v) = u$.*

The claim implies the desired property of Z because by hypothesis $X \setminus B_r(u)$ has at least three unbounded components.

We remark that if A and B are two distinct connected components of $X \setminus B_r(u)$ and if $a \in A \cap V$ and $b \in B \cap V$ then

$$d(a, b) \geq d(a, u) - r + d(b, u) - r \geq 2m - 2r > m$$

(the first inequality above holds because a geodesic path starting from a and ending at b must meet $B_r(u)$). To prove the above claim in the case $d(v, v_0) = m$ we have to show that $v_0 = u$. The point v_0 cannot be in a connected component of $X \setminus B_r(u)$: it is not in C by hypothesis, moreover, according to the above inequality and as we assume $d(v, v_0) = m$, it is not in any other component. We conclude that $v_0 \in B_r(u)$. But $r < m$ and V is m-separated, which implies that $u = v_0$.

From now on, we assume that $d(v, v_0) \geq m + 1$. We denote by x the point on the chosen geodesic segment g_v between v and v_0 at distance $m+1$ from v. To prove the above claim we need two intermediate steps.

Step 1. *Either $x \in C$ or $d(x, u) \leq 2r + 1$ (or both).*

To prove it, we define $l = d(x, B_r(u))$. If $x \in X \setminus (C \cup B_r(u))$ we have

$$m + 1 = d(x, v) \geq l + m - r$$

(the above inequality is clear: a geodesic path starting from x and ending at v must meet $B_r(u)$). Hence $r + 1 \geq l$. This shows that

$$d(x, u) = l + r \leq 2r + 1$$

and proves Step 1.

Step 2. *Let D be a component of $X \setminus B_r(u)$ other than C. If $v' \in V \cap D$ then*

$$d(x, u) < d(x, v').$$

To prove it, we first assume that $x \in C$. We have

$$d(x, v') \geq d(u, x) - r + d(u, v') - r \geq d(u, x) + m - 2r > d(u, x).$$

If $x \notin C$, we can assume, thanks to Step 1, that $d(x,u) \le 2r+1$. But

$$d(x,v') \ge d(u,v') - d(u,x) \ge m - (2r+1) > 2r+1$$

because $m > 4r+2$. This proves Step 2.

Let v be such that $d(v_0,v) > m$. Let $w = Z(v)$. The point w cannot be in C because we have

$$d(v_0,w) \le m + d(v_0,x) < m+1+d(v_0,x) = d(v_0,v)$$

and v realizes the distance between v_0 and the set $V \cap C$. Thanks to the minimality condition

$$d(x,V) = d(x,w)$$

on w, Step 2 (applied with $v' = w$) shows that w cannot be in any other component of $X \setminus B_r(u)$. Hence $w \in B_r(u)$. That is, $w = u$. ∎

To conclude the proof of the theorem, we consider the graph X' obtained from the metric space V as follows. The set of vertices of X' is V. The edges of X' are the couples of points $u,v \in V$ such that $d_X(u,v) \le 2m+1$. Notice that $\deg(X') < \infty$. As V is a maximal m-separated subset of X, the graph X' is connected and quasi-isometric to X (the choice of the constant $2m+1$ allows associating with each path of X between two points u,v of V a path in X' of the same length or shorter, joining u to v). According to Lemma 5.1 and Proposition 2.1, the graph X' has the strong isoperimetry property. Proposition 4.1 implies that so does X.

6. The marriage lemma. The aim of this section, which is independent of the preceding ones, is to give a self-contained proof of Proposition 6.2 which is the reciprocal of Proposition 2.1. See [CGH] for equivalent conditions in terms of pseudogroups and paradoxical decompositions. Lemma 6.1 below is the version we need of the marriage lemma of P. Hall (cf. [Kri]).

Recall that in a partially ordered set (poset), a subset which is totally ordered is called a *chain*. An *antichain* in a poset P is a subset $A \subset P$ such that no two elements are comparable, i.e. if $a,b \in A$ and $a \le b$ then $a = b$.

PROPOSITION 6.1 (Dilworth). *Let P be a finite poset. Let d be the maximal cardinality of an antichain of P. Then P is a disjoint union of d chains.*

Proof. If $|P| = 1$ this is true. Assume this is true for all posets of cardinality less than $|P|$. We consider two cases.

CASE 1: *There is an antichain $A \subset P$ of maximal cardinality such that both sets*

$$P^+ = \{x \in P : \exists a \in A : x \ge a\},$$
$$P^- = \{x \in P : \exists a \in A : x \le a\}$$

strictly contain A.

Notice that $P^+ \cap P^- = A$ because A is an antichain and that $P^+ \cup P^- = P$ because A is an antichain of maximal cardinality. The antichain A is a fortiori of maximal cardinality, say d, in the poset P^+. Hence by the induction hypothesis we can decompose P^+ into disjoints chains $T_1, \ldots, T_d$. The same is true for $A \subset P^-$ and we can decompose P^- into disjoints chains $B_1, \ldots, B_d$. Each intersection $A \cap T_i$ is a singleton. The same is true for $A \cap B_i$. After a permutation of the indices of the B_i we get d chains $T_1 \cup B_1, \ldots, T_d \cup B_d$ which partition P.

CASE 2: *Any antichain $A \subset P$ of maximal cardinality either contains only maximal elements or contains only minimal elements.*

This implies that an antichain of maximal cardinality coincides either with the set μ of minimal elements of P or with the set $\mathcal{M}$ of maximal elements in P. Let $M = \max\{|\mu|, |\mathcal{M}|\}$. Let $m_0 \in \mu$ and $M_0 \in \mathcal{M}$ be such that $m_0 \leq M_0$ (the case $m_0 = M_0$ is not excluded). Let A be an antichain of maximal cardinality in the poset $P \setminus \{m_0, M_0\}$. Since either $\mu \setminus \{m_0\}$ or $\mathcal{M} \setminus \{M_0\}$ has cardinality $M - 1$, we deduce that $|A| \geq M - 1$.

We claim that $|A| \leq M - 1$. Otherwise A would be an antichain of maximal cardinality in P, hence A would coincide with μ or with $\mathcal{M}$. But this is absurd since neither m_0 nor M_0 belongs to A. By the induction hypothesis $P \setminus \{m_0, M_0\}$ is a disjoint union of chains $C_1, \ldots, C_{M-1}$. Hence P is the disjoint union of the chains $C_1, \ldots, C_{M-1}, \{m_0, M_0\}$. ∎

Let $X(A, B)$ be a *bipartite graph*, that is, the set of vertices decomposes as $V(X) = A \sqcup B$ and the edges satisfy $E(X) \subset A \times B$. If $\Omega \subset A$ we define

$$R(\Omega) = \{b \in B : \exists a \in \Omega : (a, b) \in E(X)\}.$$

We will use the notation $R_X(\Omega)$ when specifying the graph is needed.

LEMMA 6.1 (The marriage lemma of P. Hall). *Let $X(A, B)$ be a finite bipartite graph. Let $n \in \mathbb{N}$. The following conditions are equivalent.*

1. $|R(\Omega)| \geq n|\Omega|$, $\forall \Omega \subset A$.
2. *There exists a set of injection(s) $i_s : A \to B$, $s = 1, \ldots, n$, with $(a, i_s(a)) \in E(X)$ for all $a \in A$ and $1 \leq s \leq n$ and with $i_s(A) \cap i_t(A) = \emptyset$ if $s \neq t$.*

Proof. Notice that the second condition implies the first one. It is enough to prove the lemma in the special case $n = 1$. The general case is obtained by induction on n as follows. Assume the lemma is true for 1 and n. We want to show that it is true for $n + 1$. By hypothesis

$$|R(\Omega)| \geq (n + 1)|\Omega|, \quad \forall \Omega \subset A,$$

in particular $|R(\Omega)| \geq |\Omega|$. The induction hypothesis implies that there exists an injection $i : A \to B$ with $(a, i(a)) \in E(X)$. Let Y be the graph obtained from X by removing the set of vertices $i(A)$ and all edges with a vertex in $i(A)$. The graph $Y = Y(A, B \setminus i(B))$ is bipartite. If Ω is a subset of A then

$$R_X(\Omega) = R_Y(\Omega) \sqcup i(A),$$

hence $|R_Y(\Omega)| \geq n|\Omega|$. The induction hypothesis applied to Y implies the existence of n injections $i_1, \ldots, i_n$ from A into $B \setminus i(A)$. Thus the $n+1$ injections $i, i_1, \ldots, i_n$ from A into B have the required properties.

The bipartite graph $X(A, B)$ defines a partial order on $V(X)$: $a < b$ if and only if $(a, b) \in E(X)$. Let us prove that B is an antichain of maximal cardinality. Let $U \subset V(X)$ be an antichain. We have

$$|U| = |U \cap A| + |U \cap B|.$$

By hypothesis $|R(U \cap A)| \geq |U \cap A|$. Hence

$$|U| \leq |R(U \cap A)| + |U \cap B|.$$

As U is an antichain, the sets $R(U \cap A)$ and $U \cap B$ are two disjoint subsets of B. This proves that $|U| \leq |B|$. By Proposition 6.1 the set $V(X)$ can be partitioned into $|B|$ chains. Let $a \in A$. Let b be the unique element of B contained in the same chain as a. We define $i : A \to B$ by $i(a) = b$. ■

LEMMA 6.2. *Let A and B be countable sets and let $X(A, B)$ be a bipartite graph with $V(X) = A \sqcup B$ and $E(X) \subset A \times B$. Assume that the degree of each vertex of X is finite. Assume that $R(A) = B$ and $|R(\Omega)| \geq 2|\Omega|$ for all finite subsets $\Omega \subset A$. Then there exists a map $f : B \to A$ such that $(f(b), b) \in E(X)$ for all $b \in B$ and $|f^{-1}(a)| \geq 2$ for all $a \in A$.*

Proof. Let $A = \{a_1, a_2, \ldots\}$. Let $A_n = \{a_1, \ldots, a_n\}$ and $B_n = R(A_n)$. Let X_n be the finite bipartite graph whose vertices are $V(X_n) = A_n \sqcup B_n$ and whose edges are $E(X_n) = (A_n \times B_n) \cap E(X)$. By hypothesis, if $\Omega \subset A_n$ then $R(\Omega) \subset B_n$ and $|R(\Omega)| \geq 2|\Omega|$. By Lemma 6.1, for each $n \in \mathbb{N}$ there exist two injections $i_n, j_n : A_n \to B_n$ such that $(a, i_n(a)) \in E(X)$ and $(a, j_n(a)) \in E(X)$ for all $a \in A_n$ and such that $i_n(A_n) \cap j_n(A_n) = \emptyset$.

As B contains the disjoint union $i_n(A_n) \sqcup j_n(A_n)$ we can define an application $f_n : B \to A$ to be the inverse of i_n on $i_n(A_n)$ and the inverse of j_n on $j_n(A_n)$. On the complementary set, we choose for each $b \in B$ an image point $a \in A$ such that $(a, b) \in E(X)$. By construction, $|f_n^{-1}(a)| \geq 2$ for all $a \in A_n$. As the vertices of B have finite degree and as B_n is finite, it follows that for each $n \in \mathbb{N}$ the family of maps

$$\{f_k|_{B_n}\}_{k \in \mathbb{N}}$$

is finite. Hence we can choose for each $n \in \mathbb{N}$ an infinite subset $I_n \subset \mathbb{N}$ such that if $k, l \in I_n$ then $f_k|_{B_n} = f_l|_{B_n}$. Moreover, we can choose the I_n such that $I_n \subset I_m$ if $n \geq m$.

We define the required map $f : B \to A$ as follows. Let $b \in B$. Choose $n \in \mathbb{N}$ such that $b \in B_n$. Choose $k \in I_n$ and define $f(b) = f_k(b)$. From the properties of the sets I_n the map f is well defined and does not depend on the choices of n and $k \in I_n$. Notice that $(f(b), b) \in E(X)$ for all $b \in B$.

Let $a \in A$. We now check that $|f^{-1}(a)| \geq 2$. Let $n \in \mathbb{N}$ with $a \in A_n$. Let $k \in I_n$ be large enough so that $A_n \subset A_k$. As $|f_k^{-1}(x)| \geq 2$ for all $x \in A_k$, let $b \neq b'$ be two points in B_k such that $f_k(b) = f_k(b') = a$. Notice that $R(\{a\}) \subset R(A_n) = B_n$, hence $b, b' \in B_n$ and as $k \in I_n$ by definition of f we have $f(b) = f_k(b) = a$ and $f(b') = f_k(b') = a$. ∎

Before we prove the reciprocal of Proposition 2.1 we recall equivalent formulations of the strong isoperimetric inequality for a connected graph X with uniformly bounded degree. We denote the set of vertices of X by V and in what follows we denote by

$$N_k(\Omega) = \{x \in V : d(x, \Omega) \leq k\}$$

the k-neighborhood of Ω in the discrete metric space V (with metric induced by the path metric of X). The *k-boundary* of Ω is

$$\partial_k \Omega = N_k(\Omega) \setminus \Omega.$$

We say that X *satisfies a strong isoperimetric inequality* if

$$\exists k \geq 1,\ \exists \varepsilon > 0,\ \forall \Omega \subset V, \qquad |\partial_k \Omega| \geq \varepsilon |\Omega|.$$

Notice that as we assume that X is connected with uniformly bounded degree, the existence of a $k \geq 1$ in the above condition implies that the condition holds for all $k \geq 1$. An obviously equivalent formulation of the strong isoperimetric inequality is

$$\exists k \geq 1,\ \exists \varepsilon > 0,\ \forall \Omega \subset V : \quad |N_k(\Omega)| \geq (1 + \varepsilon)|\Omega|.$$

Iterating the operation of taking the k-neighborhood, we get

$$|N_{nk}(\Omega)| \geq (1 + \varepsilon)^n |\Omega|.$$

This shows that if a connected graph with uniformly bounded degree satisfies a strong isoperimetric inequality then for any $m > 0$ there exists k large enough such that

$$|N_k(\Omega)| \geq m|\Omega|$$

for all $\Omega \subset V$. In the following we will need $m = 2$.

Proposition 6.2. *Let X be a connected graph with uniformly bounded degree. Let V be the set of vertices of X. If X satisfies a strong isoperimetric* *inequality then there exists a map $Z : V \to V$ and an integer $k \in \mathbb{N}$ such that*

$$d(Z(v), v) \le k, \quad |Z^{-1}(v)| \ge 2, \quad \forall v \in V.$$

Proof. According to the above discussion we can choose $k \in \mathbb{N}$ large enough such that for all $\Omega \subset V$,

$$|N_k(\Omega)| \ge 2|\Omega|.$$

Let A and B be two copies of the set V of vertices. Consider the bipartite graph $Y(A, B)$ whose vertices are $V(Y) = A \sqcup B$ and whose edges are

$$E(Y) = \{(x, y) \in A \times B : d_X(x, y) \le k\}.$$

The above inequality says that $|R(\Omega)| \ge 2|\Omega|$ for all $\Omega \subset A$. Lemma 6.2 implies the existence of a map $Z : V \to V$ with the required properties. ∎

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