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PARTIAL INTEGRAL OPERATORS IN ORLICZ SPACES WITH MIXED NORM

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Let T and S be two nonempty sets equipped with σ-algebras $\mathfrak{A}(T)$ and $\mathfrak{A}(S)$ and separable σ-finite measures μ and ν, respectively. We assume throughout that μ and ν are atom-free, although some of our results also hold in a more general setting. Let $l : T\times S\times T \to \mathbb{R}$ and $m : T\times S\times S \to \mathbb{R}$ be given measurable functions. The operators

$$Lx(t,s) = \int_T l(t,s,\tau)x(\tau,s)\,d\mu(\tau) \tag{1}$$

and

$$Mx(t,s) = \int_S m(t,s,\sigma)x(t,\sigma)\,d\nu(\sigma) \tag{2}$$

are called *partial integral operators*, inasmuch as the function x is integrated only with respect to one variable, while the other variable is "frozen". The integrals in (1) and (2) are meant in the Lebesgue–Radon sense.

Partial integral operators arise in various fields of applied mathematics, mechanics, engineering, physics, and biology (see e.g. [3, 6–9, 14, 15]).

Since partial integral operators act on functions of two variables, it is natural to study them in *spaces with mixed norm*. For the case of *Lebesgue spaces* this was carried out in [10]. In this paper we propose a parallel approach for the case of *Orlicz spaces*. Passing from Lebesgue to Orlicz spaces is always a useful device if one encounters nonlinear partial integral equations containing nonlinearities of non-polynomial, e.g. exponential, growth.

The plan of this paper is as follows. In the first section we recall some results on the so-called ideal spaces with mixed norm and partial integral

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operators in them. While these results are quite complicated in the abstract setting of ideal spaces, they become more transparent in the Orlicz space setting, as we will show in the second section. In the third section we illustrate our results for the special case of Lebesgue spaces which provide, of course, the most important example in applications.

1. Ideal spaces with mixed norm. Let U and V be two ideal spaces (i.e., L_∞-Banach lattices) with full support [20] over the domains T and S, respectively. We suppose throughout that the spaces U and V are *perfect*, which means that their norms have the Fatou property [20]. Examples of perfect ideal spaces are Lebesgue spaces and Orlicz spaces, as well as Lorentz and Marcinkiewicz spaces which arise in the theory of interpolation of linear operators [2, 13]. The *space with mixed norm* $[U \to V]$ consists, by definition, of all measurable functions $x : T \times S \to \mathbb{R}$ for which the norm

$$\|x\|_{[U\to V]} = \|s \mapsto \|x(\cdot, s)\|_U\|_V \tag{3}$$

is finite. Similarly, the space with mixed norm $[U \leftarrow V]$ is defined by the norm

$$\|x\|_{[U\leftarrow V]} = \|t \mapsto \|x(t, \cdot)\|_V\|_U. \tag{4}$$

Both $[U \to V]$ and $[U \leftarrow V]$ are ideal spaces. If they are *regular* (which means that all their elements have an absolutely continuous norm, see [20]), they are also examples of *tensor products* of U and V. In fact, for any $u \in U$ and $v \in V$ the function w defined by $w(t, s) = u(t)v(s)$ belongs to both $[U \to V]$ and $[U \leftarrow V]$ and satisfies

$$\|w\|_{[U\to V]} = \|w\|_{[U\leftarrow V]} = \|u\|_U\|v\|_V. \tag{5}$$

The most prominent example is of course given by the Lebesgue spaces $[L_p \to L_q]$ and $[L_q \leftarrow L_p]$ ($1 \le p, q \le \infty$) defined by the mixed norms

$$\|x\|_{[L_p\to L_q]} = \begin{cases} \left\{ \int_S \left[\int_T |x(t,s)|^p \, dt \right]^{q/p} ds \right\}^{1/q} & \text{if } 1 \le p, q < \infty, \\ \operatorname*{ess\,sup}_{s\in S} \left[\int_T |x(t,s)|^p \, dt \right]^{1/p} & \text{if } 1 \le p < \infty,\ q = \infty, \\ \left[\int_S \operatorname*{ess\,sup}_{t\in T} |x(t,s)|^q \, ds \right]^{1/q} & \text{if } 1 \le q < \infty,\ p = \infty, \\ \operatorname*{ess\,sup}_{(t,s)\in T\times S} |x(t,s)| & \text{if } p = q = \infty, \end{cases} \tag{6}$$

and

$$(7)\qquad \|x\|_{[L_p\leftarrow L_q]} = \begin{cases} \Big\{ \int\limits_T \Big[\int\limits_S |x(t,s)|^q\, ds \Big]^{p/q} ds \Big\}^{1/p} & \text{if } 1 \le p, q < \infty, \\ \Big[\int\limits_T \operatorname*{ess\,sup}_{s\in S} |x(t,s)|^p\, dt \Big]^{1/p} & \text{if } 1 \le p < \infty,\ q = \infty, \\ \operatorname*{ess\,sup}_{t\in T} \Big[\int\limits_S |x(t,s)|^q\, ds \Big]^{1/q} & \text{if } 1 \le q < \infty,\ p = \infty, \\ \operatorname*{ess\,sup}_{(t,s)\in T\times S} |x(t,s)| & \text{if } p = q = \infty. \end{cases}$$

These spaces are of fundamental importance in the description of kernels of bounded linear integral operators in L_p (see e.g. [18, 19]) and have been studied, for example, in [1]. Some results on general ideal spaces with mixed norm may be found in [4, 5].

In what follows, we shall describe conditions for the operators (1) and (2) to act between spaces with mixed norm. For $t \in T$ and $s \in S$, consider the families $L(s)$ and $M(t)$ of linear integral operators defined by

$$(8)\qquad L(s)u(t) = \int\limits_T l(t,s,\tau)u(\tau)\, d\mu(\tau) \qquad (s \in S)$$

and

$$(9)\qquad M(t)v(s) = \int\limits_S m(t,s,\sigma)v(\sigma)\, d\nu(\sigma) \qquad (t \in T).$$

Given two ideal spaces W_1 and W_2 over the same domain Ω, the *multiplicator space* W_1/W_2 consists, by definition, of all measurable functions w on Ω for which the norm

$$(10)\qquad \|w\|_{W_1/W_2} = \sup\{\|ww_2\|_{W_1} : \|w_2\|_{W_2} \le 1\}$$

is finite. In particular, the space $W' := L_1/W$ is called the *associate space* of an ideal space W. For example, in the case of Lebesgue spaces over a bounded domain we have

$$(11)\qquad L_{p_1}/L_{p_2} = \begin{cases} L_{p_1p_2/(p_2-p_1)} & \text{if } p_1 < p_2, \\ L_\infty & \text{if } p_1 = p_2, \\ \{0\} & \text{if } p_1 > p_2. \end{cases}$$

In particular, $(L_p)' = L_{p'}$ with $1/p + 1/p' = 1$.

The following lemma gives acting conditions for the partial integral operators (1) and (2) in terms of acting conditions for the operator families (8) and (9). As usual, we write $\mathfrak{L}(X,Y)$ for the space of all bounded linear operators between two Banach spaces X and Y; in particular, $\mathfrak{L}(X,X) =: \mathfrak{L}(X)$.

Lemma 1. *Let U_1 and U_2 be two ideal spaces over T, and V_1 and V_2 two ideal spaces over S. Suppose that the linear integral operator* (8) *maps U_1 into U_2, for each $s \in S$, and that the map $s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}$ belongs* *to* V_2/V_1. *Then the partial integral operator* (1) *acts between the spaces* $X = [U_1 \to V_1]$ *and* $Y = [U_2 \to V_2]$ *and satisfies*

$$\|L\|_{\mathfrak{L}(X,Y)} \le \|s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}\|_{V_2/V_1}. \tag{12}$$

Similarly, if the linear integral operator (9) *maps* V_1 *into* V_2, *for each* $t \in T$, *and the map* $t \mapsto \|M(t)\|_{\mathfrak{L}(V_1,V_2)}$ *belongs to* U_2/U_1, *then the partial integral operator* (2) *acts between the spaces* $X = [U_1 \leftarrow V_1]$ *and* $Y = [U_2 \leftarrow V_2]$ *and satisfies*

$$\|M\|_{\mathfrak{L}(X,Y)} \le \|t \mapsto \|M(t)\|_{\mathfrak{L}(V_1,V_2)}\|_{U_2/U_1}. \tag{13}$$

Proof. Without loss of generality, we only prove the first statement. Given $x \in X = [U_1 \to V_1]$, for almost all $s \in S$ we have

$$\|Lx(\cdot,s)\|_{U_2} \le \|L(s)\|_{\mathfrak{L}(U_1,U_2)}\|x(\cdot,s)\|_{U_1},$$

hence, by the definition of the multiplicator space V_2/V_1,

$$\begin{aligned}\|Lx\|_Y = \|s \mapsto \|Lx(\cdot,s)_{U_2}\|_{V_2} &\le \|s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}\|x(\cdot,s)\|_{U_1}\|_{V_2}\\ &\le \|s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}\|_{V_2/V_1}\|x\|_X.\end{aligned}$$

This shows that the operator (1) acts between X and Y and satisfies (12). ∎

Interestingly, in the case $V_2/V_1 = L_\infty$ the conditions of Lemma 1 are also necessary for the operator (1) to act between $X = [U_1 \to V_1]$ and $Y = [U_2 \to V_2]$. In fact, considering the operator (1) on the "separated" functions $x(t,s) = u(t)v(s)$, where $u \in U_1$ and $v \in V_1$, we see that, by the obvious relation $Lx(t,s) = v(s)L(s)u(t)$,

$$\sup_{\|u\|_{U_1}\le 1} \|s \mapsto \|v(s)L(s)u\|_{U_2}\|_{V_2} \le \|L\|_{\mathfrak{L}(X,Y)}\|v\|_{V_2/V_1}. \tag{14}$$

In case $V_2/V_1 = L_\infty$ this means exactly that

$$\|s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}\|_{V_2/V_1} \le \|L\|_{\mathfrak{L}(X,Y)}, \tag{15}$$

i.e. equality holds in (12). Analogous statements are valid, of course, for the operator (2) in case $U_2/U_1 = L_\infty$. For example, the equalities

$$\|L\|_{\mathfrak{L}([L_{p_1}\to L_q],[L_{p_2}\to L_q])} = \|s \mapsto \|L(s)\|_{\mathfrak{L}(L_{p_1},L_{p_2})}\|_{L_\infty}$$

and

$$\|M\|_{\mathfrak{L}([L_p\leftarrow L_{q_1}],[L_p\leftarrow L_{q_2}])} = \|t \mapsto \|M(t)\|_{\mathfrak{L}(L_{q_1},L_{q_2})}\|_{L_\infty}$$

are true for $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$.

To state the first theorem, some notation is in order. Given two ideal spaces X and Y, we denote by $\mathfrak{N}_l(X,Y)$ the linear space of all measurable functions $l : T \times S \times T \to \mathbb{R}$ with finite norm

$$\|l\|_{\mathfrak{N}_l(X,Y)} = \sup_{\|x\|_X\le 1} \Big\|(t,s) \mapsto \int_T |l(t,s,\tau)x(\tau,s)|\,d\mu(\tau)\Big\|_Y. \tag{16}$$

Similarly, $\mathfrak{R}_m(X,Y)$ denotes the linear space of all measurable functions $m : T \times S \times S \to \mathbb{R}$ with finite norm

$$\|m\|_{\mathfrak{R}_m(X,Y)} = \sup_{\|x\|_X \le 1} \Big\|(t,s) \mapsto \int_S |m(t,s,\sigma)x(t,\sigma)|\, d\nu(\sigma)\Big\|_Y. \tag{17}$$

Denote by $\theta = (\theta_1, \theta_2, \theta_3)$ an arbitrary permutation of the arguments $(t,s,\tau) \in T \times S \times T$, or $(t,s,\sigma) \in T \times S \times S$. Given three ideal spaces W_1, W_2, and W_3, we denote by $[W_1, W_2, W_3; \theta]$ the ideal space of all functions w of three variables for which the norm

$$\|w\|_{[W_1,W_2,W_3;\theta]} := \|\theta_3 \mapsto \|\theta_2 \mapsto \|\theta_1 \mapsto w(\theta_1,\theta_2,\theta_3)\|_{W_1}\|_{W_2}\|_{W_3}$$

is defined and finite. Recall that a linear operator A between two ideal spaces is called *regular* [19] if A may be represented as a difference of two positive operators. Building on classical results on linear integral operators, the following theorem was proved in [10]:

THEOREM 1. *Let U_1 and U_2 be two ideal spaces over T, and V_1 and V_2 two ideal spaces over S. Suppose that $l \in [U_2, V_2/V_1, U_1'; \theta]$ for some $\theta = (\theta_1, \theta_2, \theta_3)$. Then the partial integral operator* (1) *acts between X and Y, is regular, and satisfies*

$$\|l\|_{\mathfrak{R}_l(X,Y)} \le \|l\|_{[U_2,V_2/V_1,U_1';\theta]}. \tag{18}$$

Here the spaces X and Y have to be chosen according to the formula

$$\begin{cases} X = [U_1 \leftarrow V_1],\ Y = [U_2 \leftarrow V_2] & \text{if } \theta = (s,t,\tau) \text{ or } \theta = (s,\tau,t), \\ X = [U_1 \to V_1],\ Y = [U_2 \to V_2] & \text{if } \theta = (t,\tau,s) \text{ or } \theta = (\tau,t,s), \\ X = [U_1 \leftarrow V_1],\ Y = [U_2 \to V_2] & \text{if } \theta = (t,s,\tau), \\ X = [U_1 \to V_1],\ Y = [U_2 \leftarrow V_2] & \text{if } \theta = (\tau,s,t). \end{cases}$$

Similarly, if $m \in [U_2/U_1, V_2, V_1'; \theta]$ for some $\theta = (\theta_1, \theta_2, \theta_3)$, then the partial integral operator (2) *acts between X and Y, is regular, and satisfies*

$$\|m\|_{\mathfrak{R}_m(X,Y)} \le \|m\|_{[U_2/U_1,V_2,V_1';\theta]}. \tag{19}$$

Here the spaces X and Y have to be chosen according to the formula

$$\begin{cases} X = [U_1 \leftarrow V_1],\ Y = [U_2 \leftarrow V_2] & \text{if } \theta = (s,\sigma,t) \text{ or } \theta = (\sigma,s,t), \\ X = [U_1 \to V_1],\ Y = [U_2 \to V_2] & \text{if } \theta = (t,s,\sigma) \text{ or } \theta = (t,\sigma,s), \\ X = [U_1 \leftarrow V_1],\ Y = [U_2 \to V_2] & \text{if } \theta = (\sigma,t,s), \\ X = [U_1 \to V_1],\ Y = [U_2 \leftarrow V_2] & \text{if } \theta = (s,t,\sigma). \end{cases}$$

2. Orlicz spaces with mixed norm. Of course, the formulation of Theorem 1 is very clumsy, and its hypotheses are hard to verify. We are now going to show that the assertion of Theorem 1 can be made more explicit in case of Orlicz spaces; this is the main part of the paper.

For an exhaustive self-contained account of the theory and applications of Orlicz spaces we refer to the monographs [11] and [16]; let us just recall some basic notions and results which we will need in what follows.

Given a bounded domain Ω and a Young function $M : \mathbb{R} \to [0, \infty)$, the Orlicz space $L_M = L_M(\Omega)$ is defined by one of the (equivalent) norms

$$\|u\|_{L_M} = \inf \Big\{ k : k > 0,\ \int_\Omega M[|x(\omega)|/k]\, d\mu(\omega) \le 1 \Big\} \tag{20}$$

or

$$|||u|||_{L_M} = \inf_{0<k<\infty} \frac{1}{k} \Big[1 + \int_\Omega M[k|x(\omega)|]\, d\mu(\omega) \Big]. \tag{21}$$

We will use the norm (20) in the sequel and always write $d\omega$ rather than $d\mu(\omega)$. Given two Young functions M and N, we write $M \preceq N$ if there exist $k > 0$ and $u_0 \ge 0$ such that

$$M(u) \le N(ku) \quad (u \ge u_0).$$

Moreover, we write $M \prec N$ if

$$\lim_{u\to\infty} \frac{M(u)}{N(ku)} = 0$$

for every $k > 0$. Of course, in case $M(u) = |u|^p$ and $N(u) = |u|^q$ ($1 \le p$, $q < \infty$) we have $M \preceq N$ if and only if $p \le q$, and $M \prec N$ if and only if $p < q$. In general, one can show that $M \preceq N$ is equivalent to the fact that L_N is continuously imbedded in L_M, and $M \prec N$ is equivalent to the fact that L_N is absolutely continuously imbedded in L_M (i.e., the unit ball of L_N is an absolutely bounded subset of L_M). Moreover, the inclusions $L_\infty \subseteq L_M \subseteq L_1$ are true for any Orlicz space over a bounded domain.

Let $U = L_M(T)$ and $V = L_N(S)$ be two Orlicz spaces. We are interested in the Orlicz spaces with mixed norm $[U \to V]$ and $[U \leftarrow V]$ defined by (3) and (4), respectively. These spaces are perfect ideal spaces. They are regular if and only if the Young functions M and N satisfy a Δ_2-condition [11]. If $M_2, N_2 \preceq M_1, N_1$ then the inclusions

$$[L_{M_1}(T) \to L_{N_1}(S)] \subseteq [L_{M_2}(T) \to L_{N_2}(S)],$$
$$[L_{M_1}(T) \leftarrow L_{N_1}(S)] \subseteq [L_{M_2}(T) \leftarrow L_{N_2}(S)]$$

are obvious. Moreover, the inclusions

$$L_M(T \times S) \subseteq [L_1(S) \to L_M(T)], \qquad [L_M(T) \leftarrow L_1(S)] \subseteq L_1(T \times S) \tag{22}$$

follow from the Jensen integral inequality

$$M\left(\frac{1}{\mu(\Omega)}\int_{\Omega} x(\omega)\,d\omega\right) \le \frac{1}{\mu(\Omega)}\int_{\Omega} M(x(\omega))\,d\omega$$

and the definition of the norm in L_M. In fact, for $x \in L_M$ and $k > 0$ sufficiently large we have

$$\int_S M\left[\frac{1}{k}\int_T |x(t,s)|\,dt\right]ds = \int_S M\left[\frac{1}{\mu(T)}\int_T \frac{\mu(T)|x(t,s)|}{k}\,dt\right]ds$$
$$\le \frac{1}{\mu(T)}\int_S\int_T M\left[\frac{\mu(T)|x(t,s)|}{k}\right]dt\,ds < \infty.$$

Consequently, $x \in [L_1 \to L_M]$, and hence the left inclusion in (22) is proved. The right inclusion is proved analogously.

LEMMA 2. *Let M_i and N_i $(i = 1, 2)$ be Young functions satisfying*

$$N_2(u)M_2(vw) \le a + N_1(k_1uv)M_1(k_2w) \quad (v \ge v_0), \tag{23}$$

where a, k_1, k_2, v_0 are positive constants, and let $L_{M_i} = L_{M_i}(T)$ and $L_{N_i} = L_{N_i}(S)$ $(i = 1, 2)$. Then

$$[L_{M_1} \leftarrow L_{N_1}] \subseteq [L_{M_2} \to L_{N_2}].$$

Proof. Put $X := [L_{M_1} \leftarrow L_{N_1}]$ and $Y := [L_{M_2} \to L_{N_2}]$. By virtue of (23) we can find a constant c such that $\|z\|_{L_{M_1}} \le 1/k_2$ implies

$$\int_T M_2(v_0 z(t))\,dt \le c. \tag{24}$$

Fix some positive function u_0 in the space $[L_{M_1} \leftarrow L_{N_1}] \cap [L_{M_2} \to L_{N_2}]$, and denote by E_{u_0} the linear space of all $x \in L_1(T \times S)$ with finite norm

$$\|x\|_{E_{u_0}} = \inf\{\lambda : |x(t,s)| \le \lambda u_0(t,s)\}.$$

Now, for $x \in E_{u_0}$ with $\|x\|_X \le (k_1k_2)^{-1}$ and all $\lambda > 1$ we have

$$1 \le \frac{1}{\lambda}\int_T M_2\left[\lambda \frac{x(t,s)}{\|x(\cdot,s)\|_{L_{M_2}}}\right]dt.$$

Since $\|x(t,\cdot)\|_{L_{N_1}} \le (k_1k_2)^{-1}$, from (24) we get

$$\int_T M_2(k_1 v_0 \|x(t,\cdot)\|_{L_{N_1}})\,dt \le c.$$

Consequently, our hypothesis (23) implies that

$$\begin{aligned}
&\int_S N_2\left[\frac{1}{\lambda}\|x(\cdot,s)\|_{L_{M_2}}\right]ds \\
&\quad\le \frac{1}{\lambda}\int_S\int_T N_2\left[\frac{1}{\lambda}\|x(\cdot,s)\|_{L_{M_2}}\right]M_2\left[\lambda\frac{x(t,s)}{\|x(\cdot,s)\|_{L_{M_2}}}\right]dt\,ds \\
&\quad\le \frac{1}{\lambda}\int_S\int_T N_2\left[\frac{1}{\lambda}\|x(\cdot,s)\|_{L_{M_2}}\right] \\
&\qquad\times M_2\left[\max\left\{v_0,\frac{\lambda x(t,s)}{k_1\|x(\cdot,s)\|_{L_{M_2}}\|x(t,\cdot)\|_{L_{N_1}}}\right\}k_1\|x(t,\cdot)\|_{L_{N_1}}\right]dt\,ds \\
&\quad\le \frac{1}{\lambda}\int_S\int_T N_2\left[\frac{1}{\lambda}\|x(\cdot,s)\|_{L_{M_2}}\right]M_2(v_0k_1\|x(t,\cdot)\|_{L_{N_1}})\,dt\,ds+\frac{a}{\lambda}\mu(T)\nu(S) \\
&\qquad+\frac{1}{\lambda}\int_S\int_T N_1\left[\frac{x(t,s)}{\|x(t,\cdot)\|_{L_{N_1}}}\right]M_1(k_1k_2\|x(t,\cdot)\|_{L_{N_1}})\,dt\,ds \\
&\quad\le \frac{c}{\lambda}\int_S N_2\left[\frac{1}{\lambda}\|x(\cdot,s)\|_{L_{M_2}}\right]ds+\frac{1}{\lambda}(a\mu(T)\nu(S)+1).
\end{aligned}$$

Putting now $\lambda := a\mu(T)\nu(S)+c+1$ in the last inequality, we obtain

$$\int_S N_2\left[\frac{\|x(\cdot,s)\|_{L_{M_2}}}{a\mu(T)\nu(S)+c+1}\right]ds\le 1,$$

hence

$$\|x\|_Y\le a\mu(T)\nu(S)+c+1$$

by the definition of the norm (20). Since the last inequality holds for all functions $x\in E_{u_0}$ with $\|x\|_X\le(k_1k_2)^{-1}$, we conclude that

$$\|x\|_Y\le k_1k_2(a\mu(T)\nu(S)+c+1)\|x\|_X$$

for any $x\in E_{u_0}$. Furthermore, for arbitrary $n\in\mathbb{N}$ we then have

$$\|\min\{|x|,nu_0\}\|_Y\le k_1k_2(a\mu(T)\nu(S)+c+1)\|x\|_X$$

for $x\in X$. Finally, since the space Y is perfect we see that

$$\|x\|_Y\le k_1k_2(a\mu(T)\nu(S)+c+1)\|x\|_X$$

for all $x\in X$, as claimed. ∎

From Lemma 2 it follows, in particular, that $[L_M\to L_M]$ is always isomorphic to $[L_M\leftarrow L_M]$. As was shown in [17], $L_M(T\times S)$ is isomorphic to $[L_M\to L_M]$ if and only if the inequalities

$$M(u)M(v)\le a_1+b_1M(k_1uv),\qquad M(k_2uv)\le a_2+b_2M(u)M(v) \tag{25}$$

hold for some constants $a_i,b_i,k_i>0$ $(i=1,2)$.

Let us now consider some acting conditions for the operators (1) and (2) in Orlicz spaces with mixed norm. As in the case of Lebesgue spaces, the study of partial integral operators in these spaces is more convenient than in ordinary Orlicz spaces.

Given two Young functions M_1 and M_2 with $M_1 \prec M_2$, we denote by $M_1 : M_2$ the Young function defined by

$$(M_1 : M_2)(u) = \sup\{M_1(uv) - M_2(v) : 0 < v < \infty\}. \tag{26}$$

In case $M_i(u) = |u|^{p_i}$ $(i = 1, 2)$ with $p_1 < p_2$ this gives just $(M_1 : M_2)(u) = \text{const} \cdot |u|^{p_1 p_2/(p_2 - p_1)}$. In general, one may show that the multiplicator space with norm (10) of two Orlicz spaces L_{M_1} and L_{M_2} is precisely, up to equivalence of norms,

$$L_{M_1}/L_{M_2} = \begin{cases} L_{M_1:M_2} & \text{if } M_1 \prec M_2, \\ L_\infty & \text{if } M_1 \preceq M_2 \text{ but } M_1 \not\prec M_2, \\ \{0\} & \text{if } M_2 \prec M_1. \end{cases} \tag{27}$$

In particular, the associate space $L'_M = L_1/L_M$ of an Orlicz space L_M coincides with the Orlicz space $L_{M'}$ generated by the (associate) Young function

$$M'(u) = \sup\{|uv| - M(v) : 0 < v < \infty\}.$$

In what follows we suppose that M_i and N_i are given Young functions, $U_i = L_{M_i}(T)$, and $V_i = L_{N_i}(S)$ $(i = 1, 2)$. First of all, from Lemma 1 and the explicit formula (27) for the multiplicator space of two Orlicz spaces we get the following

THEOREM 2. *Let $V = L_\infty(S)$ if $N_2 \preceq N_1$ and $N_2 \not\prec N_1$, and $V = L_{N_2:N_1}$ if $N_2 \prec N_1$. Suppose that the operator* (8) *acts between U_1 and U_2 for each $s \in S$, and the map $s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}$ belongs to V. Then the operator* (1) *acts between $X = [U_1 \to V_1]$ and $Y = [U_2 \to V_2]$.*

Similarly, let $U = L_\infty(T)$ if $M_2 \preceq M_1$ and $M_2 \not\prec M_1$, and $U = L_{M_2:M_1}(S)$ if $M_2 \prec M_1$. Suppose that the operator (9) *acts between V_1 and V_2 for each $t \in T$, and the map $t \mapsto \|M(t)\|_{\mathfrak{L}(V_1,V_2)}$ belongs to U. Then the operator* (2) *acts between $X = [U_1 \leftarrow V_1]$ and $Y = [U_2 \leftarrow V_2]$.*

We recall that the conditions $\|L(\cdot)\|_{\mathfrak{L}(U_1,U_2)} \in V$ and $\|M(\cdot)\|_{\mathfrak{L}(V_1,V_2)} \in U$ are necessary for the operators (1) and (2), respectively, to act in the indicated spaces.

If the hypotheses of Theorem 2 are satisfied, then the following estimates are direct consequences of (12) and (13):

$$\|L\|_{\mathfrak{L}(X,Y)} \le \|s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}\|_{L_\infty} \qquad (N_2 \preceq N_1,\ N_2 \not\prec N_1), \tag{28}$$

$$\|L\|_{\mathfrak{L}(X,Y)} \le c_N \|s \mapsto \|L(s)\|_{\mathfrak{L}(U_1,U_2)}\|_{L_{N_1:N_2}} \qquad (N_2 \prec N_1), \tag{29}$$

$$\|M\|_{\mathfrak{L}(X,Y)} \le \|t \mapsto \|M(t)\|_{\mathfrak{L}(V_1,V_2)}\|_{L_\infty} \qquad (M_2 \preceq M_1,\ M_2 \not\prec M_1), \tag{30}$$

$$\|M\|_{\mathfrak{L}(X,Y)} \le c_M \|t \mapsto \|M(t)\|_{\mathfrak{L}(V_1,V_2)}\|_{L_{M_1:M_2}} \qquad (M_2 \prec M_1); \tag{31}$$

here c_N denotes the imbedding constant of $L_{N_2:N_1} \hookrightarrow L_{N_2}/L_{N_1}$, and c_M denotes the imbedding constant of $L_{M_2:M_1} \hookrightarrow L_{M_2}/L_{M_1}$. In particular, the estimate (28) holds if $N_1 = N_2$, and the estimate (30) holds if $M_1 = M_2$. Moreover, the following theorem is true.

THEOREM 3. *Suppose that the operator* (8) *acts in* $L_M(T)$ *for each* $s \in S$ *and* $\|L(\cdot)\|_{\mathfrak{L}(L_M)} \in L_\infty$, *while the operator* (9) *acts in* $L_M(S)$ *for each* $t \in T$ *and* $\|M(\cdot)\|_{\mathfrak{L}(L_M)} \in L_\infty$. *Then the operators* (1) *and* (2) *act in each of the spaces* $X = [L_M \to L_M]$ *and* $Y = [L_M \leftarrow L_M]$. *Moreover, the estimates*

$$\|L\|_{\mathfrak{L}(X)} \le \|s \mapsto \|L(s)\|_{\mathfrak{L}(L_M)}\|_{L_\infty}, \qquad \|M\|_{\mathfrak{L}(Y)} \le \|t \mapsto \|M(t)\|_{\mathfrak{L}(L_M)}\|_{L_\infty}$$

are true. If, in addition, the Young function M *satisfies the inequalities* (25), *then the operators* (1) *and* (2) *act in the Orlicz space* $L_M(T \times S)$ *as well.*

Proof. For the proof it suffices to remark that X is isomorphic to Y and, under the additional hypothesis (25), X is also isomorphic to $L_M(T \times S)$. The assertion then follows from Theorem 2. ∎

We suppose now that $M_i = N_i$ $(i = 1, 2)$ and $M_2 \preceq M_1$. Let $V = L_{M_2:M_1}(S)$ if $M_2 \prec M_1$, and $V = L_\infty(S)$ otherwise. Similarly, we define U with S replaced by T. The following theorem is a straightforward generalization of Theorem 3.

THEOREM 4. *Suppose that the operator* (8) *acts between* $L_{M_1}(T)$ *and* $L_{M_2}(T)$ *for each* $s \in S$ *and* $\|L(\cdot)\| \in V$, *while the operator* (9) *acts between* $L_{M_1}(S)$ *and* $L_{M_2}(S)$ *for each* $t \in T$ *and* $\|M(\cdot)\| \in U$. *Then the operators* (1) *and* (2) *act between* $X \in \{[L_{M_1} \to L_{M_1}], [L_{M_1} \leftarrow L_{M_1}]\}$ *and* $Y \in \{[L_{M_2} \to L_{M_2}], [L_{M_2} \leftarrow L_{M_2}]\}$. *If, in addition, the Young functions* M_1 *and* M_2 *satisfy* (25), *then the operators* (1) *and* (2) *act between the Orlicz spaces* $L_{M_1}(T \times S)$ *and* $L_{M_2}(T \times S)$ *as well.*

A crucial hypothesis in the above theorems is the action of the linear integral operators (8) and (9) between suitable Orlicz spaces. Some simple and effectively verifiable acting conditions for such operators between Orlicz spaces are well known (see e.g. [11] or [16]).

Theorems 2–4 above do not contain regularity conditions for the operators (1) and (2). A simple regularity condition may be obtained, however, by means of the general Theorem 1. In fact, according to Lemma 2 the inclusions $[L_{M_i} \leftarrow L_{N_i}] \subseteq [L_{M_i} \to L_{N_i}]$ and $[L_{M_i} \to L_{N_i}] \subseteq [L_{M_i} \leftarrow L_{N_i}]$ are true if the conditions

$$N_i(u)M_i(v\omega) \le a_i + N_i(b_i uv)M_i(c_i\omega) \quad (v \ge v_i) \tag{A$_i$}$$

and

$$M_i(u)N_i(v\omega) \le \overline{a}_i + M_i(\overline{b}_i uv)N_i(\overline{c}_i\omega) \quad (v \ge \overline{v}_i), \tag{B$_i$}$$

are satisfied, where $a_i, b_i, c_i, v_i, \overline{a}_i, \overline{b}_i, \overline{c}_i$, and $\overline{v}_i$ are positive constants $(i = 1, 2)$. Applying Theorem 1 to our choice of U_i and V_i $(i = 1, 2)$, and using again the explicit formula (27) for the multiplicator spaces L_{M_2}/L_{M_1} and L_{N_2}/L_{N_1}, we arrive at the following result:

THEOREM 5. *Suppose that* $N_2 \preceq N_1$ *and*

$$l \in [L_{M_2}, V, L_{M_1'}; \theta], \tag{32}$$

where $V = L_\infty(S)$ *if* $N_2 \not\prec N_1$, *and* $V = L_{N_2:N_1}(S)$ *otherwise. Then the partial integral operator* (1) *acts between* X *and* Y *according to Table* 1 *below and is regular.*

Similarly, suppose that $M_2 \preceq M_1$ *and*

$$m \in [U, L_{N_2}, L_{N_1'}; \theta], \tag{33}$$

where $U = L_\infty(T)$ *if* $M_2 \not\prec M_1$, *and* $U = L_{M_2:M_1}(T)$ *otherwise. Then the partial integral operator* (2) *acts between* X *and* Y *according to Table* 2 *below and is regular.*

X	$[L_{M_1} \to L_{N_1}]$	$[L_{M_1} \to L_{N_1}]$	$[L_{M_1} \leftarrow L_{N_1}]$	$[L_{M_1} \leftarrow L_{N_1}]$
Y	$[L_{M_2} \to L_{N_2}]$	$[L_{M_2} \leftarrow L_{N_2}]$	$[L_{M_2} \to L_{N_2}]$	$[L_{M_2} \leftarrow L_{N_2}]$
(t, s, τ)	B_1	B_1, B_2		B_2
(t, τ, s)		B_2	A_1	A_1, B_2
(s, t, τ)	B_1, A_2	B_1	A_2	
(s, τ, t)	B_1, A_2	B_1	A_2	
(τ, s, t)	A_2		A_1, A_2	A_1
(τ, t, s)		B_2	A_1	A_1, B_2

Table 1

X	$[L_{M_1} \to L_{N_1}]$	$[L_{M_1} \to L_{N_1}]$	$[L_{M_1} \leftarrow L_{N_1}]$	$[L_{M_1} \leftarrow L_{N_1}]$
Y	$[L_{M_2} \to L_{N_2}]$	$[L_{M_2} \leftarrow L_{N_2}]$	$[L_{M_2} \to L_{N_2}]$	$[L_{M_2} \leftarrow L_{N_2}]$
(t, s, σ)		B_2	A_1	A_1, B_2
(t, σ, s)		B_2	A_1	A_1, B_2
(s, t, σ)	A_2		A_1, A_2	A_1
(s, σ, t)	B_1, A_2	B_1	A_2	
(σ, t, s)	B_1	B_1, B_2		B_2
(σ, s, t)	B_1, A_2	B_1	A_2	

Table 2

We point out that the inclusions (32) and (33) are usually checked by majorant techniques. For example, (32) is certainly satisfied if

$$|l(t,s,\tau)| \le \sum_{i=1}^{n} a_i(t) b_i(s) c_i(\tau),$$

where $a_i \in L_{M_2}(T)$, $b_i \in V$, and $c_i \in L_{M_1'}$. Finally, since $[L_M \to L_M]$ is isomorphic to $[L_M \leftarrow L_M]$, from Theorem 5 we get the following

THEOREM 6. *Suppose that* $l \in [L_M, L_\infty, L_{M'}; \theta']$ *for some* $\theta' = (\theta_1', \theta_2', \theta_3')$ *and* $m \in [L_\infty, L_M, L_{M'}; \theta'']$ *for some* $\theta'' = (\theta_1'', \theta_2'', \theta_3'')$. *Then the partial integral operators* (1) *and* (2) *are regular in the spaces* $[L_M \to L_M]$ *and* $[L_M \leftarrow L_M]$. *If, in addition, the Young function* M *satisfies* (25), *then the operators* (1) *and* (2) *are regular in the Orlicz space* $L_M(T \times S)$ *as well.*

3. Lebesgue spaces with mixed norm. Choosing $M_i(u) = |u|^{p_i}$ and $N_i(u) = |u|^{q_i}$ $(i = 1, 2)$ in the results of the preceding section, we immediately get a series of analogous results in Lebesgue spaces. Since this is straightforward, we do not carry out the details. Let us just see what Theorem 5 and, in particular, Tables 1 and 2 look like in the setting of Lebesgue spaces.

THEOREM 7. *Let* $1 \le p_1, p_2, q_1, q_2 \le \infty$. *Suppose that* $q_1 \ge q_2$ *and*

$$l \in [L_{p_2}, L_{q_1 q_2/(q_1 - q_2)}, L_{p_1/(p_1-1)}; \theta].$$

Then the partial integral operator (1) *acts between* X *and* Y, *is regular, and satisfies*

$$\|l\|_{\mathfrak{R}_l(X,Y)} \le \|l\|_{[L_{p_2}, L_{q_1 q_2/(q_1-q_2)}, L_{p_1/(p_1-1)}; \theta]}, \tag{34}$$

provided one of the conditions of Table 3 *below holds.*

Similarly, suppose that $p_1 \ge p_2$ *and*

$$m \in [L_{p_1 p_2/(p_1 - p_2)}, L_{q_2}, L_{q_1/(q_1-1)}; \theta].$$

Then the partial integral operator (2) *acts between* X *and* Y, *is regular, and satisfies*

$$\|m\|_{\mathfrak{R}_m(X,Y)} \le \|m\|_{[L_{p_1 p_2/(p_1-p_2)}, L_{q_2}, L_{q_1/(q_1-1)}; \theta]}, \tag{35}$$

provided one of the conditions of Table 4 *below holds.*

X Y	$[L_{M_1} \to L_{N_1}]$ $[L_{M_2} \to L_{N_2}]$	$[L_{M_1} \to L_{N_1}]$ $[L_{M_2} \leftarrow L_{N_2}]$	$[L_{M_1} \leftarrow L_{N_1}]$ $[L_{M_2} \to L_{N_2}]$	$[L_{M_1} \leftarrow L_{N_1}]$ $[L_{M_2} \leftarrow L_{N_2}]$
(t,s,τ)	$p_1 \ge q_1$	$p_1 \ge q_1, p_2 \ge q_2$		$p_2 \ge q_2$
(t,τ,s)		$p_2 \ge q_2$	$p_1 \le q_1$	$p_1 \le q_1, p_2 \ge q_2$
(s,t,τ)	$p_1 \ge q_1, p_2 \le q_2$	$p_1 \ge q_1$	$p_2 \le q_2$	
(s,τ,t)	$p_1 \ge q_1, p_2 \le q_2$	$p_1 \ge q_1$	$p_2 \le q_2$	
(τ,s,t)	$p_2 \le q_2$		$p_1 \le q_1, p_2 \le q_2$	$p_1 \le q_1$
(τ,t,s)		$p_2 \ge q_2$	$p_1 \le q_1$	$p_1 \le q_1, p_2 \ge q_2$

Table 3

X	$[L_{M_1} \to L_{N_1}]$	$[L_{M_1} \to L_{N_1}]$	$[L_{M_1} \leftarrow L_{N_1}]$	$[L_{M_1} \leftarrow L_{N_1}]$
Y	$[L_{M_2} \to L_{N_2}]$	$[L_{M_2} \leftarrow L_{N_2}]$	$[L_{M_2} \to L_{N_2}]$	$[L_{M_2} \leftarrow L_{N_2}]$
(t,s,σ)		$p_2 \geq q_2$	$p_1 \leq q_1$	$p_1 \leq q_1, p_2 \geq q_2$
(t,σ,s)		$p_2 \geq q_2$	$p_1 \leq q_1$	$p_1 \leq q_1, p_2 \geq q_2$
(s,t,σ)	$p_2 \leq q_2$		$p_1 \leq q_1, p_2 \leq q_2$	$p_1 \leq q_1$
(s,σ,t)	$p_1 \geq q_1, p_2 \leq q_2$	$p_1 \geq q_1$	$p_2 \leq q_2$	
(σ,t,s)	$p_1 \geq q_1$	$p_1 \geq q_1, p_2 \geq q_2$		$p_2 \geq q_2$
(σ,s,t)	$p_1 \geq q_1, p_2 \leq q_2$	$p_1 \geq q_1$	$p_2 \leq q_2$	

Table 4

The following is, of course, parallel to Theorem 6:

THEOREM 8. *Let* $1 \leq p \leq \infty$. *Suppose that* $l \in [L_p, L_\infty, L_{p/(p-1)}; \theta']$ *for some* $\theta' = (\theta'_1, \theta'_2, \theta'_3)$, *and* $m \in [L_\infty, L_p, L_{p/(p-1)}; \theta'']$ *for some* $\theta'' = (\theta''_1, \theta''_2, \theta''_3)$. *Then the partial integral operators* (1) *and* (2) *are regular in* L_p *and satisfy*

$$\|l\|_{\mathfrak{R}_l(L_p,L_p)} \leq \|l\|_{[L_p,L_\infty,L_{p/(p-1)};\theta']} \tag{36}$$

and

$$\|m\|_{\mathfrak{R}_m(L_p,L_p)} \leq \|m\|_{[L_\infty,L_p,L_{p/(p-1)};\theta'']}. \tag{37}$$

Finally, let us make some remarks on the sharpness of the hypotheses given in Theorems 5–8. As in the case of ordinary integral operators [11, 12], boundedness and regularity conditions for partial integral operators which are both necessary and sufficient are not known in the Lebesgue space L_p for $1 < p < \infty$, let alone in general Orlicz spaces. However, the classical sufficient conditions are also necessary in L_p for the "extreme" cases $p = 1$ or $p = \infty$. This is also true for the conditions given in Theorems 7 and 8 above. The notation simplifies in this case (recall that $1' = \infty$ and $\infty' = 1$, by definition), and it is easy to formulate the corresponding theorem.

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