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FLOWS ON INVARIANT SUBSETS AND COMPACTIFICATIONS OF A LOCALLY COMPACT GROUP

BY

A. T. LAU (EDMONTON, ALBERTA), P. MILNES (LONDON, ONTARIO) AND J. S. PYM (SHEFFIELD)

1. Introduction. A natural way to construct a flow associated with a locally compact group G is to take a compact subset V of G and ask for G to act on V by conjugation, $x \mapsto \psi_g(x) := gxg^{-1}$ for $x \in V$, $g \in G$ (and we shall also consider the case when V is not compact). Obvious examples are when V is a compact normal subgroup and when V is a compact invariant neighbourhood of the identity (a group which possesses such a neighbourhood is called an *IN group*). In this paper we study such flows. In particular, we investigate the relationships between these flows and the $\mathcal{LC}$ compactification $G^{\mathcal{LC}}$ of G which is described in Theorem 0 below. (We are using the terminology of [1].) Our perspective on this subject is in keeping with Lawson's programme (see for example [7]) of exploiting the methods of compact semigroup theory in topological dynamics.

A key feature of a flow is its enveloping semigroup. In the above context, each ψ_g is a continuous map from V to itself. When the set V^V of all maps from V to V is given the product (or pointwise) topology, it becomes compact, and the operation of composition makes it a semigroup in which all maps $f \mapsto f \circ h$ ($h \in V^V$) are continuous. The maps ψ_g ($g \in G$) are in the *topological centre* $\Lambda(V^V) := \{f \in V^V \mid h \mapsto f \circ h \text{ is continuous}\}$, and so $\Sigma(G, V) = (\psi_G)^- \subset V^V$ is a compact semigroup called the *enveloping semigroup* of the flow (G, V).

We shall call the flow (G, V) *distal* if $\Sigma(G, V)$ is a group. Rosenblatt [11] calls a group distal if its action on itself by inner automorphisms has the property that $g_\alpha s g_\alpha^{-1} \to e$ for any net $\{g_\alpha\}$ and any s in G implies $s = e$. This is the same as our concept when V is an invariant neighbourhood of

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the identity (Proposition 5 below). Rosenblatt's main conclusion was that if a distal group is almost connected, then it has polynomial growth. At about the same time, Ruppert [12] showed that a group satisfying Rosenblatt's distality condition has this cancellation property: if $\nu s = \nu t$ for any s, t in G and ν in $G^{\mathcal{LC}}$, then $s = t$.

Our Section 2 discusses basic properties of the flows (G, V). We discover that $\Sigma(G, V)$ is to a large extent independent of V. Thus, for example, if G is σ-compact, then for any two compact invariant neighbourhoods U, V of the identity which generate G, we have $\Sigma(G, U) = \Sigma(G, V)$. In this case it makes sense to talk of the flow given by the action of G on itself, and $\Sigma(G, G)$ is the same compact semigroup as the others. We give several conditions equivalent to distality. These illustrate the relationship between $\Sigma(G, V)$ and $G^{\mathcal{LC}}$; for example, $\Sigma(G, V)$ is a group if and only if V is normal in $G^{\mathcal{LC}}$ (that is, $\nu V = V\nu$ for all $\nu \in G^{\mathcal{LC}}$). When G is an *SIN group* (that is, every neighbourhood of the identity contains an invariant neighbourhood) and V is a neighbourhood of the identity in G, it is easy to prove that $\Sigma(G, V)$ is actually a topological group. Although IN and SIN groups are closely related, the enveloping semigroups for IN groups can exhibit a wide variety of behaviour. This we show in a series of results in §4, which is devoted to the study of our ideas in the special setting of semidirect products.

In §3 we consider the structure of $G^{\mathcal{LC}}$. For some normal subgroups N of G (including compact ones), $N\mu$ is a right simple semigroup for any idempotent μ in $G^{\mathcal{LC}}$. Algebraically, therefore, $N\mu = E \times H$, where E is the left zero semigroup ($ef = e$ for all $e, f \in E$) consisting of the idempotents in $N\mu$, and H is a group. Under the mapping $s \mapsto s\mu$, the preimage of E in N is a normal subgroup and the preimage of H is a subgroup, and N is the semidirect product of these two subgroups. Thus, each idempotent of $G^{\mathcal{LC}}$ determines a semidirect product decomposition of N. Here in fact H is algebraically isomorphic to $\mu N\mu$, and $E = \{s \in N \mid \mu s\mu = \mu\}$. Topologically the situation is not so simple. We give examples to illustrate the possibilities that (i) H is dense in $N\mu$, and (ii) H is not dense in $N\mu$.

We refer the reader to [1] for terminology and results which are not explained in our paper and for which other references are not given. We should, however, state some facts about $G^{\mathcal{LC}}$ which will be continually used in our work.

THEOREM 0. *$G^{\mathcal{LC}}$ is the largest semigroup compactification of the locally compact group G in the sense that if ϕ is any continuous homomorphism of G into a compact right topological semigroup H for which $\phi(G) \subset \Lambda(H)$, then ϕ extends to a continuous homomorphism of $G^{\mathcal{LC}}$ to H. If $\phi(G)$ is dense in H, then $\phi(G^{\mathcal{LC}}) = H$. G can be regarded as homeomorphically* *embedded in* $G^{\mathcal{LC}}$. *Moreover*,

(i) *the multiplication* $G \times G^{\mathcal{LC}} \to G^{\mathcal{LC}}$ *is jointly continuous*, *and*

(ii) *for any* $\nu \in G^{\mathcal{LC}}$ *the map* $s \mapsto s\nu$, $G \to G^{\mathcal{LC}}$, *is a continuous injection*, *and so a homeomorphism on each compact subset of* G. ∎

These results can be found in [1], in particular Theorems 4.5.7 (for (i), using Ellis's joint continuity theorem) and Lemma 4.8.9 (for (ii), using a result of Veech [13]).

The *distal compactification* $G^{\mathcal{D}}$ of G is the largest semigroup compactification of G which is a group. It can also be described as the largest continuous quotient of $G^{\mathcal{LC}}$ which is a group. The *almost periodic compactification* $G^{\mathcal{AP}}$ of G is the largest compactification which is a topological group. When G is commutative, it can be realised as the group $(\widehat{G}_{\mathrm{d}})\widehat{\ }$, where $\widehat{G}$ is the Pontryagin dual of G, and the suffix d means that the group $\widehat{G}$ is given its discrete topology.

2. Flows determined by inner actions. We now describe more precisely a general setting for our work. Let G be a locally compact group acting on itself by inner automorphisms $\psi_g : s \mapsto gsg^{-1}$, $g, s \in G$, and let $O(s) = \{\psi_g(s) \mid g \in G\}$ be the orbit of s. We then define the "compact conjugacy class subgroup" G_C of G by $G_C = \{s \in G \mid \overline{O(s)} \text{ is compact}\}$, which is invariant under the inner action of G, hence normal. We shall be interested in restricting the inner action of G to G_C, and even further, to compact invariant subsets V of G_C.

If we consider $\psi_G|_{G_C}$ as a subset of $G_C^{G_C}$, then the closure $(\psi_G)^-$ of ψ_G is the *enveloping semigroup* $\Sigma(G, G_C)$ of (G, G_C), which is compact (even if G_C is not) because $\Sigma(G, G_C)$ is the same as the closure of ψ_G in $\Pi\{\overline{O(s)}^{\overline{O(s)}} \mid s \in G_C\}$, which is compact. The continuous homomorphism $g \mapsto \psi_g$, $G \to \Sigma(G, G_C)$, extends to a continuous homomorphism, denoted by $\nu \mapsto \psi_\nu$, from $G^{\mathcal{LC}}$ onto $\Sigma(G, G_C)$ (using Theorem 0). When ψ_G is restricted further to a compact invariant subset $V \subset G_C$, the enveloping semigroup $\Sigma(G, V)$ of the flow (G, V) is a homomorphic image of $\Sigma(G, G_C)$, in fact just the restriction of $\Sigma(G, G_C)$ to V. (We use the same symbol ψ_ν for $\psi_\nu|_{G_C}$ and $\psi_\nu|_V$.) The cases that will interest us most are when $V \subset G_C$ is a compact normal subgroup of G, and when G is an IN (or even SIN) group and V is a compact invariant neighbourhood of the identity e of G. We note that G_C is open if and only if G is IN (Liukkonen [8], Corollary 2.2).

Recall that G can be identified with its canonical image in $G^{\mathcal{LC}}$, so that for $\nu \in G^{\mathcal{LC}}$ and $s \in G$ the product νs is defined as an element of $G^{\mathcal{LC}}$. If, in addition, $s \in G_C$, then $\psi_\nu(s)$ is an element of $\overline{O(s)}$, so that $\psi_\nu(s)\nu$ is also defined in $G^{\mathcal{LC}}$. The second conclusion in the next lemma was established in [6] when V is a compact normal subgroup of G_C.

Lemma 1. *For any* $\nu \in G^{\mathcal{LC}}$ *and* $s \in G_C$, $\psi_\nu(s)\nu = \nu s$. *Hence* $\nu V \subset V\nu$ *for any closed invariant* $V \subset G_C$; *also*, $\nu s = \nu s \nu$ *if* ν *is idempotent.*

Proof. Let $g_\alpha \to \nu \in G^{\mathcal{LC}}$, so that $\psi_{g_\alpha}(s) \to \psi_\nu(s) \in \overline{O(s)}$. Then

$$\nu s = \lim_\alpha g_\alpha s = \lim_\alpha g_\alpha s g_\alpha^{-1} g_\alpha = \lim_\alpha \psi_{g_\alpha}(s) g_\alpha = \psi_\nu(s)\nu,$$

since multiplication $\overline{O(s)} \times G^{\mathcal{LC}} \subset G \times G^{\mathcal{LC}} \to G^{\mathcal{LC}}$ is jointly continuous. The third conclusion follows readily, as does the second since $\overline{O(s)} \subset V$. ∎

Let $V \in G_C$ be a compact invariant subset with $e \in V$, so that (G, V) is a flow. Then each $V^n \subset G_C$ is also invariant, as is the subset $G_1 = \bigcup_{n=1}^\infty V^n \subset G_C$ (G_1 perhaps failing to be compact). The proof of the next result requires little beyond taking note of the restriction homomorphism $\Sigma(G, G_1) \to \Sigma(G, V^n)$, $n \in \mathbb{N}$, and the fact that each $T \in \Sigma(G, V)$ extends naturally to V^2.

Proposition 2. *The enveloping semigroups* $\Sigma(G, V^n)$, *for* $n \in \mathbb{N}$, *and* $\Sigma(G, G_1)$ *are all isomorphic.* ∎

We can now give several characterizations of distal flows in terms of the $\mathcal{LC}$ compactification of G.

Theorem 3. *Let* $V \subset G_C$ *be a compact invariant subset of the locally compact group* G. *The following statements* (i)–(vi) *are equivalent*:

(i) *The flow* (G, V) *is distal.*

(ii) V *is normal in* $G^{\mathcal{LC}}$ (*i.e.*, $\nu V = V\nu$ *for all* $\nu \in G^{\mathcal{LC}}$).

(iii) *There is an idempotent* μ *in the minimal ideal of* $G^{\mathcal{LC}}$ *for which* $\mu V = V\mu$.

(iv) *There is an idempotent* μ *in the minimal ideal of* $G^{\mathcal{LC}}$ *for which* ψ_μ *is the identity mapping.* (*In this case*, $\mu s = s\mu$ *for all* $s \in V$, *i.e.*, μ *is in the centralizer of* V.)

(v) *There is an idempotent* μ *in the minimal ideal of* $G^{\mathcal{LC}}$ *for which the map* $s \mapsto \mu s$ *is injective on* V.

(vi) V^n *is normal in* $G^{\mathcal{LC}}$ *for all* $n \in \mathbb{N}$.

When $V = N$ *is a compact normal subgroup of* G, (i)–(vi) *are also equivalent to* (vii) *and* (viii), *and* (ix) *implies* (i)–(viii).

(vii) *There is a minimal idempotent* $\mu \in G^{\mathcal{LC}}$ *for which* $N\mu$ *is a group.*

(viii) *There is a minimal idempotent* $\mu \in G^{\mathcal{LC}}$ *for which* $N\mu$ *contains just one idempotent.*

(ix) *The natural map from* G *to* $G^{\mathcal{D}}$ *is injective on* N; *equivalently*, $\mathcal{D}(G)$ *separates the points of* N.

Proof. (i)⇒(ii). Let $\nu \in G^{\mathcal{LC}}$. Since $\{\psi_\eta \mid \eta \in G^{\mathcal{LC}}\}$ is the group $\Sigma(G, V) \subset V^V$, we have $\psi_\nu(V) = V$, so from Lemma 1, $V\nu = \psi_\nu(V)\nu = \nu V$.

(ii)⇒(iii) is trivial.

(iii)⇒(iv). From $\mu V = V\mu$ and the lemma we get $V\mu = \psi_\mu(V)\mu$. Since the map $g \mapsto g\mu$, $G \to G^{\mathcal{LC}}$, is injective (Theorem 0), we have $V = \psi_\mu(V)$. Thus ψ_μ is surjective and since $(\psi_\mu)^2 = \psi_{\mu^2} = \psi_\mu$, we conclude that ψ_μ is the identity map on V.

(iv)⇒(v). Lemma 1 tells us that when ψ_μ is the identity, $\mu s = \psi_\mu(s)\mu = s\mu$ for $s \in V$, and $s \mapsto s\mu$ is injective by Theorem 0.

(v)⇒(iii). By Lemma 1 again, we have $\mu\psi_\mu(V) = \mu\psi_\mu(V)\mu = \mu^2 V = \mu V$, so $\psi_\mu(V) = V$ if $s \mapsto \mu s$ is injective. Then Lemma 1 implies $\mu V = V\mu$.

(iv)⇒(i). Since ψ_μ is the identity on V,

$$\begin{aligned}\Sigma(G,V) &= \{\psi_\eta \mid \eta \in G^{\mathcal{LC}}\} \\ &= \{\psi_\mu\psi_\eta\psi_\mu \mid \eta \in G^{\mathcal{LC}}\} = \{\psi_\eta \mid \eta \in \mu G^{\mathcal{LC}}\mu\} \subset V^V;\end{aligned}$$

but general semigroup theory ([1], §1.2) tells us that for any idempotent μ in the minimal ideal of $G^{\mathcal{LC}}$, $\mu G^{\mathcal{LC}}\mu$ is a group with μ as its identity.

The equivalence of (iii) and (vi) follows from Proposition 2.

Let N be a compact normal subgroup of G and take $V = N$.

(iii)⇒(vii). $\mu N = N\mu$ implies that for any $s \in N$ we have $\mu s = \mu s\mu = s\mu$, and it is then easy to see that $\mu N\mu$ is a group.

(vii)⇒(viii) is trivial.

(viii)⇒(iii). Since μ is idempotent, it is a right identity for $N\mu$, and so from $\mu N \subset N\mu$ (Lemma 1) we have $\mu N\mu = \mu N$. Suppose (iii) is false: $\mu N \neq N\mu$. Then there exists an $s \in N$ with $s\mu \in N\mu \setminus \mu N$. If $t = \psi_\mu(s)$, we have $t\mu = \mu s \in \mu N$, and also $t^{-1}\mu \in \mu N$; to see this, note that $t^{-1}\mu = t^{-1}\mu s\mu s^{-1}\mu = t^{-1}t\mu s^{-1}\mu = \mu s^{-1}\mu$. So $(st^{-1}\mu)^2 = st^{-1}\mu st^{-1}\mu = st^{-1}t\mu t^{-1}\mu = st^{-1}\mu$ is an idempotent in $N\mu$. It is not μ because $\mu t\mu = t\mu$, but $(st^{-1}\mu)t\mu = s\mu t^{-1}\mu t\mu = s\mu$.

(ix)⇒(ii). This is Theorem 16(b) of [6]. ∎

The conclusions of Theorem 3 hold equally well for closed invariant subsets (and subgroups) V of G_C; these are just unions of compact $\overline{O(s)}$'s. Thus, if $G = G_C$ and (G, G_C) is distal (i.e., $(G, \overline{O(s)})$ is distal for all $s \in G_C$), then G is normal in $G^{\mathcal{LC}}$, a consequence of which is that the maximal subgroups in a minimal left ideal $L \subset G^{\mathcal{LC}}$ are dense in L. To see this, let μ be any idempotent in L. Thus $L = G^{\mathcal{LC}}\mu$, and $\mu G^{\mathcal{LC}}\mu$ is a maximal group in L. This contains $\mu G\mu$, which is equal to $G\mu^2 = G\mu$ since G is normal, and $G\mu$ is dense in $G^{\mathcal{LC}}\mu$ by continuity.

The equivalence of (i) and (v) of the theorem is very close to a result of Ruppert (Theorem 4.11 of [12]). He says that, under a mild separability condition, the group G is distal in the sense of Rosenblatt (our definition of that term is given just before Theorem 5 below) if and only if $s \mapsto \nu s$ is injective on G for each $\nu \in G^{\mathcal{LC}}$. With our methods, we can obtain injectivity on the whole of G only when $G = G_C$.

A particularly simple case occurs when G is an SIN group. If V is any compact symmetric invariant neighbourhood of the identity of G, then the flow (G, V) is equicontinuous (simply because for any invariant neighbourhood $U \subset V$, $\psi_g(U) = gUg^{-1} = U$ for all $g \in G$). This immediately gives the following proposition.

PROPOSITION 4. *If G is an SIN group and V is an invariant neighbourhood of the identity then $\Sigma(G, V)$ is a compact topological group whose topology coincides with that of uniform convergence on V.* ■

Proposition 4 is essentially known. If G is SIN and is generated by a compact symmetric neighbourhood V of the identity, then $G = G_C$ (because each $s \in G$ is contained in the compact invariant set V^n for some n), that is, G belongs to the class $[\mathrm{FC}]^-$ ([10], page 530). A group in $[\mathrm{FC}]^-$ is SIN if and only if the closure of the group of inner automorphisms in $\mathrm{Aut}(G)$ is compact in the Braconnier topology ([4], §26.3; see also [10], page 530). Under the present conditions, the latter group coincides with $\Sigma(G, V)$ (see Proposition 2).

It might be thought that because IN groups are simply the extensions of compact groups by SIN groups ([9], page 718), the structures of their associated enveloping semigroups would be only a little more complicated. This is far from the case, as the groups we study in §4 illustrate.

In our next proof, we need the basic characterisation of distal flows due to Ellis [2]. In our situation, it says that a flow (G, V) is distal if for each net $\{g_\alpha\}$ in G and any s, t in V, the relation $\lim_\alpha g_\alpha s g_\alpha^{-1} = \lim_\alpha g_\alpha t g_\alpha^{-1}$ implies $s = t$. We shall call the group G *distal in the sense of Rosenblatt* [11] if this condition holds with $V = G$ (even though G may not be compact).

We next consider how enveloping semigroups vary with the compact symmetric invariant subset $V \subset G$. We have seen in Proposition 2 that the enveloping semigroup $\Sigma(G, X)$ will be the same for $X = V$ or $X = G_1 = \bigcup_{n=1}^{\infty} V^n$. Furthermore, if (G, V) is distal so that V is normal in $G^{\mathcal{LC}}$, then G_1 is also normal in $G^{\mathcal{LC}}$.

THEOREM 5. *Let G be an IN group and V a compact invariant neighbourhood of the identity e of G. Then G acts distally on V if and only if G acts distally on every compact invariant neighbourhood of the identity e of G. The flow (G, V) is distal if and only if G is Rosenblatt distal.*

Proof. Suppose that the group G acts distally on V. Let W be any compact invariant neighbourhood of e; suppose that $\lim_\alpha \psi_{g_\alpha}(u) = \lim_\alpha \psi_{g_\alpha}(v)$ for some net $\{g_\alpha\}$ in G and $u, v \in W$. Then $\lim_\alpha \psi_{g_\alpha}(uv^{-1}) = e \in G$, so that $\psi_{g_\beta}(uv^{-1})$ lies in V for some value β of the index α. Thus

$\lim_\alpha \psi_{g_\alpha g_\beta^{-1}}(\psi_{g_\beta}(uv^{-1})) = e = \lim_\alpha \psi_{g_\alpha g_\beta^{-1}}(e)$, whence $\psi_{g_\beta}(uv^{-1}) = e$ by distality on V, so that $u = v$ as required.

The same proof with W replaced by G shows that if (G, V) is distal then G is Rosenblatt distal. The converse of the last statement is obvious. ∎

If V and W are compact invariant neighbourhoods of the identity in an IN group G, each of which generates G, then Proposition 2 implies that $\Sigma(G, V)$ and $\Sigma(G, W)$ are isomorphic. The situation can be different if the neighbourhoods V and W do not generate G.

EXAMPLE 6. *There is an IN group G for which the enveloping semigroups $\Sigma(G, V)$ are not the same for all invariant neighbourhoods V of the identity*, *even though the flows (G, V) have the additional property of being equicontinuous.*

Proof. Take $G = \mathbb{T} \times \mathbb{Z}^2$ with multiplication

$$(w, m, n)(w', m', n') = (ww'e^{inm'}, m + m', n + n'). \tag{m}$$

The centre $V = Z(G) = (\mathbb{T}, 0, 0)$ is a compact invariant neighbourhood of $e = (1, 0, 0) \in G$, and $\Sigma(G, V)$ is just the identity. $W = (\mathbb{T}, 0, 0) \cup (\mathbb{T}, 1, 0)$ is another compact invariant neighbourhood of e and

$$\psi_{(w,m,n)}(w', 1, 0) = (w, m, n)(w', 1, 0)(w^{-1}e^{inm}, -m, -n) = (w'e^{in}, 1, 0),$$

so $\Sigma(G, W) \cong \mathbb{T}$ and the homomorphism $G \to \mathbb{T}$ sends (w, m, n) to e^{in}. Yet another compact invariant neighbourhood of e is $W' = \bigcup\{(\mathbb{T}, i, j) \mid i, j \in \{0, \pm 1\}\}$; W' generates G and $\Sigma(G, W') \cong \mathbb{T}^2$ with $(w, m, n) \mapsto (e^{in}, e^{-im})$, so we also have $\Sigma(G, G) \cong \mathbb{T}^2$. ∎

There is a contrast between the group in Example 6 and the closely related group $G' = \mathbb{T} \times \mathbb{R}^2$, also with multiplication (m). Obviously, G' contains G as a normal subgroup, but here every compact invariant neighbourhood V is generating, so that Proposition 2 applies. In fact, $\Sigma(G', V) \cong \Sigma(G', G') \cong \mathbb{R}^{\mathcal{AP}} \times \mathbb{R}^{\mathcal{AP}}$, and if a is the homomorphism from $\mathbb{R}$ into $\mathbb{R}^{\mathcal{AP}}$, then the homomorphism $G \to \mathbb{R}^{\mathcal{AP}} \times \mathbb{R}^{\mathcal{AP}}$ sends (w, x, y) to $(a(x), a(-y))$. A proof of these assertions can be constructed on the same lines as the one given below for Proposition 12.

When the action of G on a compact invariant subset V is distal, $\Sigma(G, V)$ is a group and there is a continuous surjective homomorphism $\sigma : G^{\mathcal{D}} \to \Sigma(G, V)$. We should then expect the action $s \mapsto \nu \cdot s$ of $\Sigma(G, V)$ on V to be given by conjugation by the elements of $G^{\mathcal{D}}$. This is essentially so, as the next proposition says, but there are a couple of details which make the result look less transparent than the informal statement we have just given.

Proposition 7. *Let* (G,V) *be a distal flow, with* V *a compact invariant subset of* G. *Suppose that the canonical map* $G \to G^{\mathcal{D}}$ *is injective on* V, *so that* V *is homeomorphically imbedded in* $G^{\mathcal{D}}$ (*as well as in* $G^{\mathcal{LC}}$). *Then* $\sigma(\nu)s = \nu s \nu^{-1}$ *for* $\nu \in G^{\mathcal{D}}$, $s \in V$.

Proof. Lemma 1 tells us that for $\xi \in G^{\mathcal{LC}}$ and $s \in V$ we have $\psi_\xi(s)\xi = \xi s$. The canonical homomorphism $G^{\mathcal{LC}} \to G^{\mathcal{D}}$ is surjective, but by hypothesis injective on V which contains $\psi_\xi(s)$. If we apply this homomorphism to the above equality, choosing ξ to be a preimage of ν, we see that $\psi_\xi(s)\nu = \nu s$. We are now in the group $G^{\mathcal{D}}$, so we may take inverses to get $\psi_\xi(s) = \nu s \nu^{-1}$. Applying the map σ now gives $\nu \cdot s = \nu s \nu^{-1}$. ∎

3. Algebraic structure in $G^{\mathcal{LC}}$. The group G is homeomorphically embedded in $G^{\mathcal{LC}}$ as a dense subgroup. The natural expectation would be that for any idempotent μ in $G^{\mathcal{LC}}$, $G\mu$ would also be a subgroup isomorphic with G. This is often far from the case: $G\mu$ can contain a large number of idempotents. We shall now see how this comes about, and that each idempotent in $G^{\mathcal{LC}}$ determines a semidirect product decomposition of each normal subgroup of G_C.

We begin with a result about idempotents in $G^{\mathcal{LC}}$. For $\nu \in G^{\mathcal{LC}}$ and $V \subset G_C$ we write $I_V(\nu) = \{s\nu \mid s \in V \text{ and } s\nu \text{ is an idempotent in } G^{\mathcal{LC}}\}$.

Lemma 8. *Let* V *be a closed invariant subset of* G_C.

(i) *Let* μ *be an idempotent in* $G^{\mathcal{LC}}$ *and let* $s \in G_C$. *Then the following three statements are equivalent*:

(1) $s\mu$ *is an idempotent.*
(2) $\psi_\mu(s) = e$.
(3) $\mu s = \mu$.

In this situation, $I_V(\mu) = \{s\mu \mid s \in V,\ \psi_\mu(s) = e\}$.

(ii) *Let* μ *and* μ' *be idempotents in* $G^{\mathcal{LC}}$. *Then either* $I_V(\mu) \cap I_V(\mu') = \emptyset$ *or* $I_V(\mu) = I_V(\mu')$.

Now let $\nu \in G^{\mathcal{LC}}$ *and* $s \in V$.

(iii) *If* $s\nu$ *is an idempotent and* $s' \in G_C$, *then the following two statements are equivalent*:

(1) $s'\nu$ *is an idempotent.*
(2) $\psi_\nu(s') = \psi_\nu(s)$.

In this situation, $I_V(s\nu) = \{s'\nu \mid s' \in V,\ \psi_\nu(s') = \psi_\nu(s)\}$.

(iv) $s\nu$ *is an idempotent if and only if* νs *is an idempotent.*

(v) *If* $s\nu$ *is an idempotent, then the idempotent* νs *is in* $I_{VV^{-1}}(s\nu)$; *furthermore*, $\psi_\nu^2(s) = \psi_\nu(s)$ *and*

$$\circledast \qquad \nu s = \psi_\nu(s)\nu = \psi_\nu^2(s)\nu = \nu\psi_\nu(s).$$

P r o o f. To prove (i) use Lemma 1 to see that

$$(s\mu)(s\mu) = s(\mu s)\mu = s(\psi_\mu(s)\mu)\mu = s\psi_\mu(s)\mu.$$

Then the injectivity of $s' \mapsto s'\mu$, $G \to G^{\mathcal{LC}}$ (Theorem 0), shows that (1), (2) and (3) are equivalent.

(ii) is easy, since $s\mu = s'\mu'$ is equivalent to $\mu = s^{-1}s'\mu'$.

(iii) Writing $s'\nu = s's^{-1}(s\nu)$, we see from (i) that $s'\nu$ being idempotent is equivalent to $e = \psi_{s\nu}(s's^{-1}) = s\psi_\nu(s's^{-1})s^{-1}$, i.e., $\psi_\nu(s') = \psi_\nu(s)$.

(iv) The map $\eta \mapsto s^{-1}\eta s$ is an automorphism of $G^{\mathcal{LC}}$, and so preserves idempotents.

(v) Note that $\nu s = \psi_\nu(s)\nu = \psi_\nu(s)s^{-1}(s\nu) \in I_{VV^{-1}}(s\nu)$, so the first conclusion is established. Then by (i), $e = \psi_{s\nu}(\psi_\nu(s)s^{-1}) = s\psi_\nu^2(s)\psi_\nu(s^{-1})s^{-1}$, so $\psi_\nu^2(s) = \psi_\nu(s)$. The first and last equalities in $\circledast$ follow from Lemma 1. ∎

We now come to our principal theorem about structures in $G^{\mathcal{LC}}$.

THEOREM 9. *Let* G *be a locally compact group and let* $N \subset G_C$ *be a normal subgroup of* G. *Let* μ *be an idempotent in* $G^{\mathcal{LC}}$. *Then* N *is a semidirect product* $N_1 \times K_1$ *of its normal subgroup* $N_1 = \psi_\mu^{-1}(e)$ *and the subgroup* $K_1 = \{s \in N \mid s\mu = \mu s\mu\}$.

In $G^{\mathcal{LC}}$, $N\mu$ *is a left group*, $N_1\mu$ *is the set of idempotents in* $N\mu$, *and* $K_1\mu = \mu K_1\mu$ *is isomorphic to each of the maximal subgroups of* $N\mu$ (*so that* $N\mu = (N_1\mu) \times (K_1\mu)$ *algebraically*). *The subgroups* N_1 *and* K_1 *need not be closed in* N, *but* $\theta : s \mapsto s\mu$ *is continuous*, 1-1 *and a homomorphism from* K_1 *to* $K_1\mu$. *Also*, θ *is a homeomorphism if* N *is compact.*

P r o o f. Take $s \in N$. From Lemma 1, there is an $s' \in N$ such that $\mu s = s'\mu$. Thus we have $(N\mu)s\mu = N\mu s\mu = Ns'\mu^2 = N\mu$, so that $N\mu$ is left simple. From Lemma 8, the set of idempotents in $N\mu$ is $N_1\mu$ where $N_1 = \{s \in N \mid \psi_\mu(s) = e\}$ is a normal subgroup since ψ_μ is a homomorphism. Since $\mu N \subset N\mu$ (Lemma 1 again), the subgroup $\mu N\mu$ of $N\mu$ is equal to μN. If we write $K_1 = \{s \in N \mid \mu s = s\mu = \mu s\mu\}$, then K_1 is a subgroup of N and the map $s \mapsto s\mu$ is an isomorphism of K_1 onto $K_1\mu \cong \mu N\mu$. The structure theory for left simple semigroups tells us that $N\mu = (N_1\mu) \times (K_1\mu)$, so that $N_1 \cap K_1 = \{e\}$; N is the semidirect product of N_1 and K_1.

The map $s \mapsto s\mu$ is continuous and injective on $K_1 \subset G$ (Theorem 0), and a homeomorphism of K_1 when N is compact. It is also a homomorphism on K_1 since the idempotent μ commutes with the elements of K_1.

Examples to show that K_1 and N_1 need not be closed are provided by Proposition 14(i) below. ∎

4. $\mathcal{LC}$-compactifications of some semidirect products. We shall now examine our results in the context of semidirect products $G = N \times K$, where K acts on the compact group N, the multiplication is given by $(s,t)(s',t') = (st(s'), tt')$, and the topology for G is the product topology; then $N \times \{e\} \cong N$ is a normal subgroup of G, so $N \subset G_C$, and $K \cong G/N$. Proposition 11 presents examples of distal flows for which the enveloping semigroups can be explicitly determined. In Proposition 12 and Example 13 the flows are non-distal, but in the former we see that for an idempotent μ in $G^{\mathcal{LC}}$ the group μN can be dense in $N\mu$, while in the latter it need not be.

Observe that, in the same way as for the flow (G,N) above, there is a continuous homomorphism $\zeta \mapsto \psi_\zeta$ of $K^{\mathcal{LC}}$ onto the enveloping semigroup $\Sigma(K,N)$ of the flow (K,N).

We first collect some information about compactifications of semidirect products.

Proposition 10. *Let $G = N \times K$ be a semidirect product as above with N compact.*

(i) *The flow (G,N), $(g,s) \mapsto gsg^{-1}$, is distal if and only if the flow (K,N), $(t,s) \mapsto t(s)$, is distal.*

(ii) *$G^{\mathcal{LC}} \cong N \times K^{\mathcal{LC}}$ and the product in $N \times K^{\mathcal{LC}}$ is given by*

$$(s,\zeta)(s',\zeta') = (s\psi_\zeta(s'), \zeta\zeta').$$

(iii) *Every idempotent $\mu \in G^{\mathcal{LC}}$ has the form $\mu = (s,\xi) \in N \times K^{\mathcal{LC}}$, where ξ is an idempotent in $K^{\mathcal{LC}}$ and $\psi_\xi(s) = e$.*

(iv) *Every minimal left ideal $L \subset G^{\mathcal{LC}}$ has the form $L = N \times \mathfrak{L}$, where $\mathfrak{L}$ is a minimal left ideal in $K^{\mathcal{LC}}$.*

Proof. We omit the details of this proof. For (i) we point out that, if $g = (s,t) \in G$, then $gs'g^{-1} = (\mathcal{I}_s \circ t)(s')$, where $\mathcal{I}_s$ is the inner automorphism $s' \mapsto ss's^{-1}$ of N. Part (ii) is Theorem 5.2.11 in [1], and (iv) comes from it. (iii) is closely related to Lemma 8(i). ∎

Remarks. One can consider extensions that are more general than semidirect products. (See, for example, [6] and the references there for extension formalism.) If the extension $G = N \times K$ is central, the cocycle plays no role in the flow (G,N); we still have $gs'g^{-1} = (\mathcal{I}_s \circ t)(s')$ for $g = (s,t)$, as in the semidirect product case, and the conclusion of Lemma 8(i) still holds. The group $\mathbb{T} \times \mathbb{Z}^2$ of Example 6 and the group $\mathbb{T} \times \mathbb{R}^2$ which follows that example are central extensions; for them, and for all of our semidirect product examples, N is abelian, so the inner automorphisms $\mathcal{I}_s$ of N are trivial. However, statements (ii) and (iii) of Lemma 8 must be substantially altered for the central extension setting. We do not know an example of a non-central extension $N \times K$ of a compact group N. ∎

PROPOSITION 11. *Let K be a discrete abelian group with compact dual group $\widehat{K}$, let K act on* (*the direct product*) $N = \mathbb{T} \times \widehat{K}$ *by* $t : (w, \widehat{t}) \mapsto (w\widehat{t}(t), \widehat{t})$, *and let $G = N \times K$ be the semidirect product. Then*

(i) $\Sigma(K, N) \cong K^{\mathcal{AP}}$, *and*
(ii) N *is normal in* $G^{\mathcal{LC}}$.

Proof. (i) Recall that $K^{\mathcal{AP}} \cong (\widehat{K}_d)\widehat{\ }$. Then if $\{t_\alpha\} \subset K$ converges to $\chi \in K^{\mathcal{AP}}$, we have $t_\alpha(w, \widehat{t}) = (w\widehat{t}(t_\alpha), \widehat{t}) \to (w\chi(\widehat{t}), \widehat{t})$ for all $(w, \widehat{t}) \in N$, from which it is not hard to see that $\Sigma(K, N) \cong K^{\mathcal{AP}}$.

(ii) Since $K^{\mathcal{AP}}$ is a group, the flow (K, N) is distal. Hence (G, N) is distal and N is normal in $G^{\mathcal{LC}}$ by Theorem 3. ∎

REMARK. The flows (K, N) and (G, N) in the previous theorem are seldom equicontinuous; for example, they are not if $K = \mathbb{Z}$ and $\widehat{K} = \mathbb{T}$. Nevertheless, we still have an enveloping semigroup which is a topological group, but this is in the pointwise topology and not in the topology of uniform convergence on V as it was in Proposition 4. ∎

The next proposition expands on ideas used in [6], Example 18. For it H and K are topological groups with H compact and K infinite discrete, and G is the semidirect product $N \times K = H^K \times K$ with multiplication $(f, t)(f', t') = (fR_t f', tt')$; here we are thinking of H^K as the set of all functions from K into H, so $R_t f'(t_1) = f'(t_1 t)$. In this setting the action of K on N is never distal so there exist idempotents $\mu \in G^{\mathcal{LC}}$ for which $\mu N \subsetneqq N\mu$; nonetheless, μN is always dense in $N\mu$. In the rest of this paper, βK denotes the Stone–Čech compactification of the discrete space K.

PROPOSITION 12. *Let H, K and $G = N \times K = H^K \times K$ be as above. Then*

(i) $\Sigma(K, N) \cong \beta K$,
(ii) N *is not normal in* $G^{\mathcal{LC}}$, *and*
(iii) μN *is dense in* $N\mu$ *for every idempotent* $\mu \in G^{\mathcal{LC}}$.

Proof. First note that each $f \in N = H^K$ extends to an $\widetilde{f} \in \mathcal{C}(\beta K, H)$, the space of all continuous functions from βK into H; in fact, $N \cong \widetilde{N} := \mathcal{C}(\beta K, H)$. It is then clear how the transformations $\psi_\zeta \in \Sigma(K, N)$, $\zeta \in \beta K$, act on N: if $f \in N$ and $t_\alpha \to \zeta$ in βK, we have $R_{t_\alpha} f \to \psi_\zeta(f)$ (pointwise), so

$$[\psi_\zeta(f)](t) = \lim_\alpha R_{t_\alpha} f(t) = \lim_\alpha f(tt_\alpha) = \lim_\alpha \widetilde{f}(tt_\alpha) = \widetilde{f}(t\zeta).$$

(i) Let $\nu, \eta \in \beta K$ with $\eta \neq \nu$, so there is a set $V \subset K \subset \beta K$ with $\eta \in \overline{V}$ and $\nu \notin \overline{V}$ (closure in βK). Choose a member $h \in H$, $h \neq e$, and define a function $f \in N$ by $f(t) = h$ if $t \in V$, $f(t) = e$ otherwise; then $\widetilde{f}(\eta) \neq \widetilde{f}(\mu)$, so

$$\psi_\eta(f) = \widetilde{f}(\cdot\,\eta) \neq \widetilde{f}(\cdot\,\nu) = \psi_\nu(f)$$

and the homomorphism $\nu' \mapsto \psi_{\nu'}$, $\beta K \to \Sigma(K, N)$, is 1-1, hence an isomorphism.

(ii) Since $\Sigma(K, N) \cong \beta K$, which is not a group, the flow (K, N) is not distal, nor is (G, N), and N is not normal in $G^{\mathcal{LC}}$ (Theorem 3).

(iii) Let $\mu = (\mathfrak{f}, \xi)$ be an idempotent in $G^{\mathcal{LC}} \cong N \times \beta K$, where ξ is an idempotent in βK and $\mathfrak{f} \in N$ with $\psi_\xi(\mathfrak{f}) = e$, and let

$$g \in N\mu = \{(f, e)(\mathfrak{f}, \xi) \mid f \in N\} = \{(f, \xi) \mid f \in N\},$$

$g = (f_1, \xi)$, say. We approximate g by members of

$$\mu N = \{(\mathfrak{f}, \xi)(f, e) \mid f \in N\} = \{(\mathfrak{f}\psi_\xi(f), \xi) \mid f \in N\} \subset N\mu$$

as follows: for any finite $A \subset K$ we find an $f \in N$ such that $(\mathfrak{f}\psi_\xi(f))|_A = f_1|_A$.

So, let L be the left ideal $\beta K\,\xi \subset \beta K$. From Theorem 0, the map $t \mapsto t\xi$, $K \to L$, is injective, and the total disconnectedness of βK gives a continuous function $\mathcal{F} \in \mathcal{C}(L, H)$ such that $\mathcal{F}(t\xi) = [\mathfrak{f}^{-1} f_1](t) = \mathfrak{f}(t)^{-1} f_1(t)$ for all $t \in A$. Then defining $f(t) = \mathcal{F}(t\xi)$ for all $t \in N$, we get the required function f, since $\widetilde{f}(\nu) = \mathcal{F}(\nu\xi)$ for all $\nu \in \beta K$; so $[\psi_\xi(f)](t) = \widetilde{f}(t\xi) = \mathcal{F}(t\xi\xi) = \mathcal{F}(t\xi) = [\mathfrak{f}^{-1} f_1](t)$ for all $t \in A$. ∎

Part (iii) of Proposition 12 raises the intriguing

QUESTION. What conditions does one require on a compact normal subgroup $N \subset G$ to ensure that μN is dense in $N\mu$ for every idempotent $\mu \in G^{\mathcal{LC}}$?

We do not know the answer. In the present setting of semidirect products $G = N \times K$, the action of K on N must not be distal if we want $\mu N \subsetneqq N\mu$. Here is an example where μN is not dense in $N\mu$. It appears in [3] (II.5.5) as an example due to Furstenberg of a proximal flow.

EXAMPLE 13. *There is a group G with a compact normal subgroup N' with the property that for all minimal idempotents $\mu \in G^{\mathcal{LC}}$, $\mu N'$ is a singleton and so is not dense in $N'\mu$.*

P r o o f. Let N be the almost periodic compactification $\mathbb{R}^{\mathcal{AP}} \cong (\mathbb{R}_d)^\wedge$ of the additive real numbers $\mathbb{R}$, and let $K = (\mathbb{R}^+, \times)$ act on N in the natural way,

$$[a(\chi)](x) := \chi(ax) \quad \text{for } a \in K,\ \chi \in N \text{ and } x \in \mathbb{R},$$

so that we have the semidirect product $G = N \times K$. Then, for $n \in \mathbb{N} \subset \mathbb{R}^+$, $[n(\chi)](x) = \chi(x)^n$. Now there is a net $\{n_\alpha\} \subset \mathbb{N}$ such that $w^{n_\alpha} \to 1$ for all $w \in \mathbb{T}$. It follows that $\Sigma(K,N)$ has a zero, onto which every minimal idempotent $\xi \in K^{\mathcal{LC}}$ must map, namely $\psi_\xi(N) = \{e\} \subset N$. Thus $\mu = (\chi,\xi)$ is a minimal idempotent in $G^{\mathcal{LC}}$ for all $\chi \in N$, and $\mu(N,1)$ is the singleton μ. ∎

REMARKS. 1. If $\mathfrak{L}$ is a minimal left ideal in $K^{\mathcal{LC}}$ so that $L = N \times \mathfrak{L}$ is a minimal left ideal in $G^{\mathcal{LC}}$, we have $\mu L = (\chi, \mathfrak{L})$, a maximal subgroup of $L = N \times \mathfrak{L}$ that is not dense in L.

2. This example raises a general question about the maximal subgroups G' of a minimal left ideal L in $G^{\mathcal{LC}}$. Is it possible for G' to be trivial, that is, for L to consist entirely of idempotents? ∎

Part (iii) of the next proposition shows that in the setting of Proposition 12, writing $N = H^K$ as a semidirect product $N_1 \times K_1$ is very much the same as the decomposition of $\mathbb{C}$-valued bounded functions on a group in [5] (Theorem 3–4). For this we identify $N = H^K$ with $\widetilde{N} = \mathcal{C}(\beta K, H)$ and need the definition of a *minimal function* $f \in H^K$. Such a function is characterized by the existence of a minimal idempotent $\xi \in \beta K$ for which $f(\cdot) = \widetilde{f}(\cdot\,\xi)$, where $\widetilde{f}$ is the continuous extension of f to βK. (See [1], §4.8, for more details.)

PROPOSITION 14. *As in Proposition* 12, *let* $G = N \times K = H^K \times K$, *and identify* $N = H^K$ *with* $\widetilde{N} = \mathcal{C}(\beta K, H)$. *Suppose that* μ *is a minimal idempotent in* $G^{\mathcal{LC}}$, *so* $\mu = (\mathfrak{f},\xi) \in L = N \times \mathfrak{L} \subset N \times \beta K \cong G$, *where* ξ *is an idempotent in a minimal left ideal* $\mathfrak{L} \subset \beta K$ *and* $\mathfrak{f} \in N$ *with* $\psi_\xi(\mathfrak{f}) = e \in N$. *As in Theorem* 9, *let* $N_1 = \psi_\mu^{-1}(e)$ *and* $K_1 = \{s \in N \mid s\mu = \mu s \mu\}$.

(i) *The subgroups* N_1 *and* K_1 *of* N *are not closed.*

(ii) *We have* $N_1 = \{f \in N \mid \widetilde{f}(\nu) = e \in H$ *for all* $\nu \in \mathfrak{L}\}$, *and for any other minimal idempotent* $\mu' = (\mathfrak{f}', \xi') \in L = N \times \mathfrak{L}$, *the corresponding subgroup* N_1' *coincides with* N_1.

(iii) $\psi_\xi(N) = \{\widetilde{f}(\cdot\,\xi) \mid f \in N\}$ *is a maximal subgroup of* H-*valued minimal functions on* K, *and* $\psi_\xi(N) \neq \psi_{\xi'}(N)$ *if* ξ *and* ξ' *are different minimal idempotents in* $\mathfrak{L}$; *also*, $K_1 = \mathfrak{f}\,\psi_\xi(N)\mathfrak{f}^{-1}$.

Proof. (i) N_1 contains $\{f \in N \mid \lim_{t\to\infty} f(t) = e \in H\}$, which is dense in N, and Proposition 12(iii) says that $\mu N = \mu N \mu$ is dense in $N\mu$, hence also $K_1 = \{s \in N \mid s\mu = \mu N \mu\}$ is dense in N.

(ii) Observing that e denotes here first the identity of N, then that of K, we have

$$\begin{aligned} N_1 = \psi_\mu^{-1}(e) &= \{f \in N \mid (\mathfrak{f},\xi)(f,e) = (\mathfrak{f}\psi_\xi(f),\xi) = (\mathfrak{f},\xi) = \mu\} \\ &= \{f \in N \mid \psi_\xi(f) = e \in N\} = \{f \in N \mid \widetilde{f}(\,\cdot\,\xi) = e \in N\} \\ &= \{f \in N \mid \widetilde{f}(\nu) = e \in H \text{ for all } \nu \in \mathfrak{L}\} \end{aligned}$$

(since $(K\xi)^- = \mathfrak{L}$). The last claim of (ii) follows from this.

(iii) Note first that $\psi_\xi(N) = \{\widetilde{f}(\cdot\,\xi) \mid f \in N\}$ consists of minimal H-valued functions on K, and indeed is a maximal subgroup in N of such functions.

To see that $\psi_\xi(N) \neq \psi_{\xi'}(N)$ if $\xi \neq \xi'$ are minimal idempotents in $\mathfrak{L}$, take an $F \in \mathcal{C}(L,H)$ with $F(\xi) \neq F(\xi')$ and define $f \in N$ by $f(t) = F(t\xi)$; then $f \in \psi_\xi(N) = \{\widetilde{f_1}(\cdot\,\xi) \mid f_1 \in N\}$ and $f \notin \psi_{\xi'}(N)$, since if $f = F(\cdot\,\xi) = \widetilde{f_2}(\cdot\,\xi')$ for some $f_2 \in N$, then $F(\nu\xi) = \widetilde{f_2}(\nu\xi')$ for all $\nu \in \beta K$, so $F(\xi) = F(e\xi) = \widetilde{f_2}(e\xi') = \widetilde{f_2}(\xi'\xi') = F(\xi'\xi) = F(\xi')$, which is a contradiction.

To establish the last claim, note that an $f \in K_1$ must satisfy

$$(f,e)\mu = (f,e)(\mathfrak{f},\xi) = (f\mathfrak{f},\xi) = \mu(f,e) = (\mathfrak{f},\xi)(f,e) = (\mathfrak{f}\,\psi_\xi(f),\xi),$$

i.e., $f\mathfrak{f} = \mathfrak{f}\,\psi_\xi(f)$. Thus $K_1 = \{f \in N \mid f = \mathfrak{f}\,\psi_\xi(f)\mathfrak{f}^{-1}\} = \mathfrak{f}\,\psi_\xi(N)\mathfrak{f}^{-1}$. ∎

Notes 1. If we consider $G = \mathbb{Z}_2^{\mathbb{Z}} \times \mathbb{Z}$ and want to write $N = \mathbb{Z}_2^{\mathbb{Z}}$ as $N_1 \times K_1$, there are $2^{\mathfrak{c}}$ choices for N_1, since $\beta\mathbb{Z}$ has $2^{\mathfrak{c}}$ minimal left ideals $L = \beta K\xi$ ([1], 4.12.6). For each fixed choice of L, and so of N_1, there are $2^{\mathfrak{c}}$ choices for $K_1 = \psi_\xi(N)$, since $\beta\mathbb{Z}$ also has $2^{\mathfrak{c}}$ minimal right ideals, i.e., L contains $2^{\mathfrak{c}}$ minimal idempotents ξ. Suppose that $\mathbb{Z}_2$ is replaced with a non-abelian group such as the symmetric group S_3, so that the functions $\mathfrak{f} \in N_1$ will have a role to play in determining the K_1's (Proposition 14(iii)). Thus we are considering $\mathrm{S}_3^{\mathbb{Z}} \times \mathbb{Z}$, and for fixed choices of $N_1 = \ker\varepsilon = \{f \in N \mid \psi_\xi(f) = e \in N\}$ and maximal subgroup $\psi_\xi(N)$ of minimal functions, the subgroup N_1 must be very large; for example, it contains

$$\{f \in N = \mathrm{S}_3^{\mathbb{Z}} \mid \lim_{n\to\infty} f(n) = e \in \mathrm{S}_3\}$$

if ξ is at ∞. In such a case, there is a plethora of K_1's of the form $\mathfrak{f}\,\psi_\xi(N)\mathfrak{f}^{-1}$ corresponding to the various $\mathfrak{f}$'s in N_1; often these subgroups $\mathfrak{f}\,\psi_\xi(N)\mathfrak{f}^{-1} \subset N$ do not consist of minimal functions.

2. In the general case $G = H^K \times K$, let us consider what happens if we work with another minimal left ideal $\mathfrak{L}' \subset \beta K$, $\mathfrak{L}' \neq \mathfrak{L}$. Then we get the same maximal subgroups in N of minimal functions on K that we got from $\mathfrak{L}$. For let $\xi' \in \mathfrak{L}'$ be the minimal idempotent that is in the same minimal right ideal as the minimal idempotent $\xi \in \mathfrak{L}$, so that $\xi\xi' = \xi'$ and $\xi'\xi = \xi$; suppose also that $f = \widetilde{f}(\cdot\,\xi) \in \psi_\xi(N)$. Then $\widetilde{f}(\nu) = \widetilde{f}(\nu\xi)$ for all $\nu \in \beta K$, and in particular for $\nu = t\xi'$, so that $\widetilde{f}(t\xi') = \widetilde{f}(t\xi'\xi) = \widetilde{f}(t\xi) = f(t)$. Thus $\psi_\xi(N) = \psi_{\xi'}(N) = \{\widetilde{f}(\cdot\,\xi') \mid f \in N\}$, which implies that the subgroup K_1' from ξ' is equal to K_1 (from ξ) if H is abelian. However, as has been indicated, the subgroups N_1' and N_1 can be quite different even for abelian H; if H is not abelian, then a function $\mathfrak{f}' \in N_1'$ can give a $K_1' = \mathfrak{f}'\psi_{\xi'}(N)\mathfrak{f}'^{-1}$ that is quite different from any K_1 coming from L.

3. The situation is very simple for Example 13, namely $G = N \times K = \mathbb{R}^{\mathcal{AP}} \times \mathbb{R}^+$; for every minimal idempotent $\mu \in G^{\mathcal{LC}}$, we have $N_1 = \ker\varepsilon = N$,

and $K_1 = \{e\}$. On the other hand, when the action of K on N is distal, we always have $N_1 = \{e\}$ and $K_1 = N$. ∎

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Department of Mathematics
University of Alberta
Edmonton, Alberta T6G 2G1
Canada
E-mail: tlau@math.ualberta.ca

Department of Mathematics
University of Western Ontario
London, Ontario N6A 5B7
Canada
E-mail: milnes@uwo.ca

Department of Mathematics
University of Sheffield
Sheffield S10 2TN, England
E-mail: j.pym@sheffield.ac.uk

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Added in proof. With regard to the sentence before Proposition 11, we now have an example of a non-central extension $N \times K$ with N compact. It is modified from a discrete 6-dimensional nilpotent group.