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HOCHSCHILD COHOMOLOGY OF PIECEWISE HEREDITARY ALGEBRAS

BY

DIETER HAPPEL (CHEMNITZ)

Let Λ be a finite-dimensional algebra over an algebraically closed field k. The category of (finite length) Λ-modules is denoted by $\operatorname{mod}\Lambda$. We denote by $D^{\mathrm{b}}(\Lambda)$ the bounded derived category of complexes over $\operatorname{mod}\Lambda$. We say that Λ is *piecewise hereditary of type* $\mathcal{H}$ if there exists a hereditary abelian category $\mathcal{H}$ such that $D^{\mathrm{b}}(\Lambda)$ is triangle-equivalent to the bounded derived category $D^{\mathrm{b}}(\mathcal{H})$ of complexes over $\mathcal{H}$. It is easily seen that for all $X, Y \in \mathcal{H}$, $\operatorname{Hom}_{\mathcal{H}}(X, Y)$ and $\operatorname{Ext}^1_{\mathcal{H}}(X, Y)$ are k-vector spaces which are finite-dimensional over k and the composition is bilinear over k. We refer to these as the *basic properties* of $\mathcal{H}$. The type of Λ is of course only defined up to derived equivalence. This class of algebras has been previously studied in several articles (compare for example [H1], [HRS2], [HR2] and the references in those).

The aim of this note is to show that results in [HR1] and previous results on Hochschild cohomology [H2] allow the computation of the Hochschild cohomology of piecewise hereditary algebras, and to apply these results to show that certain hereditary categories will not admit a tilting complex whose endomorphism algebra is representation-finite. This gives an alternative proof of results previously established by Meltzer [M]. In the first section we recall some of the relevant properties of piecewise hereditary algebras. The second section contains the computations of the Hochschild cohomology and some consequences.

1. Relevant properties of piecewise hereditary algebras. Let $\mathcal{H}$ be a hereditary abelian category and let Λ be a piecewise hereditary algebra of type $\mathcal{H}$. In this section we collect some properties of piecewise hereditary algebras. These results are contained in [H1], [HRS], [HR1] or [HR2].

First we have to recall some of the relevant terminology. A finite-dimensional algebra is said to be *representation-finite* if there are only finitely many isomorphism classes of indecomposable Λ-modules. Otherwise we say

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that Λ is *representation-infinite*. Also, Λ is said to be *representation-directed* if no indecomposable Λ-module lies on a cycle; by definition a *cycle* is a sequence

$$X_0 \xrightarrow{f_0} X_1 \to \ldots \to X_{r-1} \xrightarrow{f_{r-1}} X_r \simeq X_0$$

where $r \geq 1$, X_i is indecomposable for $0 \leq i \leq r$ and $f_i \in \mathrm{Hom}_\Lambda(X_{i-1}, X_i)$ is non-zero and non-invertible for $0 \leq i \leq r-1$. Note that a representation-directed algebra necessarily is representation-finite [R]. A representation-finite algebra Λ is said to be *simply connected* if it satisfies the separation condition (compare [BLS]).

We say that a finite-dimensional algebra Λ is *tilting-cotilting equivalent* to a finite-dimensional algebra Γ if there exists a sequence $\Lambda_0, \ldots, \Lambda_m$ of finite-dimensional algebras and a sequence ${}_{\Lambda_0}T_0, \ldots, {}_{\Lambda_{m-1}}T_{m-1}$ of tilting modules or cotilting modules such that $\Lambda = \Lambda_0$, $\Lambda_i = \mathrm{End}_{\Lambda_{i-1}} T_{i-1}$ for $0 < i \leq m$ and $\Gamma = \Lambda_m$.

Finally, a finite-dimensional algebra Γ is called a *quasitilted algebra* if there exists a hereditary abelian category $\mathcal{H}$ satisfying our basic properties and a tilting object $T \in \mathcal{H}$ such that $\Gamma = \mathrm{End}\, T$. Recall that $T \in \mathcal{H}$ is said to be a *tilting object* if $\mathrm{Fac}\, T = \{X \in \mathcal{H} \mid \mathrm{Ext}^1(T, X) = 0\}$, where $\mathrm{Fac}\, T$ is the full subcategory of $\mathcal{H}$ containing the epimorphic images of finite direct sums of indecomposable summands of T. Quasitilted algebras can also be characterized homologically as was shown in [HRS1]. In fact, a finite-dimensional algebra Λ is quasitilted if and only if the global dimension of Λ is at most two and each indecomposable Λ-module X satisfies either $\mathrm{proj.dim}_\Lambda X \leq 1$ or $\mathrm{inj.dim}_\Lambda X \leq 1$.

THEOREM 1.1. *Let Λ be a piecewise hereditary algebra of type $\mathcal{H}$. Then*:

(i) *Λ is a factor algebra of a finite-dimensional hereditary algebra.*
(ii) *Λ is tilting-cotilting equivalent to a quasitilted algebra.*
(iii) *If Λ is representation-finite, then Λ is representation-directed.*

2. Hochschild cohomology. Let us briefly recall the two main examples of connected hereditary abelian categories containing a tilting object.

First of all let H be a finite-dimensional connected hereditary k-algebra. Then $\mathrm{mod} H$ is trivially such an example. If H is in addition basic we may assume that $H = k\vec{\Delta}$ where $k\vec{\Delta}$ denotes the path algebra of a finite quiver $\vec{\Delta}$ without oriented cycles. For later purposes we recall the following notation. The set of vertices of $\vec{\Delta}$ is denoted by Δ_0 and the set of arrows is denoted by Δ_1. For an arrow α in $\vec{\Delta}$ we denote by $s(\alpha)$ the starting point and by $e(\alpha)$ the end point of α, where vertices are interpreted as idempotents in $k\vec{\Delta}$. Let α be an arrow in $\vec{\Delta}$; then $\nu(\alpha) = \dim_k s(\alpha)(k\vec{\Delta})e(\alpha)$. Moreover, let n be the number of vertices in $\vec{\Delta}$.

Secondly, let $p = (p_1, \ldots, p_t)$ with $p_i > 1$ be a weight sequence of natural numbers and let $\lambda = (1 = \lambda_3, \ldots, \lambda_t)$ with $\lambda_i \in k \setminus \{0\}$ be a set of distinct parameters. Let $\mathbb{X}(p, \lambda)$ be a weighted projective line over k of type (p, λ) in the sense of [GL]. Let $\mathcal{H}(p, \lambda) = \operatorname{coh} \mathbb{X}(p, \lambda)$ be the category of coherent sheaves on $\mathbb{X}(p, \lambda)$. Then $\mathcal{H}(p, \lambda)$ is a hereditary abelian category satisfying our basic properties. Given the weight sequence $p = (p_1, \ldots, p_t)$ we define $d_p = (t-2) - \sum_{i=1}^{t} 1/p_i$. If $d_p < 0$ then the corresponding sheaf category is of domestic type. It is known that in this case the sheaf category is derived-equivalent to the module category of a tame hereditary algebra. So one may assume that $d_p \geq 0$ in order to distinguish the two cases. In particular we will assume that $t \geq 3$.

It is known that $D^{\mathrm{b}}(\mathcal{H}(p, \lambda))$ is triangle-equivalent to $D^{\mathrm{b}}(C(p, \lambda))$, where $C(p, \lambda)$ is a canonical algebra of type (p, λ) in the sense of [R]. In fact, there is a tilting object $T \in \mathcal{H}(p, \lambda)$ such that $\operatorname{End} T = C(p, \lambda)$. For our purposes the following description of the canonical algebras is important: They are one-point extensions of the path algebra of the quiver $\vec{\Delta}$ in Figure 1 by an indecomposable module M whose dimension vector is defined by $\dim_k \operatorname{Hom}_{k\vec{\Delta}}(P(\omega), M) = 2$ and $\dim_k \operatorname{Hom}_{k\vec{\Delta}}(P(i), M) = 1$ for all other indecomposable projective $k\vec{\Delta}$-modules. But we have to require that M as a representation of $\vec{\Delta}$ has the property that the one-dimensional subspaces $M(i)$ for $1 \leq i \leq t$ of $M(\omega)$ are pairwise different. The number of arrows in the different branches is $p_i - 1$ for $1 \leq i \leq t$.

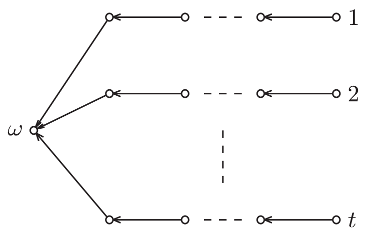

Fig. 1

We now present the results on the Hochschild cohomology of piecewise hereditary algebras. We denote by $H^i(\Lambda)$ the ith cohomology space (i.e. $H^i(\Lambda) = \operatorname{Ext}^i_{\Lambda^{\mathrm{e}}}(\Lambda, \Lambda)$, where Λ^{e} denotes the enveloping algebra).

THEOREM 2.1. *Let Λ be a connected piecewise hereditary algebra. Then $H^0(\Lambda) \simeq k$ and $H^i(\Lambda) = 0$ for $i \geq 3$.*

Proof. By 1.1(i) we know that Λ is a factor algebra of a finite-dimensional hereditary algebra. In other words, there is no oriented cycle in the quiver of Λ, hence Λ has trivial center and so $H^0(\Lambda) \simeq k$. Moreover, we know by 1.1(ii) that Λ is tilting-cotilting equivalent to a quasitilted algebra

Γ. By the tilting invariance of Hochschild cohomology [H2] we infer that $H^i(\Lambda) \simeq H^i(\Gamma)$. Since $\mathrm{gl.dim}\,\Gamma \le 2$ we have $H^i(\Gamma) = 0$ for $i \ge 3$, for $\mathrm{proj.dim}_{\Gamma^e}\Gamma = \mathrm{gl.dim}\,\Gamma$. Hence we have shown the assertion.

In the next two results we deal with the two main cases of hereditary categories mentioned above. We keep the notation introduced above. The first of these results follows easily as in the previous theorem from 1.1 and the calculations in [H2] of the Hochschild cohomology of finite-dimensional hereditary algebras.

THEOREM 2.2. *Let Λ be a connected piecewise hereditary algebra of type* $\mathrm{mod}\,H$ *for a basic connected finite-dimensional hereditary k-algebra $H = k\vec{\Delta}$, where $\vec{\Delta}$ is a finite quiver without cycles. Then $H^0(\Lambda) = k$, $\dim_k H^1(\Lambda) = 1 - n + \sum_{\alpha\in\Delta_1}\nu(\alpha)$ and $H^i(\Lambda) = 0$ for $i \ge 2$.*

COROLLARY 2.3. *Let Λ be a connected piecewise hereditary algebra of type* $\mathrm{mod}\,H$ *for a basic connected finite-dimensional hereditary k-algebra $H = k\vec{\Delta}$, where $\vec{\Delta}$ is a finite quiver without cycles. If Λ is representation-finite, then Λ is simply connected if and only if the underlying graph Δ of $\vec{\Delta}$ is a tree.*

Proof. By 1.1(iii) we know that Λ is representation-directed. It was shown in [H2] that a representation-directed algebra is a simply connected algebra if and only if $H^1(\Lambda) = 0$. Thus Λ is simply connected if and only if $1-n+\sum_{\alpha\in\Delta_1}\nu(\alpha) = 0$. It is straightforward to see that $1-n+\sum_{\alpha\in\Delta_1}\nu(\alpha) = 0$ if and only if Δ is a tree.

THEOREM 2.4. *Let Λ be piecewise hereditary of type* $\mathrm{coh}\,\mathbb{X}(p,\lambda)$ *with $d_p \ge 0$. Then $H^0(\Lambda) \simeq k$, $H^1(\Lambda) = 0$ and $\dim_k H^2(\Lambda) = t-3$.*

Proof. Since $D^{\mathrm{b}}(\Lambda)$ is triangle-equivalent to $D^{\mathrm{b}}(\mathrm{coh}\,\mathbb{X}(p,\lambda))$ and $D^{\mathrm{b}}(\mathrm{coh}\,\mathbb{X}(p,\lambda))$ is triangle-equivalent to $D^{\mathrm{b}}(C(p,\lambda))$ we infer by [Ri] that $H^i(\Lambda) \simeq H^i(C(p,\lambda))$ for all $i \ge 0$. Now we consider $C(p,\lambda)$ as above to be the one-point extension $k\vec{\Delta}[M]$, where $\vec{\Delta}$ is the quiver as above. In [H2] we constructed a long exact sequence for the Hochschild cohomology of a one-point extension. Using this and the fact that $k\vec{\Delta}$ is a hereditary algebra and that $\mathrm{End}\,M \simeq k$ we infer that we obtain an exact sequence

$$0 \to H^1(\Lambda) \to H^1(k\vec{\Delta}) \to \mathrm{Ext}^1_{k\vec{\Delta}}(M,M) \to H^2(\Lambda) \to 0.$$

Since Δ is a tree we infer that $H^1(k\vec{\Delta}) = 0$, hence $H^1(\Lambda) = 0$. Thus $\dim_k H^2(\Lambda) = \dim_k \mathrm{Ext}^1_{k\vec{\Delta}}(M,M) = t-3$. The last equality is easily established by evaluating the Tits form (see for example [R]) at the dimension vector of M and using the fact that $\mathrm{End}\,M \simeq k$.

The first part of the next corollary was previously shown in [M] by using entirely different methods such as the transitivity of the braid group action on the set of complete exceptional sequences.

COROLLARY 2.5. *Let Λ be piecewise hereditary of type* $\operatorname{coh}\mathbb{X}(p,\lambda)$ *with* $d_p \geq 0$.

(i) *If* $t \geq 4$, *then* Λ *is representation-infinite.*

(ii) *If* Λ *is representation-finite*, *then* Λ *is simply connected.*

Proof. We start by showing (i). If $t \geq 4$ we infer by 2.4 that $H^2(\Lambda) \neq 0$. If Λ is representation-finite, then Λ is representation-directed by 1.1(iii). But then it was shown in [H2] that $H^i(\Lambda) = 0$ for $i \geq 2$, a contradiction.

The second assertion follows as in 2.3 by using the fact that Λ is simply connected if and only if $H^1(\Lambda) = 0$. The latter holds by 2.4.

Note that there are examples of representation-finite piecewise hereditary algebras of type $\operatorname{coh}\mathbb{X}(p,\lambda)$ with $d_p \geq 0$. However, the question is open for which weight sequences there actually exists a representation-finite piecewise hereditary algebra of the corresponding type. The related problem in the case when the piecewise hereditary algebra is of type $\operatorname{mod} H$ for a basic connected finite-dimensional hereditary k-algebra H is also unsolved.

It is also an open question if there exists a connected piecewise hereditary algebra Λ with $H^1(\Lambda) \neq 0 \neq H^2(\Lambda)$.

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Fakultät für Mathematik
Technische Universität Chemnitz
D-09107 Chemnitz, Germany
E-mail: happel@mathematik.tu-chemnitz.de

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