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NOTE ON THE TRANSITIVITY OF PURE ESSENTIAL EXTENSIONS

BY

LASZLO FUCHS (NEW ORLEANS, LOUISIANA), LUIGI SALCE (PADOVA) AND PAOLO ZANARDO (PADOVA)

1. Introduction. It is well known that the theory of pure-injective modules over arbitrary rings runs, in many respects, parallel to the theory of injective modules. As a matter of fact, pure-injectivity can be viewed as injectivity in an appropriate category; see e.g. Gruson–Jensen [GJ]. In view of this, one is tempted to believe that everything true for "essential" extensions will carry over, mutatis mutandis, to the analogous notion of "pure-essential" extensions (this concept is instrumental in establishing the existence of pure-injective hulls). Our main goal in this note is to show that this is false; in fact, the transitivity of pure-essential extensions is a rare phenomenon even among Prüfer domains: only over rank one discrete valuation domains (abbreviated: DVR) is the transitivity of pure-essential extensions enjoyed by all modules (see Theorem 6).

Throughout, R will denote a commutative domain and Q its field of quotients, mostly viewed as an R-module.

Recall that a submodule A of the R-module B is called *pure* (in P. Cohn's sense) if every finite system of equations over A,

$$\sum_{j=1}^{n} r_{ij}x_j = a_i \in A \quad (i = 1, \ldots, m)$$

with coefficients $r_{ij} \in R$, is solvable in A provided that it admits a solution in B. This is equivalent to saying that, for every R-module M, the map $M \otimes_R A \to M \otimes_R B$ induced by the embedding $A \to B$ is injective (see Bourbaki [B]). A weaker version of purity, introduced by Warfield (see e.g. [FS]), is especially useful over domains: A is said to be *relatively divisible* (or an *RD-submodule*) in B if

$$rB \cap A = rA \quad \text{for all } r \in R.$$

Manifestly, purity implies relative divisibility. Warfield [W] proved that over

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Prüfer domains (and among the domains, only for Prüfer domains) the notions of purity and relative divisibility coincide.

The crucial notion of pure-essential submodule was introduced for abelian groups by Maranda [M] and extended to modules over general rings independently by Kiełpiński [K] and Stenström [S] (see also [W]). A module B is said to be a *pure-essential extension* of its submodule A (and A *pure-essential* in B) if A is pure in B and, for each non-zero submodule K of B,

$$K \cap A = 0 \quad \text{implies} \quad (A \oplus K)/K \text{ is not pure in } B/K.$$

The *pure-injective hull* $PE(M)$ of a module M is then a maximal pure-essential extension of M, or, to put it in a different way, a pure-injective module in which M is pure-essential. The notions of *RD-essential extension* and of *RD-injective hull* $\widehat{M}$ of M are defined analogously.

That in general pure-essential extensions do not have the transitive property, has been observed for modules over valuation domains in Exercise 2 of [FS, XI.2]. However, no hint for a concrete example was given there. This note started as a response to an inquiry by Professor José Luis Gómez Pardo who asked for a counterexample of the failing transitivity. This failure resulted in incorrect proofs in publications on pure-injective modules; see the forthcoming paper by Gómez Pardo and Guil Asensio [GG]. (We thank the referee for calling our attention to this paper and for other useful comments.)

We feel it will serve a better understanding of the situation if we are aware of the limitations of transitivity, and if particular cases are known in which transitivity can be used safely. Though the problem makes sense for modules over arbitrary rings, already the case of domains will give the particular flavor of the problem involved.

In this note, we deal primarily with modules over commutative domains R, and show that the transitivity of RD-essential extensions holds for all R-modules if and only if R is a DVR. Hence the same conclusion holds for modules over Prüfer domains if relative divisibility is replaced by purity.

We prove two additional results, both on valuation domains R. For such domains, we show that the notion "pure-essential" is transitive for torsion-free modules. We also investigate finitely generated R-modules, and prove that for finitely generated modules transitivity holds exactly over almost maximal valuation domains.

We refer to the monograph [FS] for basic facts on this topic.

2. Domains for which RD-essential extensions are transitive. It is an easy exercise to prove that in verifying that an RD-submodule A is RD-essential in B, it is enough to check that for non-zero cyclic submodules K of B, $A \cap K = 0$ implies that $(A \oplus K)/K$ is no longer an RD-submodule of B/K.

The proof of the necessity of the main result is based on the following easy observation; it is our main tool in designing counterexamples to transitivity.

LEMMA 1. *Suppose that there exist* R*-modules* A, B *such that* $A < B < \widehat{A}$, *where* A *is RD-essential in* B *and* $\widehat{B}$ *is not isomorphic to* $\widehat{A}$. *Then* A *is not RD-essential in* $\widehat{B}$.

Proof. The RD-embedding of A in $\widehat{B}$ extends, by [FS, XI.1.4], to an embedding of $\widehat{A}$ in $\widehat{B}$. By hypothesis, this cannot be epic. Consequently, A cannot be RD-essential in $\widehat{B}$, since $\widehat{A}$ is a maximal RD-essential extension of A, by [FS, XI.1.7]. ■

The following lemma collects conditions we will find useful in testing the RD-essential property.

LEMMA 2. *Let* R *be an arbitrary domain.*

(a) *If an essential submodule is an RD-submodule*, *then it is RD-essential as well.*

(b) *Let* S *be a multiplicatively closed subset of* R $(0 \notin S)$, *and* A *an* R_S*-submodule of the* R_S*-module* B. *Then* A *is an RD-essential* R_S*-submodule of* B *if and only if it is an RD-essential* R*-submodule of* B.

(c) *Let* A *be an RD-submodule of the* R*-module* B. *If there exists a multiplicatively closed subset* S *of* R *such that* $\bigcap_{s\in S} sA$ *is essential in* $\bigcap_{s\in S} sB$ *and* B/A *is* S*-divisible*, *then* A *is RD-essential in* B.

(d) *For an RD-submodule* A *of a torsion-free* R*-module* B *to be RD-essential in* B, *it is necessary and sufficient that for each non-zero RD-submodule* H *of* B, $A \cap H = 0$ *implies that* $A \oplus H$ *is not an RD-submodule in* B.

Proof. (a) is trivial.

(b) Clearly, A is relatively divisible as an R_S-submodule of B if and only if it is relatively divisible as an R-submodule. Assume that A is an RD-essential R_S-submodule of B, and let K be a non-zero R-submodule of B such that $A \cap K = 0$. Note that $K_S \neq 0$ and $A \cap K_S = 0$. Therefore $(A \oplus K_S)/K_S$ is not a relatively divisible R_S-submodule of B/K_S. Hence there exists an element $a \in A \setminus rs^{-1}A$ for suitable $r \in R$, $s \in S$ such that $a \in rs^{-1}B + K_S$. Thus $a = rs^{-1}b + kt^{-1}$ with $b \in B$, $k \in K$ and $t \in S$. But then $tsa \in rB + K$ and $tsa \notin rA$ (since $a \notin rA = rA_S$), showing that $(A \oplus K)/K$ is not a relatively divisible R-submodule of B/K. Consequently, A is an RD-essential R-submodule of B.

Conversely, assume that A is an RD-essential R-submodule of B, and let K be an R_S-submodule of B such that $A \cap K = 0$. Then $(A \oplus K)/K$ is not relatively divisible as an R-submodule of B/K, hence it is not relatively divisible as an R_S-submodule. This means that A is an RD-essential R_S-submodule of B.

(c) To show that A is RD-essential in B, it is enough to prove that for any $0 \neq b \in B \setminus A$ with $Rb \cap A = 0$, $(A \oplus Rb)/Rb$ cannot be relatively divisible in B/Rb. If $Rb \cap A = 0$, then $b \notin \bigcap_{s \in S} sB$, so there exists an $s \in S$ not dividing b in B. As B/A is S-divisible, there exist $x \in B$, $a \in A$ such that $sx = a + b$. Here a is not divisible by s, so $(A \oplus Rb)/Rb$ is not relatively divisible in B/Rb, proving that A is RD-essential in B.

(d) The necessity is obvious. To prove sufficiency, let $K \neq 0$ be a submodule of B disjoint from A, and let $K \leq H \leq B$ be such that H/K is the torsion part of B/K. Since by hypothesis $A \oplus H$ is not relatively divisible in B, $(A \oplus H)/H$ is not relatively divisible in B/H. Thus there exists an element $a + h \in rB$ for some $a \in A$, $h \in H$ and $r \in R$, such that $a \notin rA$. But h has a non-zero multiple $sh \in K$ $(s \in R)$, so $sa + K \in sr(B/K)$, and $sa \notin srA$, since A is torsion-free. This shows that A is RD-essential in B. ∎

It is obvious from part (b) that if RD-essentiality is transitive over R, it is then also transitive over its localizations R_S.

Lemmas 1 and 2 will be used in the proofs of the following two lemmas, which provide the necessity argument for the main theorem.

LEMMA 3. *If a domain R is not a Dedekind domain, then the property of being RD-essential is not transitive.*

Proof. The RD-injective hull $\widehat{D}$ of a divisible R-module D coincides with its injective hull E. Suppose D is not injective, so that we can choose a non-zero cyclic R-submodule B/D in E/D. Then D is an essential RD-submodule of B, therefore it is RD-essential in B, by Lemma 2(a). But B is not divisible, hence $\widehat{B}$ cannot be divisible either; as a result, $\widehat{B}$ is not isomorphic to E. From Lemma 1, we conclude that the transitivity of RD-essentiality fails.

Thus we have shown that, if RD-essentiality is transitive in the category of R-modules, then divisible R-modules are injective. It is well known that this property characterizes Dedekind domains among the domains. ∎

Recall that over Prüfer domains, and hence also over Dedekind and valuation domains, the notions of purity and relative divisibility coincide.

LEMMA 4. *Let R be a Dedekind domain such that the property of pure-essentiality is transitive for the R-modules. Then R is local, hence a DVR.*

Proof. Assume that R is not local. By Lemma 2(b), we can suppose R is semi-local (hence a PID), with exactly two maximal ideals pR and qR. The R-module

$$\prod_{n<\omega} R/p^nR/\bigoplus_{n<\omega} R/p^nR$$

is divisible and not torsion, so it contains a copy of Q. Consequently, there is a submodule C of the direct product $\prod_{n<\omega} R/p^nR$ for which the pure-exact sequence

$$0 \to A \to C \to Q \to 0$$

(where $A = \bigoplus_{n<\omega} R/p^nR$) is not splitting. Evidently, $p^\omega A = 0 = p^\omega C$. The localization R_q of R at the prime q satisfies $pR_q = R_q$ and $q(Q/R_q) = Q/R_q$, whence we conclude that $\mathrm{Ext}^1_R(Q/R_q, A) = 0$: indeed, if D denotes the divisible hull of A, then D/A is p-primary torsion (thus with zero q-component), so the first term in the exact sequence

$$\mathrm{Hom}_R(Q/R_q, D/A) \to \mathrm{Ext}^1_R(Q/R_q, A) \to \mathrm{Ext}^1_R(Q/R_q, D) = 0$$

vanishes. Now the exact sequence

$$0 = \mathrm{Ext}^1_R(Q/R_q, A) \to \mathrm{Ext}^1_R(Q, A) \to \mathrm{Ext}^1_R(R_q, A) \to 0$$

guarantees the existence of the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & R_q & \to & 0 \\ & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \to & A & \to & C & \to & Q & \to & 0 \end{array}$$

where β and γ are the inclusions and the top row is a non-splitting pure-exact sequence. A is pure-essential in B by Lemma 2(c) (setting $S = \{p^n \mid n \in \mathbb{N}\}$). Since the first Ulm submodule $A^1 = \bigcap_{r\in R\setminus 0} rA$ of A vanishes, the pure-injective hull $\widehat{A}$ of A is its completion $\widetilde{A}$ in the R-topology (which is equal to the p-adic topology) (see [F, Thm. 41.9]). But A dense in C implies $\widetilde{C} = \widetilde{A}$. Since $B^1 = 0$, $\widetilde{B} = \widehat{B}$. In view of [F, Thm. 39.8], $\widetilde{B}$ is isomorphic to $\widetilde{A} \oplus \widetilde{R}_q$, so not isomorphic to $\widetilde{C} = \widehat{C}$. Lemma 1 shows that transitivity fails in this case. Thus R must be local. ∎

The last lemma should be compared with Exercises 5 and 7 in Section 41 of [F]: they claim the transitivity of pure-essential extensions for abelian groups. Consequently, the statements of these exercises are incorrect: transitivity does not even hold for abelian groups!

For the sufficiency part of the main theorem we will require the following lemma.

LEMMA 5. *Let R be a DVR, and A a pure submodule of the R-module B. A is pure-essential in B if and only if the following two conditions are satisfied*:

(i) *the first Ulm submodule B^1 of B is an essential extension of A^1*;
(ii) *B/A is a divisible module.*

Proof. The sufficiency is a particular case of Lemma 2(c) with $S = R\setminus\{0\}$. For necessity, assume A pure-essential in B. Note that $B^1 \cap A = A^1$

holds by purity. Then (i) follows from the fact that if we factor out any submodule of B^1 disjoint from A, the image of A will retain its purity. To prove (ii), by way of contradiction, suppose that B/A contains a non-zero basic submodule, and hence a pure cyclic submodule $\neq 0$, say X/A for some submodule X of B. Since cyclic modules are pure-projective, we obtain $X = A \oplus Y$ for some cyclic $Y \neq 0$. Therefore, $(A \oplus Y)/Y = X/Y$ would be pure in B/Y, contradicting the hypothesis of A being pure-essential in B. ∎

It is worthwhile pointing out that in the proof of Lemma 5 the hypothesis that R is a valuation domain ensures the existence of a basic submodule, while the DVR-property of R guarantees that non-zero basic submodules have non-zero cyclic summands.

We are now ready to prove our main result mentioned in the introduction.

THEOREM 6. *A domain R has the property that "RD-essential" is transitive for R-modules exactly if R is a DVR.*

Proof. For the proof of necessity, observe that Lemma 3 implies that R must be a Dedekind domain, while Lemma 4 shows that it has to be a DVR.

Conversely, let R be a discrete rank one valuation domain. Since the properties in (i) and (ii) of Lemma 5 are evidently transitive, the "if" part of the claim is immediate. ∎

As an immediate consequence of the theorem we get the following

COROLLARY 7. *A Prüfer domain R has the property that the notion "pure-essential" is transitive if and only if R is a DVR.* ∎

3. Transitive pure-essential extensions over valuation domains.

In this section, we confine ourselves to modules over valuation domains R. We denote by P the maximal ideal of R.

By virtue of Theorem 6, pure-essentiality is in general not transitive for modules over valuation domains; this has already been observed in Exercise 2 in [FS, XI.2]. However, we intend to show that torsion-free modules behave nicer: transitivity does hold for torsion-free modules over arbitrary valuation domains.

The next lemma characterizes pure-essential extensions of torsion-free modules over valuation domains.

LEMMA 8. *Suppose $A < C$ are torsion-free modules over a valuation domain R. The module A is pure-essential in C if and only if A is pure in C, and for any $c \in C \setminus A$ there are $x \in C$, $a \in A$, $r \in R$ such that $rx = a + c$, where a is not divisible by r.*

P r o o f. The sufficiency is obvious in view of the remark made before Lemma 1. Assume now that A is pure-essential in C, and let $c \in C \setminus A$ satisfy $Rc \cap A = 0$. Then there are $x \in C$, $a' \in A$, $s, t \in R$ such that $sx = a' + tc$ where a' is not divisible by s. This equation implies that $s = rt$ ($r \in P$) is a proper multiple of t, hence $a' = ta$ for some $a \in A$. Division by t yields $rx = a + c$, where a is not divisible by r. ∎

PROPOSITION 9. *Pure-essentiality is transitive for torsion-free modules over valuation domains.*

P r o o f. Assume that A, B, C are torsion-free modules such that A is pure-essential in B, which is pure-essential in C. Making repeated use of Lemma 8, we show that for any $c \in C \setminus A$ there are $x \in C$, $a \in A$, $s \in R$ such that $sx = a + c$ where a is not divisible by s. If $c \in B$, then there is nothing to prove. So let $c \in C \setminus B$. As B is pure-essential in C, there are $y \in C$, $b \in B$, $r \in R$ satisfying $ry = b + c$ with r not dividing b, c. If $b \in A$ we are done. Otherwise, we can find $z \in B$, $a \in A$, $t \in R$ such that $tz = a + b$ where t does not divide a, b. Hence $ry - tz = -a + c$. If $t \mid r$ (resp. $r \mid t$), then $t \mid a$ fails (resp. $r \mid c$ fails), establishing the claim. ∎

The last proposition does not leave much room for improvement. In fact, suppose for a moment that R is a non-local Dedekind domain. With arguments similar to those used in Lemma 4, replacing A by R_p and $\prod_{n<\omega} R/p^nR$ by $\widehat{R}_p$, and using the fact that if $R_p = \widehat{R}_p$ then R is local, one can prove that the transitivity of pure-essentiality fails for torsion-free modules over such an R.

One of the easiest examples to demonstrate the failure of transitivity for pure-essential extensions over valuation domains (which was the basis of the exercise in [FS] mentioned above) is as follows.

Suppose R is a valuation domain that is not almost maximal. As shown in [FS, Chapter IX], there exists a two-generated indecomposable R-module $Y = xR + yR$ with annihilator sequence $0 < A = \operatorname{Ann} x < J = \operatorname{Ann}(y + xR)$, and generators x, y subject to the relations

$$ry = ru_r x \quad (r \in J),$$

where $\{u_r\}$ is a system of units in R such that $u_r - u_{pr} \in r^{-1}A$ for all $r \in J \setminus \{0\}$ and $p \in P$, and no $u \in R$ satisfies $u - u_r \in r^{-1}A$ for all $r \in J \setminus \{0\}$. The cyclic basic submodule $X = xR$ of Y is pure and essential in Y, hence pure-essential, by Lemma 1(a). However, X is not pure-essential in $PE(Y)$, because $PE(X) \cong S/AS$ and $PE(Y) \cong S/AS \oplus S/JS$ (see [FS, XI.5.9]).

In this example $PE(Y)$ is, in general, not finitely generated, so it does not answer the transitivity question for finitely generated modules. We need a more delicate approach; this will be provided by our final result.

THEOREM 10. *Let R be a valuation domain. Pure-essentiality is transitive for finitely generated R-modules if and only if R is an almost maximal valuation domain.*

Proof. First assume that R is almost maximal. Then finitely generated R-modules are direct sums of cyclic modules. As shown in [FS, IX.5.6], pure submodules of finite direct sums of cyclic modules are summands. We conclude that there are no proper finitely generated pure-essential extensions of finitely generated R-modules at all, hence pure-essentiality is trivially transitive for these modules.

Conversely, assume that R is not almost maximal. Consider the two-generated torsion module $Y = xR + yR$ in the example above, and pick any $a \in P \setminus J$. Let zR be a cyclic module isomorphic to $R/a^{-1}J$, and set $Z = Y \oplus zR$. We verify some properties of the following submodule of Z:

$$Y' = xR + (ay + z)R.$$

(i) Y' is a two-generated uniform module, with annihilator sequence $0 < A < a^{-1}J$; it contains $xR = X$ as pure and essential submodule. Indeed, since Z has Goldie dimension two, if xR were not essential in Y', then $Y' \cap zR \neq 0$ would hold. Thus $px + q(ay + z) = rz \neq 0$ for suitable $p, q, r \in R$. Clearly, $r \notin a^{-1}J$, so $(q - r)z = 0$ implies $q \notin a^{-1}J$; but this is impossible because of $qay \in xR$. Thus xR is essential in Y'. The rest of the claim in (i) is clear.

(ii) Y' is pure in Z. Indeed, Y' is pure in Z if and only if Y'/X (which is cyclic generated by $ay+z+X$) is pure in $Z/X \cong (Y/X) \oplus zR$. As $ay+z+X$ generates a direct summand in Z/X, Y' is pure in Z.

(iii) Y' is pure-essential in Z. For this, we prove that $wR \cap Y' = 0$ $(0 \neq w \in Z)$ implies that $(Y' \oplus wR)/wR$ is not pure in Z/wR. Write $w = sw'$ with $w' = px+qy+tz \in Z \setminus PZ$, where one of $p, q, t \in R$ is a unit. Evidently, $s \notin a^{-1}J$, since otherwise $w \in Y$ and $0 \neq wR \cap xR \leq wR \cap Y'$. We claim that t must be a unit. In fact, for all $r \in J \setminus A$, we have

$$rw' = r(p + qu_r)x \in w'R \cap xR = 0,$$

hence $p + qu_r \in r^{-1}A$. Since no unit $u \in R$ satisfies $u - u_r \in r^{-1}A$ for all $r \in J \setminus A$, it follows that $p, q \in P$. Therefore, t is a unit, and we can assume $t = 1$. Since $z + ay + (px - (a - q)y) = w'$ and $px - (a - q)y \in PY$, we deduce that $s(z + ay) + wR \in sp(Z/wR)$ for some $p' \in P$. But $s(z + ay) \notin sp'Y'$, otherwise $s(z + ay) = sp'(cx + d(ax + z))$ $(c, d \in R)$ would imply $sz = sp'dz$, whence $s(1 - p'd) \in \operatorname{Ann} z = a^{-1}J$, and consequently $s \in a^{-1}J$, a contradiction. Therefore, $(Y' \oplus wR)/wR$ is not pure in Z/wR.

One can now easily finish the proof of the theorem. Just observe that X is not pure-essential in Z, since $(X \oplus zR)/zR$ is pure in Z/zR. ■

In the global case, if R is an almost maximal Prüfer domain (i.e., R is h-local and all localizations R_P at maximal ideals P are almost maximal valuation domains, see Brandal [Br]), then pure-essentiality is transitive for finitely generated R-modules: indeed, an easy reduction to the local case works. However, it is an open question whether or not the converse holds, i.e., if the transitivity of pure-essentiality for finitely generated R-modules over a Prüfer domain R forces R to be almost maximal. The problem consists in deciding if R must be h-local; if so, then Lemma 2(b) will yield the desired conclusion.

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Department of Mathematics
Tulane University
New Orleans, Louisiana 70118
U.S.A.
E-mail: fuchs@mailhost.tcs.tulane.edu

Dipartimento di Matematica Pura ed Applicata
Università di Padova
Via Belzoni 7
35131 Padova, Italy
E-mail: salce@math.unipd.it
pzanardo@math.unipd.it

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