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# DISCONTINUOUS QUASILINEAR ELLIPTIC PROBLEMS AT RESONANCE

BY

NIKOLAOS C. KOUROGENIS AND
NIKOLAOS S. PAPAGEORGIOU (ATHENS)

In this paper we study a quasilinear resonant problem with discontinuous right hand side. To develop an existence theory we pass to a multivalued version of the problem, by filling in the gaps at the discontinuity points. We prove the existence of a nontrivial solution using a variational approach based on the critical point theory of nonsmooth locally Lipschitz functionals.

**1. Introduction.** Let $Z \subseteq \mathbb{R}^N$ be a bounded domain with $C^1$-boundary $\Gamma$. In this paper we consider the following quasilinear Dirichlet problem at resonance with discontinuities:

$$(1)\qquad \begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \\ \qquad = \lambda_1|x(z)|^{p-2}x(z) + f(z, x(z)) \quad \text{a.e. on } Z, \\ x_{|\Gamma} = 0, \quad 2 \le p < \infty. \end{cases}$$

Here $\lambda_1$ denotes the first eigenvalue of the $p$-Laplacian

$$-\Delta_p x = -\operatorname{div}(\|Dx\|^{p-2}Dx)$$

with Dirichlet boundary conditions (i.e. of $(-\Delta_p, W_0^{1,p}(Z))$). In this work we deal with the case where $f(z,x)$ has nonzero limits as $x \to \pm\infty$. This implies that the potential $F(z,x) = \int_0^x f(z,r)\,dr$ goes to infinity as $x \to \pm\infty$. This case was studied by Ahmad–Lazer–Paul [1] and Rabinowitz [9]. The case of finite limits as $x \to \pm\infty$ was examined by Thews [10], Ward [11] and Benci–Bartolo–Fortunato [3]. In the last paper, this kind of problems were called "strongly resonant". All these works deal with semilinear equations which have a continuous term $f(z,x)$.

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In our work we do not make any continuity hypotheses on the function $f(z,x)$. So problem (1) need not have a solution. In order to develop a reasonable existence theory, we need to pass to a multivalued version of (1) by, roughly speaking, filling in the gaps at the discontinuity points of $f(z,\cdot)$. So we introduce the following two functions:

$$f_1(z,x)=\varliminf_{x'\to x} f(z,x')=\lim_{\delta\downarrow 0}\operatorname*{ess\,inf}_{|x'-x|<\delta} f(z,x'),$$
$$f_2(z,x)=\varlimsup_{x'\to x} f(z,x')=\lim_{\delta\downarrow 0}\operatorname*{ess\,sup}_{|x'-x|<\delta} f(z,x').$$

Evidently, $f_1\le f_2$ and we set $\overline{f}(z,x)=[f_1(z,x),f_2(z,x)]=\{y\in\mathbb{R}: f_1(z,x)\le y\le f_2(z,x)\}$. Then instead of (1) we study the following multivalued problem:

$$\text{(2)}\qquad \begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \\ \qquad \in \lambda_1|x(z)|^{p-2}x(z)+\overline{f}(z,x(z)) \quad \text{a.e. on } Z, \\ x_{|\Gamma}=0, \quad 2\le p<\infty. \end{cases}$$

Definition. By a *solution* of (2) we mean a function $x\in W_0^{1,p}(Z)$ such that $\|Dx\|^{p-2}Dx\in W^{1,q}(Z,\mathbb{R}^N)$ and there exists $g\in L^q(Z)$ such that $g(z)\in\overline{f}(z,x(z))$ a.e. on $Z$ and

$$-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z))=\lambda_1|x(z)|^{p-2}x(z)+g(z)$$
$$\text{a.e. on } Z \text{ (here } 1/p+1/q=1).$$

Our approach to obtain a solution of problem (2) is variational, based on the critical point theory of nonsmooth, locally Lipschitz energy functionals as developed by Chang [5]. In the next section, for the convenience of the reader, we recall some basic definitions and facts of this theory.

**2. Preliminaries.** The nonsmooth critical point theory developed by Chang [5] is based on the subdifferential theory for locally Lipschitz functionals due to Clarke [6].

Let $X$ be a Banach space and $X^*$ its topological dual. A function $f: X\to\mathbb{R}$ is said to be *locally Lipschitz* if for every $x\in X$ there exists a neighbourhood $U$ of $x$ and a constant $k>0$ depending on $U$ such that $|f(z)-f(y)|\le k\|z-y\|$ for every $z,y\in U$. From convex analysis we know that a proper, convex and lower semicontinuous $g: X\to\overline{\mathbb{R}}=\mathbb{R}\cup\{+\infty\}$ is locally Lipschitz in the interior of its *effective domain* $\operatorname{dom} g=\{x\in X: g(x)<\infty\}$. The *generalized directional derivative* of $f(\cdot)$ at $x$ in the direction $y\in X$ is defined by

$$f^0(x;y)=\varlimsup_{\substack{x'\to x\\ \lambda\downarrow 0}}\frac{f(x'+\lambda y)-f(x')}{\lambda}.$$

It is easy to check that $f^0(x;\cdot)$ is sublinear and continuous (in fact, $k$-Lipschitz). So, by the Hahn–Banach theorem, it is the support function of the convex set $\partial f(x)$ given by

$$\partial f(x) = \{x^* \in X^* : (x^*, y) \le f^0(x; y) \text{ for all } y \in X\}.$$

The set $\partial f(x)$ is always nonempty, bounded and $w^*$-closed (hence $w^*$-compact) and it is called the *subdifferential* of $f(\cdot)$ at $x$. If $f(\cdot)$ is also convex, then this subdifferential coincides with the subdifferential in the sense of convex analysis. Moreover, in this case we also have $f^0(x;\cdot) = f'(x;\cdot)$ with $f'(x;\cdot)$ being the directional derivative at $x$ of the convex function $f$.

Also, if $f(\cdot)$ is strictly differentiable at $x \in X$ (in particular, if $f(\cdot)$ is continuously Gateaux differentiable at $x$), then $\partial f(x) = \{f'(x)\}$. If $f, g : X \to \mathbb{R}$ are locally Lipschitz functions then $\partial(f+g)(x) \subseteq \partial f(x) + \partial g(x)$ and $\lambda \partial f(x) = \partial(\lambda f)(x)$ for all $x \in X$ and all $\lambda \in \mathbb{R}$. Finally, if $f(\cdot)$ has a local minimum at $x \in X$, then $0 \in \partial f(x)$.

If $f : X \to \mathbb{R}$ is locally Lipschitz, then a point $x \in X$ is said to be a *critical point* of $f(\cdot)$ if $0 \in \partial f(x)$. We say that $f(\cdot)$ satisfies the *Palais–Smale condition* (*(PS)-condition*) if any sequence $\{x_n\}_{n\ge1} \subseteq X$ along which $\{f(x_n)\}_{n\ge1}$ is bounded and $m(x_n) = \min\{\|x^*\| : x^* \in \partial f(x_n)\} \to 0$ as $n \to \infty$ has a strongly convergent subsequence. Since for $f \in C^1(X)$, $\partial f(x) = f'(x)$ for all $x \in X$, we see that when $f(\cdot)$ is smooth we recover the classical (PS)-condition (see Rabinowitz [9]).

The following theorem is due to Chang [5] and extends to a nonsmooth setting the well-known "mountain pass theorem" due to Ambrosetti–Rabinowitz [2].

THEOREM 1. *If $X$ is a reflexive Banach space, $R(\cdot) : X \to \mathbb{R}$ is a locally Lipschitz functional which satisfies the $(PS)$-condition and for some $\varrho > 0$ and $y \in X$ with $\|y\| > \varrho$ we have*

$$\max\{R(0), R(y)\} \le \alpha < \beta \le \inf[R(x) : \|x\| = \varrho]$$

*then $R(\cdot)$ has a critical point $x^* \in X$ such that $R(x^*) = c \ge \beta$ is characterized by*

$$c = \inf_{\Gamma} \max_{t} R(\gamma(t))$$

*where $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0,\ \gamma(1) = y\}$.*

In problem (2) there appears the first eigenvalue $\lambda_1$ of $(-\Delta_p, W_0^{1,p}(Z))$. This is the least real number $\lambda$ for which the problem

$$\begin{cases} -\operatorname{div}(\|Dx(z)\|^{p-2} Dx(z)) = \lambda |x(z)|^{p-2} x(z) & \text{a.e. on } Z, \\ x_{|\Gamma} = 0, \end{cases}$$

has a nontrivial solution. This eigenvalue $\lambda_1$ is positive, isolated and simple (i.e. the associated eigenfunctions are constant multiples of each other). Furthermore, we have a variational characterization via the Rayleigh quotient, i.e.

$$\lambda_1 = \min[\|Dx\|_p^p/\|x\|_p^p : x \in W_0^{1,p}(Z)].$$

This minimum is realized at the normalized eigenfunction $u_1$. Note that if $u_1$ minimizes the Rayleigh quotient, then so does $|u_1|$ and hence we infer that the first eigenfuction $u_1$ does not change sign on $Z$. In fact, we can show that $u_1 \neq 0$ a.e. on $Z$ (usually we take $u_1(z) > 0$ a.e. on $Z$). For details we refer to Lindqvist [8].

**3. Existence theorems.** We start by introducing our hypotheses on the discontinuous term $f(z,x)$. Recall that a function $h : Z \times \mathbb{R} \to \mathbb{R}$ is said to be *N-measurable* if for all $x : Z \to \mathbb{R}$ measurable, $z \to h(z,x(z))$ is measurable (superpositional measurability). The hypotheses are:

H($f$): $f : Z \times \mathbb{R} \to \mathbb{R}$ is a Borel measurable function such that

(i) $f_1, f_2$ are N-measurable;
(ii) for almost all $z \in Z$ and all $x \in \mathbb{R}$, we have $|f(z,x)| \le \alpha(z)$ with $\alpha \in L^\infty(Z)$;
(iii) for almost all $z \in Z$, we have $f_1(z,x), f_2(z,x) \to f_+(z)$ as $x \to +\infty$, $f_1(z,x), f_2(z,x) \to f_-(z)$ as $x \to -\infty$ and $f_-(z) \le 0 \le f_+(z)$ with strict inequalities on sets of positive Lebesgue measure;
(iv) there exists $\mu > \lambda_1$ such that uniformly for almost all $z \in Z$ we have

$$\overline{\lim_{x\to 0}} \frac{pF(z,x)}{|x|^p} \le -\mu.$$

Let $J : W_0^{1,p}(Z) \to \mathbb{R}_+$ and $G : W_0^{1,p}(Z) \to \mathbb{R}$ be defined by

$$J(x) = \frac{1}{p}\|Dx\|_p^p \quad \text{and} \quad G(x) = \frac{\lambda_1}{p}\|x\|_p^p + \int_Z F(z,x(z))\,dz.$$

Clearly, $J(\cdot) \in C^1(W_0^{1,p}(Z))$ and is convex (thus locally Lipschitz; see Section 2) and $G(\cdot)$ is locally Lipschitz (see Chang [5]). Set $R(x) = J(x) - G(x)$. Then $R : W_0^{1,p}(Z) \to \mathbb{R}$ is locally Lipschitz.

PROPOSITION 2. *If hypotheses* H($f$) *hold*, *then* $R(\cdot)$ *satisfies the* ($PS$)-*condition.*

Proof. Let $\{x_n\}_{n\ge1} \subseteq W_0^{1,p}(Z)$ be a sequence such that $\{R(x_n)\}_{n\ge1}$ is bounded and $m(x_n) \to 0$ as $n \to \infty$. So for some $M_1 > 0$ and all $n \ge 1$ we have $|R(x_n)| \le M_1$, hence

$$-M_1 \le \frac{1}{p}\|Dx_n\|_p^p - \frac{\lambda_1}{p}\|x_n\|_p^p - \int_Z F(z,x_n(z))\,dz \le M_1. \tag{3}$$

Suppose that $\{x_n\}_{n\geq 1}$ is unbounded. Then we may assume (at least for a subsequence) that $\|x_n\|_{1,p} \to \infty$ as $n \to \infty$. Let $y_n = x_n/\|x_n\|_{1,p}$, $n \geq 1$. Then, by passing to a subsequence if necessary, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(Z), \quad y_n \to y \text{ in } L^p(Z),$$
$$y_n(z) \to y(z) \text{ a.e. on } Z, \quad |y_n(z)| \leq h(z) \text{ a.e. on } Z \text{ with } h \in L^p(Z).$$

Divide (3) by $\|x_n\|_{1,p}^p$ to obtain

$$-\frac{M_1}{\|x_n\|_{1,p}^p} \leq \frac{1}{p}\|Dy_n\|_p^p - \frac{\lambda_1}{p}\|y_n\|_p^p - \int_Z \frac{F(z,x_n(z))}{\|x_n\|_{1,p}^p}\,dz \leq \frac{M_1}{\|x_n\|_{1,p}^p}. \tag{4}$$

Note that

$$\begin{aligned}\left|\int_Z \frac{F(z,x_n(z))}{\|x_n\|_{1,p}^p}\,dz\right| &= \left|\int_Z \int_0^{x_n(z)} \frac{f(z,r)}{\|x_n\|_{1,p}^p}\,dr\,dz\right| \\ &\leq \int_Z \frac{1}{\|x_n\|_{1,p}^p} \int_0^{x_n(z)} \alpha(z)\,dr\,dz \quad \text{(using hypothesis H}(f)\text{(ii))} \\ &\leq \int_Z \frac{\alpha(z)}{\|x_n\|_{1,p}^p}|x_n(z)|\,dz \leq \frac{\|x_n\|_p}{\|x_n\|_{1,p}^p}\|\alpha\|_q \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Thus by passing to the limit as $n \to \infty$ in (4), we obtain

$$\frac{1}{p}\varliminf \|Dy_n\|_p^p = \frac{\lambda_1}{p}\|y\|_p^p. \tag{5}$$

From the weak lower semicontinuity of the norm functional we see that

$$\frac{1}{p}\|Dy\|_p^p \leq \frac{1}{p}\varliminf \|Dy_n\|_p^p. \tag{6}$$

Moreover, from the variational characterization of $\lambda_1$ (see Section 2), we have

$$\frac{\lambda_1}{p}\|y\|_p^p \leq \frac{1}{p}\|Dy\|_p^p. \tag{7}$$

Combining (5), (6) and (7), we infer that

$$\|Dy\|_p^p = \lambda_1\|y\|_p^p.$$

Since $\|y_n\|_{1,p}^p = 1$ for $n \geq 1$ and $\|y_n\|_p^p \to \|y\|_p^p$ as $n \to \infty$, we have $\|Dy_n\|_p^p \to 1 - \|y\|_p^p$ as $n \to \infty$. So using the previous relations we have $\lim \|Dy_n\|_p^p = \|Dy\|_p^p$ and we conclude that $\|y\|_{1,p} = 1$, i.e. $y \neq 0$. Without any loss of generality we will assume that $y = +u_1$ (the analysis is the same when $y = -u_1$). So $y(z) = u_1(z) > 0$ a.e. on $Z$ (see Section 2). Let $x_n^* \in \partial R(x_n)$ such that $m(x_n) = \|x_n^*\|_{-1,q}$, $n \geq 1$. The existence of such an element follows from the fact that $\partial R(x_n)$ is a nonempty weakly compact subset of $W^{-1,q}(Z)$ (see Section 2) and from the weak lower semicontinuity of the norm functional. Let $A : W_0^{1,p}(Z) \to W^{-1,q}(Z)$ be defined by

$$\langle A(x), y\rangle = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{\mathbb{R}^N}\, dz \quad \text{for all } x, y \in W_0^{1,p}(Z).$$

Here by $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,q}(Z))$. It is easy to see that the operator $A(\cdot)$ is monotone, demicontinuous (i.e. if $x_n \to x$ in $W_0^{1,p}(Z)$ as $n\to\infty$, then $A(x_n) \xrightarrow{w} A(x)$ in $W^{-1,q}(Z)$ as $n\to\infty$), hence maximal monotone. As such it has the generalized pseudomonotone property (see Browder–Hess [4]). We have

$$x_n^* = A(x_n) - \lambda_1\|x_n\|^{p-2}x_n - v_n$$

with $v_n \in \partial K(x_n)$, where $K : W_0^{1,p}(Z) \to \mathbb{R}$ is defined by

$$K(x) = \int_Z F(z, x(z))\, dz.$$

Using Theorem 2.2 of Chang [5], we have

$$\partial K(x) \subseteq \Big\{ v \in L^q(Z) : \int_Z v(z)w(z)\, dz \le K^0(x; w) \text{ for all } w \in L^p(Z)\Big\},$$

where

$$K^0(x; w) = \varlimsup_{\substack{h\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}[K(x + h + \lambda w) - K(x + h)].$$

So we have

$$K^0(x; w) = \varlimsup_{\substack{h\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}\int_Z \int_{(x+h)(z)}^{(x+h+\lambda w)(z)} f(z, r)\, dr\, dz.$$

Performing a change of variables to $r(\eta) = x(z) + h(z) + \eta\lambda w(z)$ and using Fatou's lemma we obtain

$$\begin{aligned} K^0(x; w) &= \varlimsup_{\substack{h\to 0\\ \lambda\downarrow 0}} \frac{1}{\lambda}\int_Z\int_0^1 f(z, x(z) + h(z) + \eta\lambda w(z))\lambda w(z)\, d\eta\, dz \\ &\le \int_Z \varlimsup_{\substack{h\to 0\\ \lambda\downarrow 0}} \int_0^1 f(z, x(z) + h(z) + \eta\lambda w(z))w(z)\, d\eta\, dz \\ &\le \int_{\{w>0\}} f_2(z, x(z))w(z)\, dz + \int_{\{w<0\}} f_1(z, x(z))w(z)\, dz. \end{aligned}$$

Therefore if $v \in \partial K(x)$, we have

$$\int_Z v(z)w(z)\,dz \le \int_{\{w>0\}} f_2(z,x(z))w(z)\,dz + \int_{\{w<0\}} f_1(z,x(z))w(z)\,dz \quad \text{for all } w \in L^p(Z).$$

Hence $v(z) \in [f_1(z,x(z)), f_2(z,x(z))]$ a.e. on $Z$. Thus for every $n \ge 1$ we have $f_1(z,x(z)) \le v_n(z) \le f_2(z,x_n(z))$ a.e. on $Z$.

From the choice of the sequence $\{x_n\}_{n\ge1}$ we have

$$|R(x_n)| \le M_1, \quad |\langle x_n^*, u\rangle| \le \varepsilon_n \|u\|_{1,p} \quad \text{for all } u \in W_0^{1,p}(Z) \text{ with } \varepsilon_n \downarrow 0.$$

So, taking $u = x_n$ we have

$$-M_1 p \le \|Dx_n\|_p^p - \lambda_1 \|x_n\|_p^p - p\int_Z F(z,x_n(z))\,dz \le M_1 p \tag{8}$$

and

$$-\varepsilon_n\|x_n\|_{1,p} \le -\langle A(x_n), x_n\rangle + \lambda_1\|x_n\|_p^p + \int_Z v_n(z)x_n(z)\,dz \le \varepsilon_n \|x_n\|_{1,p}. \tag{9}$$

Note that $\langle A(x_n), x_n\rangle = \|Dx_n\|_p^p$. Then adding (8) and (9), we obtain

$$-pM_1 - \varepsilon_n\|x_n\|_{1,p} \le \int_Z (v_n(z)x_n(z) - pF(z,x_n(z)))\,dz \le pM_1 + \varepsilon_n\|x_n\|_{1,p}.$$

Divide by $\|x_n\|_{1,p}$. We have

$$\frac{-pM_1}{\|x_n\|_{1,p}} - \varepsilon_n \le \int_Z \left(v_n(z)y_n(z) - \frac{pF(z,x_n(z))}{\|x_n\|_{1,p}}\right) dz \le \frac{pM_1}{\|x_n\|_{1,p}} + \varepsilon_n \tag{10}$$

Recalling that $y_n(z) \to y(z) = u_1(z) > 0$ as $n \to \infty$ for almost all $z \in Z$, we deduce that $x_n(z) \to +\infty$ as $n \to \infty$. Thus by hypothesis H($f$)(iii) we have $\int_Z v_n(z)y_n(z)\,dz \to \int_Z f_+(z)u_1(z)\,dz$. On the other hand, if we fix $z \in Z \setminus N$, $|N| = 0$ (here $|\cdot|$ denotes the Lebesgue measure on $Z$ and $N$ is the Lebesgue-null set outside of which we have $f(z,x_n(z)) \to f_+(z)$), then given $\varepsilon > 0$ we can find $n_0(z) \ge 1$ such that for all $n \ge n_0(z)$ we have $x_n(z) \ge x_{n_0}(z) > 0$ and $|f(z,x_n(z)) - f_+(z)| < \varepsilon$.

So we see that

$$\frac{pF(z,x_n(z))}{x_n(z)} = \frac{p}{x_n(z)} \int_0^{x_n(z)} f(z,r)\,dr = \frac{p}{x_n(z)} \int_0^{x_{n_0}(z)} f(z,r)\,dr + \frac{p}{x_n(z)} \int_{x_{n_0}(z)}^{x_n(z)} f(z,r)\,dr$$

implies

$$-\frac{p}{x_n(z)}x_{n_0}(z)\|\alpha\|_\infty + \frac{p}{x_n(z)}(x_n(z)-x_{n_0}(z))(f_+(z)-\varepsilon)$$
$$\leq \frac{pF(z,x_n(z))}{x_n(z)} \leq \frac{p}{x_n(z)}x_{n_0}(z)\|\alpha\|_\infty + \frac{p}{x_n(z)}(x_n(z)-x_{n_0}(z))(f_+(z)+\varepsilon).$$

Since $\varepsilon > 0$ was arbitrary, from the above inequalities we infer that

$$\frac{pF(z,x_n(z))}{x_n(z)} \xrightarrow{n\to\infty} pf_+(z) \quad \text{for all } z \in Z\setminus N,\ |N|=0.$$

Therefore

$$\int_Z \frac{pF(z,x_n(z))}{\|x_n\|_{1,p}}\,dz = \int_Z \frac{pF(z,x_n(z))}{x_n(z)}\cdot\frac{x_n(z)}{\|x_n\|_{1,p}}\,dz$$
$$= \int_Z \frac{pF(z,x_n(z))}{x_n(z)}y_n(z)\,dz \xrightarrow{n\to\infty} p\int_Z f_+(z)u_1(z)\,dz.$$

So if we pass to the limit as $n\to\infty$ in (10), we obtain

$$(1-p)\int_Z f_+(z)u_1(z)\,dz = 0, \quad \text{hence} \quad \int_Z f_+(z)u_1(z)\,dz = 0.$$

But $u_1(z)>0$ a.e. on $Z$ and $f_+(z)\geq 0$ a.e. on $Z$ with strict inequality on a set of positive Lebesgue measure. Thus $\int_Z f_+(z)u_1(z)\,dz>0$, a contradiction.

Therefore $\{x_n\}_{n\geq 1}\subseteq W_0^{1,p}(Z)$ is bounded. Hence, by passing to a subsequence if necessary, we may assume that as $n\to\infty$, $x_n \xrightarrow{w} x$ in $W_0^{1,p}(Z)$, $x_n\to x$ in $L^p(Z)$ (from the compact embedding of $W_0^{1,p}(Z)$ in $L^p(Z)$), $x_n(z)\to x(z)$ a.e. on $Z$ and $|x_n(z)|\leq \kappa(z)$ a.e. on $Z$, where $\kappa\in L^p(Z)$.

Recall that $|\langle x_n^*, u\rangle| \leq \varepsilon_n\|u\|_{1,p}$ for all $u\in W_0^{1,p}(Z)$. Now set $u=x_n-x$. We have

$$-\varepsilon_n\|x_n-x\|_{1,p} \leq \langle A(x_n), x_n-x\rangle - \frac{\lambda_1}{p}\int_Z |x_n(z)|^{p-2}x_n(z)(x_n-x)(z)\,dz$$
$$-\int_Z v_n(z)(x_n-x)(z)\,dz$$
$$\leq \varepsilon_n\|x_n-x\|_{1,p}.$$

Note that

$$\frac{\lambda_1}{p}\int_Z |x_n(z)|^{p-2}x_n(z)(x_n-x)(z)\,dz \xrightarrow{n\to\infty} 0$$

and

$$\int_Z v_n(z)(x_n-x)(z)\,dz \xrightarrow{n\to\infty} 0.$$

So we obtain

$$\lim \langle A(x_n), x_n - x\rangle = 0$$

As we already mentioned, $A$ is generalized pseudomonotone, so from the above equality we infer that $\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle$ and therefore $\|Dx_n\|_p \to \|Dx\|_p$ as $n \to \infty$. We also know that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$. Since $L^p(Z, \mathbb{R}^N)$ is uniformly convex, we deduce that $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$, hence $x_n \to x$ in $W_0^{1,p}(Z)$ as $n \to \infty$. ∎

PROPOSITION 3. *If hypotheses* H$(f)$ *hold, then there exist* $\beta_1, \beta_2 > 0$ *such that for all* $x \in W_0^{1,p}(Z)$ *we have* $R(x) \ge \beta_1 \|x\|_{1,p}^p - \beta_2 \|x\|_{1,p}^\theta$ *with* $p < \theta \le p^* = Np/(N-p)$.

Proof. By virtue of hypothesis H$(f)$(iv), given $\varepsilon > 0$ we can find $\delta > 0$ such that for almost all $z \in Z$ and all $|x| \le \delta$ we have

$$F(z,x) \le \frac{1}{p}(-\mu + \varepsilon)|x|^p. \tag{11}$$

On the other hand, by hypothesis H$(f)$(iii), for almost all $z \in Z$ and all $|x| > \delta$ we have

$$|F(z,x)| \le \|\alpha\|_\infty |x|. \tag{12}$$

From (11) and (12) it follows that we can find $\gamma > 0$, for example

$$\gamma \ge \frac{1}{\delta^\theta}(\|\alpha\|_\infty + \frac{\mu}{p}\delta^p),$$

such that for almost all $z \in Z$ and all $x \in \mathbb{R}$ we have

$$F(z,x) \le \frac{1}{p}(-\mu + \varepsilon)|x|^p + \gamma|x|^\theta, \quad p < \theta \le p^* = \frac{Np}{N-p}. \tag{13}$$

Therefore

$$\begin{aligned} R(x) &= \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p - \int_Z F(z, x(z))\,dz \\ &\ge \frac{1}{p}\|Dx\|_p^p - \frac{\lambda_1}{p}\|x\|_p^p + \frac{1}{p}(\mu - \varepsilon)\|x\|_p^p - \gamma\|x\|_\theta^\theta \quad \text{(using (13))} \\ &\ge \frac{1}{p}\|Dx\|_p^p - \frac{1}{p}(\lambda_1 - \mu + \varepsilon)\|x\|_p^p - \gamma\|x\|_\theta^\theta. \end{aligned}$$

Choose $\varepsilon > 0$ such that $\lambda_1 + \varepsilon < \mu$ and use the embedding of $W_0^{1,p}(Z)$ in $L^\theta(Z)$ (since $\theta \le p^* = Np/(N-p)$) to obtain

$$R(x) \ge \frac{1}{p}\|Dx\|_p^p - \gamma_1 \|Dx\|_p^\theta \quad \text{for some } \gamma_1 > 0. \tag{14}$$

Thus from (14) it follows that there exist $\beta_1, \beta_2 > 0$ such that

$$R(x) \ge \beta_1 \|x\|_{1,p}^p - \beta_2 \|x\|_{1,p}^\theta \quad \text{for all } x \in W_0^{1,p}(Z). \ ∎$$

Now we are ready to state and prove an existence theorem for problem (2).

THEOREM 4. *If hypotheses* H$(f)$ *hold, then problem* (2) *has a nontrivial solution.*

Proof. From Proposition 3 we know that there exist $\beta_1, \beta_2 > 0$ such that for all $x \in W_0^{1,p}(Z)$ we have

$$R(x) \geq \beta_1 \|x\|_{1,p}^p - \beta_2 \|x\|_{1,p}^\theta.$$

Thus we can find $\varrho > 0$ small enough such that $R(x) \geq \xi > 0$ for all $\|x\|_{1,p} = \varrho$. Also, $R(0) = 0$ and for $t > 0$ we have

$$R(tu_1) = \frac{t^p}{p}\|Du_1\|_p^p - \frac{\lambda_1 t^p}{p}\|u_1\|_p^p - \int_Z F(z, tu_1(z))\, dz = -\int_Z F(z, tu_1(z))\, dz,$$

since $\|Du_1\|_p^p = \lambda_1 \|u_1\|_p^p$ (Rayleigh quotient).

From the proof of Proposition 2 we know that

$$\frac{F(z, tu_1(z))}{tu_1(z)} \xrightarrow{t\to\infty} \infty \quad \text{a.e. on } Z,$$

hence

$$F(z, tu_1(z)) \xrightarrow{t\to\infty} \infty \quad \text{a.e. on } Z.$$

So for $t > 0$ large enough we have $R(tu_1) \leq 0$. Therefore we can apply Theorem 1 and obtain $x \in W_0^{1,p}(Z)$ such that $R(x) \geq \xi > 0$ and $0 \in \partial R(x)$. Evidently, $x \neq 0$. Also, we have

$$0 = A(x) - \lambda_1 |x|^{p-2} x - v$$

with $v \in \partial K(x)$, where $K : W_0^{1,p}(Z) \to \mathbb{R}$ is defined by

$$K(x) = \int_Z F(z, x(z))\, dz$$

(see the proof of Proposition 2). Recall that $v(z) \in \overline{f}(z, x(z))$ a.e. on $Z$ and so $v \in L^q(Z)$. We have $A(x) = \lambda_1 |x|^{p-2} x + v$, hence

$$\langle A(x), \phi\rangle = \lambda_1 (|x|^{p-2}x, \phi)_{pq} + (v, \phi)_{pq} \quad \text{for all } \phi \in C_0^\infty(Z).$$

Here by $(\cdot,\cdot)_{pq}$ we denote the duality brackets for the pair $(L^p(Z), L^q(Z))$. So we have

$$\int_Z \|Dx(z)\|^{p-2} (Dx(z), D\phi(z))_{\mathbb{R}^N}\, dz = \int_Z (\lambda_1 |x(z)|^{p-2} x(z) + v(z))\phi(z)\, dz \quad \text{for all } \phi \in C_0^\infty(Z).$$

From the definition of the distributional derivative we conclude that

$$-\operatorname{div}(\|Dx(z)\|^{p-2} Dx(z)) = \lambda_1 |x(z)|^{p-2} x(z) + v(z) \quad \text{a.e. on } Z,$$

hence

$$-\operatorname{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \lambda_1|x(z)|^{p-2}x(z) + \bar{f}(z, x(z)) \qquad \text{a.e. on } Z,$$

i.e. $x \in W_0^{1,p}(Z)$ is a nontrivial solution of problem (2). ∎

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Department of Mathematics
National Technical University
Zografou Campus
Athens 157 80, Greece
E-mail: npapg@math.ntua.gr

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