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# COORDINATES OF MAXIMAL ROOTS OF WEAKLY NON-NEGATIVE UNIT FORMS

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**1. Introduction and main result.** A quadratic form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is called a *unit form* provided it is of the shape

$$\chi(x_1, \ldots, x_n) = \sum_{1 \le i \le n} x_i^2 + \sum_{1 \le i < j \le n} \chi_{ij} x_i x_j$$

for all $(x_1, \ldots, x_n) \in \mathbb{Z}^n$, of course with integer coefficients $\chi_{ij}$. Such forms and their root systems are ubiquitous in many parts of mathematics: for example in Lie theory, in singularity theory as well as in the representation theory of finite-dimensional algebras. The most prominent forms are those associated with the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. These are those unit forms which are "connected", positive definite and such that for all $i < j$ we have $\chi_{ij} \le 0$. There are corresponding positive semidefinite forms which are labelled by the diagrams $\widetilde{\mathbb{A}}_n$, $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$. These diagrams are referred to as the *extended Dynkin* or *Euclidean* or *affine* diagrams. In general, the matrix of coefficients $\chi_{ij}$ which determines such a unit form is what is called an "intersection matrix" (see [Sl]).

Given a unit form $\chi : \mathbb{Z}^n \to \mathbb{Z}$, an integer vector $x = (x_1, \ldots, x_n)$ will be called a 1-*root* provided $\chi(x) = 1$. In this way we try to avoid the possible confusion: The usual root system attached to a symmetric generalized Cartan matrix consists of real and imaginary roots. The real roots are 1-roots, but in general there may be additional 1-roots. Of course, for the Dynkin and the Euclidean forms, the 1-roots are just the real roots.

Sometimes it is helpful to consider $\mathbb{Z}^n$ together with a fixed basis consisting of 1-roots. Think for example of choosing a root basis of a root system in Lie theory. In some applications such a basis will even be given intrinsically. For instance, in the representation theory of finite-dimensional algebras, $\mathbb{Z}^n$ is the Grothendieck group of finite-dimensional representations with respect

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to all exact sequences, and the simple representations provide an intrinsic basis of this lattice.

Frequently, only the 1-roots which are linear combinations of the base vectors with only non-negative coefficients are of interest. Think again of Lie theory or representation theory. After identifying the elements of the basis with the canonical base vectors, these linear combinations are just the positive vectors in the following sense: The lattice $\mathbb{Z}^n$ is partially ordered by defining $x \geq y$ if $x_i - y_i \geq 0$ for all $i = 1, \ldots, n$. A vector $x$ is called *positive* if $x > 0$.

Having in mind applications in the representation theory of algebras, one has to study unit forms which satisfy weaker positivity conditions than positive definiteness resp. semidefiniteness. Namely, a unit form $\chi$ is said to be *weakly positive* if $\chi(x) > 0$ provided $x > 0$ and is said to be *weakly non-negative* if $\chi(x) \geq 0$ provided $x \geq 0$. For consistency, in the sequel we refer to positive definite forms as positive forms and to positive semidefinite forms as non-negative forms.

Recall the following theorem of Ovsienko ([Ov1], see also [Ri]): If $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly positive unit form and $x = (x_1, \ldots, x_n)$ is a positive 1-root of $\chi$, then $x_i \leq 6$ for all $i = 1, \ldots, n$. Note that 6 is the best bound possible, as the maximal root of the root system of type $\mathbb{E}_8$ shows.

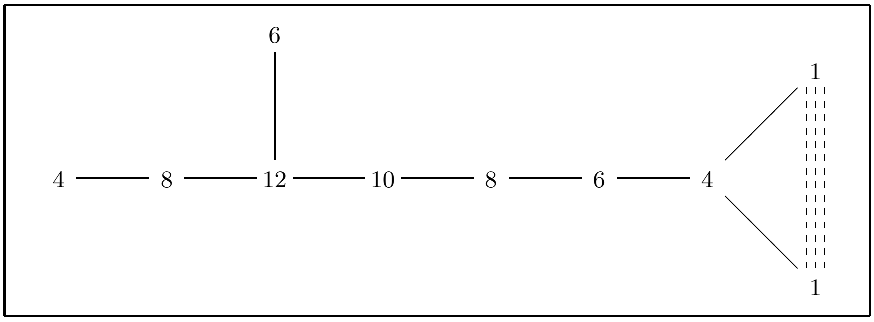
In this paper we consider the corresponding problem for weakly non-negative unit forms. A weakly non-negative unit form which is not weakly positive always has infinitely many positive 1-roots, thus there cannot exist a bound for their coordinates. On the other hand, it frequently happens that the set of positive 1-roots of a weakly non-negative unit form has maximal elements.

Main Theorem. *If $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly non-negative unit form and $x = (x_1, \ldots, x_n)$ is a maximal positive* 1*-root of $\chi$, then*

$$x_i \leq 12 \quad \textit{for all } i = 1, \ldots, n.$$

The bound 12 is the best possible, as the following example shows. For displaying the form $\chi$ we use the associated bigraph: The vertex set is $\{1, \ldots, n\}$. The vertices $i$ and $j$ are connected by $\chi_{ij}$ dotted edges if $\chi_{ij} \geq 0$ and by $|\chi_{ij}|$ solid edges if $\chi_{ij} < 0$. Moreover, we attach the coordinates $x_i$ of a maximal positive 1-root $x$ of $\chi$ to the vertices of the bigraph.

Observe that the bigraph is obtained by "glueing" two copies of the extended Dynkin diagram of type $\widetilde{\mathbb{E}}_8$ along the subdiagram $\mathbb{E}_8$ and connecting the extension vertices by three dotted edges. The 1-root $x$ is the sum of the positive radical generators of the two subforms of type $\widetilde{\mathbb{E}}_8$, thus the number 12 occurs as the sum of their maximal entries.




We will prove the theorem in Section 9 after establishing several preliminary results, some of which should be of interest on their own. Obviously, we can and will restrict ourselves to the case when $x$ is *sincere*, i.e. $x_i > 0$ for all $i = 1, \dots, n$. Note that, if $x$ is sincere, then the requirement that the unit form $\chi$ is weakly non-negative is not really necessary but follows from the maximality. Namely, it is shown in [HP] under the assumption $\chi_{ij} \geq -5$ for all $i < j$ and in [Ov2] in general that the existence of a maximal sincere positive 1-root forces a unit form to be weakly non-negative.

Our original interest in maximal positive 1-roots of weakly non-negative unit forms came from representation theory of finite-dimensional algebras and similar structures. In this context unit forms occur as Tits forms or Euler characteristics (see e.g. [Ga], [Bo], [Ri]) and their weak positivity (resp. weak non-negativity) is frequently related to finite (resp. tame) representation type.

Let us briefly review the connection between the representations of an algebra $A$ over an algebraically closed field $k$ and the weak definiteness of its Tits form $\chi_A$. To give the definition of $\chi_A$, we suppose that $A$ is basic, the ordinary quiver of $A$ is directed and $A$ has up to isomorphism exactly $n$ simple modules $S_1, \dots, S_n$. Then

$$\chi_A(x) = \sum_{i,j=1}^{n} \Big( \sum_{\nu=0}^{2} (-1)^\nu \dim_k \operatorname{Ext}_A^\nu(S_i, S_j) \Big) x_i x_j.$$

It is easy to see that this is a unit form.

An algebra $A$ has *finite representation type* if there are only finitely many isomorphism classes of indecomposable finite-dimensional $A$-modules. In order to check finite representation type for an arbitrary $A$, one may proceed as follows: Using covering theory (see [BG]) one may suppose that $A$ is simply connected. This implies that $A$ has a directed ordinary quiver and the Auslander–Reiten quiver of $A$ has a postprojective component. For an algebra $A$ of this kind it is shown in [Bo] that finite representation type is equivalent to the weak positivity of its Tits form $\chi_A$.

But the connection between the representations of $A$ and the form $\chi_A$ is even closer. Namely, the dimension vectors of the indecomposable $A$-modules are precisely the positive 1-roots of $\chi_A$. Recall that the dimension vector $x = (x_1, \ldots, x_n)$ of an $A$-module $X$ has as component $x_i$ just the multiplicity of $S_i$ in a composition series of $X$.

Concerning tame representation type the picture is not yet complete. In [Pe1] it is shown that tame type of $A$ implies weak non-negativity of the Tits form $\chi_A$. Moreover, there are interesting classes of tame algebras (see e.g. [Ri]) where the dimension vectors of the discrete indecomposable $A$-modules are just the connected positive 1-roots of $\chi_A$. If one asks in addition for the existence of sincere directing indecomposable modules, then $\chi_A$ actually has only finitely many sincere positive 1-roots. Thus our main theorem applies. We will give some more applications of our theorem for the representation theory of finite-dimensional algebras in the final section of this paper.

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## 2. Preliminaries

**2.1.** In Section 4 we will prove a reduction theorem allowing us to pass from general unit forms to so-called semigraphical forms which will be discussed in Section 6. On the other hand, the reduction theorem makes it necessary to slightly increase the class of forms we have to consider. Namely, we will have to deal with semiunit forms, where a *semiunit form* is a map $\chi : \mathbb{Z}^n \to \mathbb{Z}$,

$$x = (x_1, \ldots, x_n) \mapsto \sum_{\substack{i,j=1 \\ i \leq j}}^{n} \chi_{ij} x_i x_j,$$

such that $\chi_{ij} \in \mathbb{Z}$ and $\chi_{ii} \in \{0, 1\}$. Obviously, any unit form is also semiunit. For some definitions it will be convenient to use an even more general setup. We call a map $\chi$ as above an *integral form* if just $\chi_{ij} \in \mathbb{Z}$.

Given such a form we put $\chi_{ij} := \chi_{ji}$ for all $i < j$ and define a symmetric integral matrix $A_\chi$ with coefficients $(A_\chi)_{ij} := \chi_{ij}$ for $i \neq j$ and $(A_\chi)_{ii} := 2\chi_{ii}$. We denote by $e(1), \ldots, e(n)$ the canonical base vectors in $\mathbb{Z}^n$. The symmetric bilinear form $(-,-)_\chi : \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$, $x \mapsto \frac{1}{2}(x A_\chi x^{\mathrm{T}})$, has the following properties:

(a) $(x, x)_\chi = \chi(x)$ for all $x \in \mathbb{Z}^n$.
(b) $(e(i), e(i))_\chi = \chi_{ii}$ and $2(e(i), e(j))_\chi = \chi_{ij}$ for all $i \neq j$.
(c) $(x, y)_\chi = \frac{1}{2}(\chi(x + y) - \chi(x) - \chi(y))$ for all $x, y \in \mathbb{Z}^n$.

Whenever no confusion is possible, we omit the index $\chi$.

The *radical* of $\chi$ is defined as $\operatorname{Rad}\chi := \{x \in \mathbb{Z}^n : xA_\chi = 0\}$ whereas the *positive radical* $\operatorname{Rad}^+\chi$ consists only of all positive $x$ in $\operatorname{Rad}\chi$. The *corank* of $\chi$ is the rank of the free abelian group $\operatorname{Rad}\chi$. In analogy to 1-roots an element $x \in \mathbb{Z}^n$ is said to be a 0-*root* of $\chi$ if $\chi(x) = 0$.

**2.2.** Of course, one may also consider integral forms $\chi : \mathbb{Z}^I \to \mathbb{Z}$ for arbitrary finite sets $I$. Usually we will identify forms which only differ by renaming the vertices. But let us present one example where this use of more general index sets is appropriate. Namely, if $I$ is a subset of $\{1, \ldots, n\}$, then $\mathbb{Z}^I$ is embedded in $\mathbb{Z}^n$ in the canonical way. Obviously, for an integral (resp. semiunit, unit) form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ the *restriction* $\chi|I : \mathbb{Z}^I \to \mathbb{Z}$ defined by $\chi|I(x) = \chi(x)$ is again an integral (resp. semiunit, unit) form.

Note that we will use the notation $x|I$ for the image of $x \in \mathbb{Z}^n$ under the canonical retraction of the above-mentioned embedding $\mathbb{Z}^I \to \mathbb{Z}^n$.

A vector $x \in \mathbb{Z}^n$ is called *sincere* provided $x$ does not lie in $\mathbb{Z}^I$ for any proper subset $I$ of $\{1, \ldots, n\}$, or equivalently, $x_i \neq 0$ for all $i = 1, \ldots, n$. A semiunit form $\chi$ is said to be *sincere* if there exists a sincere positive 1-root $x$ of $\chi$.

If $x$ is a maximal positive 1-root of $\chi$ and we define $I$ as the *support* $\operatorname{supp} x := \{i : x_i \neq 0\}$ of $x$, then $x \in \mathbb{Z}^I$ and $x$ is a maximal sincere positive 1-root of the restriction $\chi|I$. This shows that it is enough to prove our main theorem for maximal sincere positive 1-roots of weakly non-negative semiunit forms.

**2.3.** If $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a unit form then we will use the well-known concept of reflections. The *reflection* $\sigma_i^\chi : \mathbb{Z}^n \to \mathbb{Z}^n$ with respect to $i \in \{1, \ldots, n\}$ is the linear map defined by $\sigma_i^\chi(x) = x - 2(e(i), x)_\chi e(i)$ and has the following properties:

(a) $(\sigma_i^\chi)^2 = \mathrm{id}_{\mathbb{Z}^n}$.

(b) $(\sigma_i^\chi(x), y)_\chi = (x, \sigma_i^\chi(y))_\chi$ for all $x, y \in \mathbb{Z}^n$.

(c) $(\sigma_i^\chi(x), \sigma_i^\chi(y))_\chi = (x, y)_\chi$ for all $x, y \in \mathbb{Z}^n$, in particular $\chi(\sigma_i^\chi(x)) = \chi(x)$ for all $x \in \mathbb{Z}^n$.

**2.4.** Let us recall once more the common way of visualizing integral forms using bigraphs. The vertex set of the bigraph $B_\chi$ of a semiunit form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is just $\{1, \ldots, n\}$. The vertices $i, j$ are connected by $|\chi_{ij} - \delta_{ij}|$ solid edges provided $\chi_{ij} - \delta_{ij} < 0$ and by $|\chi_{ij} - \delta_{ij}|$ dotted edges provided $\chi_{ij} - \delta_{ij} \geq 0$. In particular, if $\chi$ is a semiunit form, then the vertex $i$ has no loop if $\chi_{ii} = 1$ and one solid loop if $\chi_{ii} = 0$.

## 3. Basic properties of weakly non-negative semiunit forms

**3.1.** Throughout this section we suppose that $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly non-negative semiunit form. This implies immediately that $\chi_{ij} \geq -2$. Moreover, the fact that $0 \leq \chi(ne(i) + e(j)) = \chi_{jj} + n\chi_{ij}$ for all $i, j$ with $\chi_{ii} = 0$ shows that $\chi_{ij} \geq 0$.

The following lemma is quite analogous to the corresponding one for the weakly positive situation (compare [Ri, 1.0(4)]).

LEMMA. (a) *For a positive* 1*-root of a weakly non-negative semiunit form* $\chi$ *the following assertions hold*:

(a1) $2(e(i), x) \geq -2$ *for all* $i = 1, \ldots, n$.
(a2) *If* $x_i > 0$, *then* $2(e(i), x) \leq 2$.
(a3) $\chi_{ij} \leq 3$ *for all* $i \neq j$ *such that* $x_i \neq 0 \neq x_j$.

(b) *For a positive* 0*-root of a weakly non-negative semiunit form* $\chi$ *the following assertions hold*:

(b1) $2(e(i), x) \geq 0$ *for all* $i = 1, \ldots, n$.
(b2) *If* $x_i > 0$, *then* $2(e(i), x) = 0$.
(b3) $\chi_{ij} \leq 2$ *for all* $i \neq j$ *such that* $x_i \neq 0 \neq x_j$.

Proof. (a1) and (a2) follow from applying $\chi$ to $x \pm e(i)$. To prove (a3), by possibly interchanging $i, j$ we may suppose $2(x, e(i) - e(j)) \leq 0$ and obtain $0 \leq \chi(x + (e(i) - e(j)) \leq \chi(x) + \chi(e(i) - e(j)) \leq 3 + \chi_{ij}$.

(b1) and (b2) follow from applying $\chi$ to $2x \pm e(i)$, whereas the proof of (b3) is completely analogous to that of (a3). ∎

**3.2.** We call a weakly non-negative semiunit form $\chi$ *finitely sincere* provided $\chi$ is sincere and there are only finitely many sincere positive 1-roots. Using the above lemma it turns out that the finitely sincere forms are exactly the forms possessing a maximal sincere positive 1-root.

PROPOSITION. *For a weakly non-negative semiunit form* $\chi$ *the following assertions are equivalent*:

(a) *There exists a maximal sincere positive* 1*-root* $x$.
(b) $\chi$ *is sincere and* $\operatorname{Rad}^+ \chi = \emptyset$.
(c) $\chi$ *is finitely sincere.*

Proof. (a)⇒(b) and (c)⇒(a) are obvious. For (b)⇒(c) we assume that the set of sincere positive 1-roots is infinite. Hence we are able to find an infinite subset $\{x(1), x(2), \ldots\}$ satisfying $x(i) < x(i+1)$ for all $i \in \mathbb{N}$. By 3.1(a) we know $2(e(j), x(i)) \in \{0, \pm 1, \pm 2\}$ for all $i \in \mathbb{N}$ and $j = 1, \ldots, n$. Hence there exist $s < t$ such that $2(e(j), x(s)) = 2(e(j), x(t))$ for all $j = 1, \ldots, n$. Consequently, $x(t) - x(s) \in \operatorname{Rad}^+ \chi$. ∎

REMARK. *If* $\chi$ *is a weakly non-negative*, *finitely sincere semiunit form*, *then the bigraph* $B_\chi$ *of* $\chi$ *is connected and the points* $i$ *such that* $\chi_{ii} = 0$ *are characterized by* $\chi_{ij} \geq 0$ *for all* $j \neq i$.

Proof. If $B_\chi$ has two non-trivial connected components supported by the sets $I_1, I_2$ and $x$ is a sincere positive 1-root, then $x = x_1 + x_2$ with $x_i = x|I_i$. Hence $1 = \chi(x_1) + \chi(x_2)$. Without loss of generality this means $\chi(x_1) = 1$ and $\chi(x_2) = 0$. Consequently, all the vectors $x_1 + nx_2$ are sincere positive 1-roots.

To prove the second assertion we suppose $\chi_{ij} \geq 0$ for all $j \neq i$ and from the connectedness of $B_\chi$ derive the existence of some $j$ such that actually $\chi_{ij} > 0$. Therefore the assumption $\chi_{ii} = 1$ would lead to $2(e(i), x) = 2x_i + \sum_{j\neq i} \chi_{ji} x_j \geq 3$, contradicting 3.1(a). ■

**3.3.** In [Ri, 1.1(7)] it is shown that maximal sincere positive 1-roots of weakly positive forms have at most 2 exceptional vertices. This generalizes to our situation:

LEMMA. *Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be a weakly non-negative unit form with $n \geq 2$ and suppose that $x$ is a maximal sincere positive* 1*-root of $\chi$. Then one and only one of the following two situations occurs.*

(a) *There is exactly one exceptional vertex $i$ (i.e. $x_i = 2$, $2(e(i), x) = 1$ and $2(e(j), x) = 0$ for all $j \neq i$).*

(b) *There are exactly two exceptional vertices $i_1, i_2$ (i.e. $x_{i_1} = x_{i_2} = 1$, $2(e(i_1), x) = 2(e(i_2), x) = 1$ and $2(e(j), x) = 0$ for all $j \neq i_1, i_2$).*

Proof. As $\chi$ is supposed to be a unit form, the vectors $\sigma_j(x)$ are all 1-roots as well. Hence $2(e(j), x) \geq 0$ for all $j$. We consider the equation $2 = 2\chi(x) = \sum_{j=1}^n x_j 2(e(j), x)$ and assume that there exists $i$ such that $x_i = 1$, $2(e(i), x) = 2$ and $2(e(j), x) = 0$ for all $j \neq i$. Putting $\mu = x - e(i)$ and calculating $\chi(\mu) = 0$, from 3.1(b) we obtain $(e(j), \mu) = 0$ for all $j \neq i$ whereas immediately $(e(i), \mu) = (e(i), x) - \chi(e(i)) = 0$. Thus $n \geq 2$ yields $\mu \in \mathrm{Rad}^+ \chi$. This contradiction shows that only the two alternatives of the lemma can occur. ■

**3.4.** LEMMA. *Suppose $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a weakly non-negative semiunit form and $x$ is a sincere positive* 1*-root. For $r = 0, 1$ we put $I^r = \{i \in I : \chi_{ii} = r\}$ and $x^r = x|I^r$. Then one and only one of the following three situations occurs*:

(a) $\chi(x^1) = 1$ *and $\chi_{ij} = 0$ for all $i \in I^0$, $j \in I$.*

(b) $\chi(x^1) = 0$ *and there exist $i, j \in I^0$ such that $x_i = x_j = \chi_{ij} = 1$. Additionally, if $s \in I^0$, $t \in I$ and $\chi_{st} \neq 0$, then $\{s, t\} = \{i, j\}$.*

(c) $\chi(x^1) = 0$ *and there exist $i \in I^0$, $j \in I^1$ such that $x_i = x_j = \chi_{ij} = 1$. Additionally, if $s \in I^0$, $t \in I$ and $\chi_{st} \neq 0$, then $\{s, t\} = \{i, j\}$.*

*Moreover, if $x$ is maximal and $I^0 \neq \emptyset \neq I^1$, then* (c) *holds and* $\mathrm{card}(I^0) = 1$.

Proof. By 3.1 we know that $\chi_{ij} \geq 0$ for all $i \in I^0$, $j \in I$. We may suppose $I = \{1, \ldots, n\}$ and $I^0 = \{1, \ldots, m\}$ for some $m \leq n$. Consider the equation

$$1 = \Big( \sum_{i \in I^1} x_i^2 + \sum_{\substack{i,j \in I^1 \\ i<j}} \chi_{ij} x_i x_j \Big) + \sum_{\substack{i,j \in I^0 \\ i<j}} \chi_{ij} x_i x_j + \sum_{\substack{i \in I^1 \\ j \in I^0}} \chi_{ij} x_i x_j .$$

The first summand of this sum is just $\chi(x^1)$ and the second is $\chi(x^0)$. As also the third summand is non-negative and integer, exactly one of these summands has to be 1 and the others have to be 0. This leads to the three cases because $x$ is sincere.

In case (a) all the vectors $nx^0 + x^1$ and in case (b) all the vectors $x^0 + nx^1$ are positive 1-roots greater than $x$. Hence for $x$ maximal only case (c) is possible. Furthermore, if there existed $k \in I^0$, $k \neq i$, then also the vectors $x + ne(k)$ would be 1-roots greater than $x$. ■

**3.5.** We recall that a unit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is *critical* (resp. *hypercritical*) if it is not weakly positive (resp. weakly non-negative) but $\chi|J$ is weakly positive (resp. weakly non-negative) for every proper subset $J$ of $I$. Every critical unit form $\chi$ is non-negative and its radical is generated by a sincere positive vector $\mu$ (see [Ri]) which in this paper will be called the *characteristic vector* of $\chi$. For an arbitrary unit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ we denote a subset $J \subseteq I$ resp. the induced restriction $\eta := \chi|J$ as critical (resp. hypercritical) restriction provided that $\eta$ is critical (resp. hypercritical).

A weakly non-negative semiunit form $\chi$ is called 0-*sincere* if there is a sincere vector $y \in \operatorname{Rad}^+ \chi$. Note that by Lemma 3.1(b2) it would be sufficient to require only that $y$ is a positive sincere 0-root. The fact that we can shift any vector into the positive cone by adding integer multiples of $y$ shows that a 0-sincere form has to be non-negative. As observed above, any critical unit form is 0-sincere with corank 1.

If $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is an arbitrary weakly non-negative unit form, we consider the union $I^+$ of all $\operatorname{supp} \mu$ where $\mu \in \operatorname{Rad}^+ \chi$ and denote $\chi|I^+$ by $\chi^+$. By construction the form $\chi^+$ is 0-sincere. We call $\chi^+$ the 0-*sincere kernel* of $\chi$. Using the following lemma, we deduce that a vector $x \in \mathbb{Z}^I$ is a 0-root of $\chi^+$ if and only if $\operatorname{supp} x \subseteq I^+$ and $x \in \operatorname{Rad} \chi$. In particular, this shows that $\operatorname{Rad}^+ \chi = \operatorname{Rad}^+ \chi^+$.

LEMMA. *Suppose $\chi$ is a weakly non-negative semiunit form and $x \in \operatorname{Rad}^+ \chi$. If $\mu \in \mathbb{Z}^n$ is a* 0*-root of $\chi$ such that* $\operatorname{supp} \mu \subseteq \operatorname{supp} x$, *then* $\mu \in \operatorname{Rad} \chi$.

Proof. Assuming the existence of an index $i$ such that $2(e(i), \mu) \neq 0$, we may choose $\varepsilon \in \{\pm 1\}$ such that $\varepsilon 2(e(i), \mu) \geq 1$. Putting $y = e(i) - 2\varepsilon\mu$ we observe $\chi(y) = \chi(e(i)) - \varepsilon 4(e(i), \mu) \leq 1 - 2 = -1$. On the other hand, the requirement on the supports shows that $y + kx$ is positive for some $k \in \mathbb{N}$. Thus we arrive at the contradiction $0 \le \chi(y + kx) = \chi(y) = -1$. ■

## 4. The reduction theorem

**4.1.** Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form and pick $i \ne j \in \{1, \dots, n\}$. The $\mathbb{Z}$-isomorphism $R_{ij} : \mathbb{Z}^n \to \mathbb{Z}^n$ is defined on the canonical base vectors by $R_{ij}(e(k)) = e(k)$ for $k \ne j$ and $R_{ij}(e(j)) = e(j) - e(i)$. Hence for $x = (x_1, \dots, x_n) \in \mathbb{Z}^n$ the coordinates of $x' = R_{ij}(x)$ are $x'_k = x_k$ for $k \ne i$ and $x'_i = x_i - x_j$.

The map $\chi' = \chi R_{ij}^{-1}$ is called the *small reduction* of $\chi$ with respect to $(i, j)$ provided $\chi_{ij} < 0$. Note that, if $\chi$ is a unit form and $\chi_{ij} = -1$, then this is just a direct Gabrielov transformation. Immediate calculations show:

(a) $\chi'_{kl} = \chi_{kl}$ if $j \notin \{k, l\}$.
(b) $\chi'_{jk} = \chi_{jk} + \chi_{ik}$ if $j \ne k \ne i$.
(c) $\chi'_{jj} = \chi_{jj} + \chi_{ii} + \chi_{ji}$ and $\chi'_{ji} = \chi_{ji} + 2\chi_{ii}$.

**4.2.** In the following lemma we use the norm $|x| = \sum_{i=1}^n x_i$ to measure the size of vectors in the cone $C_n := \{x \in \mathbb{Z}^n : x \ge 0\}$.

Lemma. *Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form and $\chi'$ be the small reduction of $\chi$ with respect to $(i, j)$. Then the following assertions hold*:

(a) *$R_{ij}^{-1}(C_n) \subseteq C_n$, in particular $R_{ij}^{-1}$ preserves the order on $\mathbb{Z}^n$.*

(b) *$\chi R_{ij}^{-1}(x) = \chi'(x)$ for all $x \in \mathbb{Z}^n$, in particular $R_{ij}^{-1}$ maps the set of positive 1-roots (resp. 0-roots) $x'$ of $\chi'$ bijectively to the set of all positive 1-roots (resp. 0-roots) $x$ of $\chi$ satisfying $x_i \ge x_j$.*

(c) *If $x \in \mathbb{Z}^n$ is sincere positive and $x_i > x_j$, then $x' = R_{ij}(x)$ is sincere positive and $|x'| < |x|$.*

(d) *If $\chi$ is a weakly non-negative semiunit form and $\chi_{ij} < 0$ then $\chi'$ is also a weakly non-negative semiunit form.*

(e) *If $\chi$ is a weakly non-negative semiunit form with $\chi_{ij} < 0$ and $x$ is a maximal sincere positive 1-root of $\chi$ with $x_i > x_j$, then $\chi'$ is a weakly non-negative semiunit form and $x' = R_{ij}(x)$ is a maximal sincere positive 1-root of $\chi'$ satisfying $|x'| < |x|$.*

Proof. (a), (b) and (c) are obvious. In (d) the weak non-negativity of $\chi'$ is clear. By 3.1 we know $\chi_{ii} = \chi_{jj} = 1$ and $\chi_{ij} \in \{-1, -2\}$. If $\chi_{ij} = -1$, then $\chi'_{jj} = 1 + 1 - 1 = 1$. If $\chi_{ij} = -2$, then $\chi'_{jj} = 1 + 1 - 2 = 0$. Part (e) is an immediate consequence of the previous parts. ■

Note that applying this lemma to a unit form will usually lead to a semiunit form. So it is just this lemma that made it necessary to introduce semiunit forms.

**4.3.** Using the previous lemma, we can switch to another weakly non-negative, finitely sincere semiunit form with a maximal sincere 1-root $x'$ of smaller norm provided in our given root $x$ we find $i, j$ such that $\chi_{ij} > 0$ and $x_i > x_j$. We will now see that the last restriction is not essential.

For an integer form $\chi : \mathbb{Z}^n \to \mathbb{Z}$ and indices $i \neq j$ we consider the restriction $\chi' = (\chi R_{ij}^{-1})|J$ where $J = \{1, \ldots, i-1, i+1, \ldots, n\}$. The form $\chi'$ is called the *tightening* of $\chi$ with respect to $(i, j)$.

To formulate the properties of $\chi'$, let $L_i$ be the subgroup of $\mathbb{Z}^n$ consisting of all $x = (x_1, \ldots, x_n)$ such that $x_i = x_j$. We observe that the map $\Delta : \mathbb{Z}^J \to L_i$ given by $(\Delta(x'))_k = x'_k$ for $k \neq i$ and $(\Delta(x'))_i = x'_j$ is an isomorphism. The following lemma is an immediate consequence of the previous considerations of this section.

LEMMA. *Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form and $\chi'$ be the tightening of $\chi$ with respect to $(i, j)$. Then the following assertions hold*:

(a) *$\Delta(C_J) \subseteq C_n$, in particular $\Delta$ is order preserving.*

(b) *$\chi\Delta(x') = \chi'(x')$ for all $x' \in \mathbb{Z}^J$, in particular $\Delta$ maps the set of positive* 1*-roots* (*resp.* 0*-roots*) *$x'$ of $\chi'$ bijectively to the intersection of $L_i$ with the set of all positive* 1*-roots* (*resp.* 0*-roots*) *of $\chi$.*

(c) *If $x \in \mathbb{Z}^n$ is sincere positive and $x_i = x_j$, then $x' = x|J$ is sincere positive and $|x'| < |x|$.*

(d) *If $\chi$ is a weakly non-negative semiunit form and $\chi_{ij} < 0$ then $\chi'$ is also a weakly non-negative semiunit form.*

(e) *If $\chi$ is a weakly non-negative semiunit form with $\chi_{ij} < 0$ and $x$ is a maximal sincere positive* 1*-root of $\chi$ with $x_i = x_j$, then $\chi'$ is a weakly non-negative semiunit form and $x' = x|J$ is a maximal sincere positive* 1*-root of $\chi'$ satisfying $|x'| < |x|$.*

**4.4.** Sections 4.1 and 4.2 furnish the proof of our fundamental reduction theorem:

THEOREM. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly non-negative semiunit form and $x$ a maximal sincere positive* 1*-root of $\chi$. If we write $I$ as the disjoint union of two subsets $J$ and $K$, then there is a weakly non-negative semiunit form $\chi' : \mathbb{Z}^{I'} \to \mathbb{Z}$, a maximal sincere positive* 1*-root $x'$ of $\chi'$ and a monomorphism $\varphi : \mathbb{Z}^{I'} \to \mathbb{Z}^I$ with the following properties*:

(a) *$I'$ is the disjoint union of $K$ and a subset $J'$ of $J$.*

(b) *$\varphi(z) = z$ for all $z \in \mathbb{Z}^K$.*

(c) *$\varphi(C_{I'}) \subseteq C_I$, in particular $\varphi$ is order preserving.*

(d) *$\chi\varphi(z') = \chi'(z')$ for all $z' \in \mathbb{Z}^{I'}$, in particular $\varphi$ induces an injection from the set of positive* 1*-roots* (*resp.* 0*-roots*) *of $\chi'$ to the set of positive* 1*-roots* (*resp.* 0*-roots*) *of $\chi$.*

(e) $\varphi(x') = x$.
(f) $\chi_{ij} \geq 0$ *for all* $i, j \in J'$.

Proof. We apply 4.1 and 4.2 as long as we find vertices $i, j \in J$ such that $\chi_{ij} < 0$. This process has to stop since the norm of the considered maximal sincere positive 1-root always decreases. ∎

The triple $(\chi', x', \varphi)$ in the above theorem is called a *full reduction* of the pair $(\chi, x)$ with respect to the subset $J$ of $I$.

**4.5.** The reduction theorem usually decreases the number of variables occurring in the form obtained. We also need a process increasing the number of variables called doubling of vertices (see [D-Z]).

Let $\chi : \mathbb{Z}^n \to \mathbb{Z}$ be an integer form. For $i \in \{1, \ldots, n\}$ we define a new integer form $\chi^{(i)} : \mathbb{Z}^{n+1} \to \mathbb{Z}$ by $\chi^{(i)}|\{1, \ldots, n\} = \chi$, $\chi^{(i)}_{n+1,n+1} = \chi_{ii}$ and $(e(n+1), e(j))_{\chi^{(i)}} = (e(i), e(j))_\chi$ for all $j = 1, \ldots, n$. We say that $\chi^{(i)}$ is obtained from $\chi$ by *doubling* the vertex $i$. Actually, the bigraph of $\chi^{(i)}$ is constructed from the bigraph of $\chi$ by doubling the vertex $i$ thus obtaining two vertices $i$ and $n+1$. The edges between these two vertices depend on $\chi^{(i)}_{(n+1)i} = 2\chi_{ii}$. Clearly, if $\chi$ is a semiunit (resp. unit) form, then $\chi^{(i)}$ is a semiunit (resp. unit) form as well. To understand the relation of $\chi$ and $\chi^{(i)}$, it is suitable to introduce the surjective homomorphism $\pi : \mathbb{Z}^{n+1} \to \mathbb{Z}^n$ given by $(\pi(x))_k = x_k$ for $k \neq i$ and $(\pi(x))_i = x_i + x_{n+1}$.

Lemma. *If* $\chi : \mathbb{Z}^n \to \mathbb{Z}$ *is an integer form, then the form* $\chi^{(i)}$ *obtained by doubling the vertex* $i$ *has the following properties*:

(a) $\pi(C_{n+1}) = C_n$ *and therefore* $\pi$ *is order preserving. In addition, for* $0 \leq x \leq y$ *in* $\mathbb{Z}^n$ *and* $y' \geq 0$ *in* $\mathbb{Z}^{n+1}$ *such that* $\pi(y') = y$ *there exists* $x' \in \mathbb{Z}^{n+1}$ *with* $0 \leq x' \leq y'$ *and* $\pi(x') = x$.

(b) $(x, y)_{\chi^{(i)}} = (\pi(x), \pi(y))_\chi$ *for all* $x, y \in \mathbb{Z}^{n+1}$. *In particular*, $\pi$ *maps the set of positive* 1*-roots* (*resp.* 0*-roots*) *of* $\chi^{(i)}$ *surjectively to the set of positive* 1*-roots* (*resp.* 0*-roots*) *of* $\chi$.

(c) $\operatorname{Rad} \chi^{(i)} = \operatorname{Rad} \chi \oplus \mathbb{Z}(e(n+1) - e(i))$. *In addition, a vector* $x \in \mathbb{Z}^{n+1}$ *lies in* $\operatorname{Rad}^+ \chi^{(i)}$ *if and only if it can be written as* $y + q(e(n+1) - e(i))$ *where* $y \in \operatorname{Rad}^+ \chi$ *and* $q$ *is a non-negative integer such that* $q \leq y_i$.

(d) $\chi$ *is weakly non-negative if and only if* $\chi^{(i)}$ *is weakly non-negative.*

(e) $x \in C_{n+1}$ *is a maximal positive* 1*-root of* $\chi^{(i)}$ *if and only if* $\pi(x)$ *is a maximal positive* 1*-root of* $\chi$.

Proof. (a), (b), (c) and one direction of (d) are obvious. For the converse we pick $x \in C_{n+1}$ and from $\pi(x) \in C_n$ we obtain $\chi^{(i)}(x) = \chi(\pi(x)) \geq 0$. For (e) we observe that in the case of a 1-root $y$ of $\chi^{(i)}$ such that $y > x$ also $\pi(y) > \pi(x)$ holds. Conversely, if $z$ is a 1-root of $\chi$ satisfying $z > \pi(x)$, then it is easy to find $y$ such that $\pi(y) = z$ and $y > x$. ∎

REMARK. Obviously, $e(n+1)-e(i) \in \operatorname{Rad}\chi^{(i)}$. If, conversely, $\chi' : \mathbb{Z}^{n+1} \to \mathbb{Z}$ is an integer form such that $\chi'|\{1,\ldots,n\} = \chi$ and $e(n+1)-e(i) \in \operatorname{Rad}\chi'$, then $\chi' = \chi^{(i)}$.

**5. 2-layer 1-roots.** Throughout this section we suppose that $\chi : \mathbb{Z}^n \to \mathbb{Z}$ is a weakly non-negative semiunit form.

**5.1.** LEMMA. *Let $\mu$ be a positive 0-root of the weakly non-negative semiunit form $\chi$ and $x \in \mathbb{Z}^n$. Suppose there is a non-negative integer $n$ such that $x + n\mu$ is positive and sincere. If $(x,\mu)_\chi = 0$, then $\mu \in \operatorname{Rad}\chi$.*

Proof. If we assume $\mu \notin \operatorname{Rad}\chi$, then there exists $i$ such that by 3.1 we have $2(e(i),\mu) \geq 1$. Hence there is $l \in \mathbb{N}$ with $2(x + l\mu, e(i)) = 2(x, e(i)) + l2(\mu, e(i)) \geq \chi(x) + 2$. Putting $t = \max\{n, l\}$ and $y = x + t\mu$, we see that the sincere positive vector $\mu$ satisfies $2(y, e(i)) \geq \chi(x) + 2$. Observing that $(x,\mu) = 0$ implies $\chi(y) = \chi(x)$, we arrive at the contradiction $0 \leq \chi(y - e(i)) \leq \chi(x) + 1 - 2(e(i), y) \leq -1$. ∎

**5.2.** A positive 1-root $x$ of $\chi$ is called 2-*layer* if there exist positive 0-roots $\mu, \mu'$ such that $\mu + \mu' = x$.

PROPOSITION. *Suppose $x$ is a positive 1-root of the weakly non-negative semiunit form $\chi$. If $\mu$ is a positive 0-root of $\chi$ such that $\mu \notin \operatorname{Rad}\chi$ and $x > \mu$, then $x - \mu$ is a positive 0-root as well. In particular, $x$ is 2-layer.*

Proof. Without loss of generality we may suppose that $x$ is sincere. By 5.1 we obtain $(x,\mu) \neq 0$. The inequalities $0 \leq \chi(x-\mu) = \chi(x) - 2(x,\mu) = 1 - 2(x,\mu)$ and $0 \leq \chi(x + n\mu) = \chi(x) + 2n(x,\mu) = 1 + 2n(x,\mu)$ show that $2(x,\mu) = 1$. Hence $\chi(x-\mu) = \chi(x) + \chi(\mu) - 2(x,\mu) = 1 + 0 - 1 = 0$. ∎

COROLLARY. *Suppose $x$ is a maximal positive 1-root of a weakly non-negative semiunit form $\chi$. If there is a positive 0-root $\mu$ such that $x > \mu$, then $x$ is 2-layer.*

**5.3.** LEMMA. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a 0-sincere weakly non-negative semiunit form. Suppose there is $i \in I$ such that $\chi|(I \setminus \{i\})$ is a unit form. If the set $U$ of all $y \in \operatorname{Rad}^+\chi$ satisfying $y_i = 1$ is non-empty and finite, then $y_j \leq 6$ for all $y \in U$ and $j \in I$.*

Proof. We claim that $\chi' = \chi|(I \setminus \{i\})$ is weakly positive. If not, there is some positive $\mu'$ such that $i \notin \operatorname{supp}\mu'$ and $\chi(\mu') = 0$. From Lemma 3.5 we know that $\mu' \in \operatorname{Rad}^+\chi$, which contradicts the finiteness of $U$.

Let now $y \in U$. As $\chi(y - e(i)) = \chi(e(i)) = 1$, we deduce that $y - e(i)$ is a positive 1-root of the weakly positive unit form $\chi'$. ∎

THEOREM. *If $x$ is a maximal positive 2-layer 1-root of a weakly non-negative semiunit form $\chi : \mathbb{Z}^n \to \mathbb{Z}$, then $x_i \leq 12$ for all $i = 1,\ldots,n$.*

Proof. Let $x = \mu + \mu'$ with $\mu, \mu'$ positive 0-roots of $\chi$. We inductively double all points $i \in I = \{1, \ldots, n\}$. Of course, the resulting form $\overline{\chi} : \mathbb{Z}^{\overline{I}} \to \mathbb{Z}$ does not depend on the ordering chosen. We write $\overline{I} = \{1, \ldots, 2n\}$ as $I = \{1^-, \ldots, n^-\} \cup \{1^+, \ldots, n^+\}$ where $i^- = i$ and $i^+ = i+n$ for all $i = 1, \ldots, n$. Clearly, the projection map $\pi : \mathbb{Z}^{\overline{I}} \to \mathbb{Z}^I$ sending $e(i^-)$ and $e(i^+)$ to $e(i)$ has by induction the properties listed in Lemma 4.3.

We put $\mu^- = \sum_{i=1}^n \mu_i e(i^-)$, $\mu^+ = \sum_{i=1}^n \mu_i e(i^+)$ and $\overline{x} = \mu^- + \mu^+$. As $\pi(\overline{x}) = x$, part (d) of Lemma 4.3 tells us that $\overline{x}$ is a maximal positive 1-root of $\overline{\chi}$.

The sets $I^- = \operatorname{supp} \mu^-$ and $I^+ = \operatorname{supp} \mu^+$ have the property that $I^- \cup I^+ = \operatorname{supp} \overline{x}$. Consequently, $\chi' = \overline{\chi}|I^- \cup I^+$ is a finitely sincere semiunit form and $\overline{x}$ is a maximal sincere positive 1-root of $\chi'$.

Let now $(\eta, z, \varphi)$ be a full reduction of $(\chi', \overline{x})$ with respect to $I^-$ where $\eta : \mathbb{Z}^J \to \mathbb{Z}$ and $J$ is the disjoint union of $J^-$ and $I^+$. Putting $\lambda = z|J^-$, we see that $z = \lambda + \mu^+$, yielding $\varphi(\lambda) = \mu^-$ and therefore $\eta(\lambda) = \overline{\chi}\varphi(\mu^-) = \overline{\chi}(\mu^-) = \chi(\mu) = 0$. Writing this down explicitly yields $0 = \sum_{i \in J^-} \eta_{ii} \lambda_i^2 + \sum_{i<j} \eta_{ij} \lambda_i \lambda_j$. By construction all $\eta_{ij}$ are non-negative. Thus the sincerity of $\lambda$ forces all $\eta_{ij}$ to be 0. Without loss of generality there is $i^+ \in I^+$ such that $\eta_{i^+ i^+} = 1$. By Lemma 3.4 we see that $\eta|I^+$ is a unit form and $J^-$ consists of just one element $\omega$. Moreover, there is a unique $i^+ \in I^+$ such that $\eta_{\omega i^+} = 1$ and $\mu^+_{i^+} = 1 = z_\omega$.

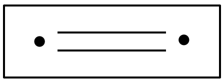
Putting $\eta' = \eta|I^+ = \overline{\chi}|I^+ = \chi|\operatorname{supp}(\mu')$, we deduce from 5.1 that $\mu^+ \in \operatorname{Rad}^+ \eta'$. The vector $\mu^+$ is contained in the set $U = \{y \in \operatorname{Rad}^+ \eta' : y_{i^+} = 1\}$. We observe that $(e(j^+), y)_\eta = 0$ and $(e(\omega), y)_\eta = y_{i^+} = 1$ for all $y \in U$ and $j^+ \in I^+$. The assumption that the set $U$ is infinite would lead to the existence of $y, y' \in U$ satisfying $y' < y$. But then $0 < y - y'$ would lie in $\operatorname{Rad}^+ \eta$, contradicting the finite sincerity of $\eta$. Hence $U$ is finite and by the above lemma we obtain $\mu_{j^+} \le 6$ for all $j^+ \in I^+$. But this shows $\mu'_j \le 6$ and by symmetry also $\mu_j \le 6$ for all $j = 1, \ldots, n$. ∎

## 6. Semigraphical forms

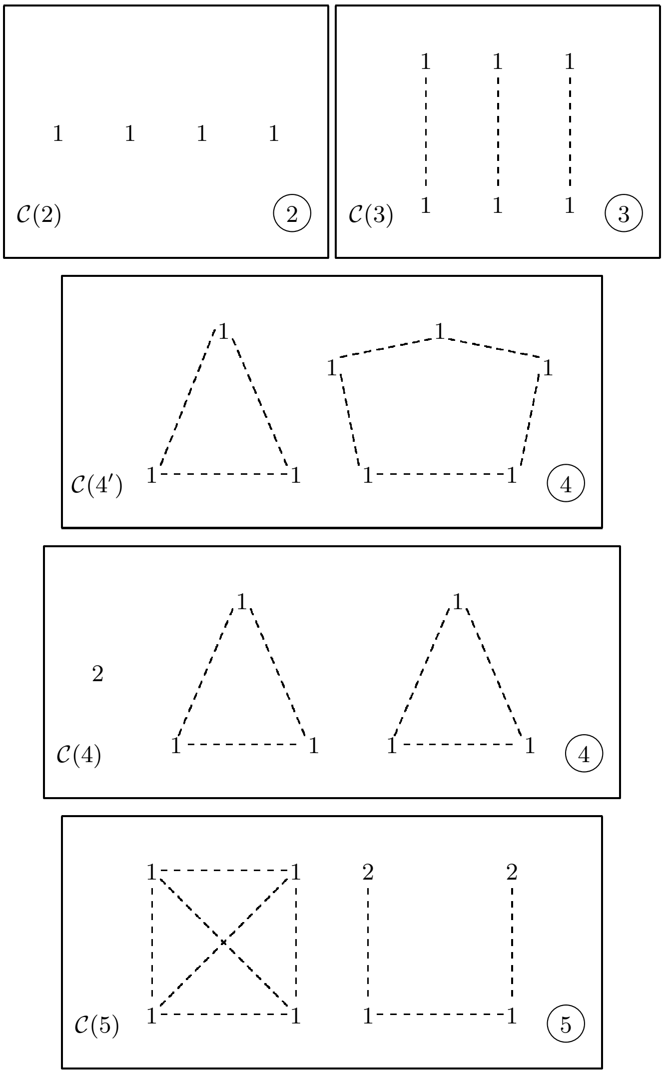
**6.1.** A semiunit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is called *semigraphical* if there exists $\omega \in I$ such that $\chi_{\omega i} < 0$ for all $i \in I$ with $i \ne \omega$ and $\chi_{ij} \ge 0$ for all $i, j \ne \omega$. An index $\omega$ as occurring in the definition is called a *center*. The center is unique provided $\operatorname{card} I > 2$.

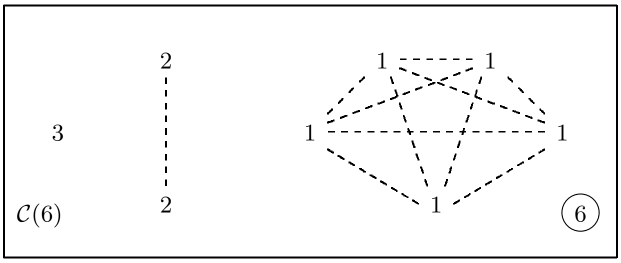
The graphical forms $\chi$ introduced in [Ri] are just the semigraphical unit forms such that $|\chi_{ij}| \le 1$ for all $i, j \in I$. Graphical forms $\chi$ with $\operatorname{card} I > 2$ are usually visualized by their reduced bigraph $B'(\chi)$, which is just the full subbigraph of $B(\chi)$ supported by the edges different from the center. Note that $B'(\chi)$ does not contain any solid edge hence is a graph with dotted edges.

**6.2.** We observe that a weakly non-negative semigraphical semiunit form by 3.1 is actually a unit form; we investigate the critical semigraphical unit forms. First we notice that the form $\mathcal{C}(1)$ given by the Kronecker bigraph



is obviously a critical semigraphical unit form. As shown in [Ri] there are up to isomorphism exactly 6 critical graphical forms, namely $\mathcal{C}(2), \dots, \mathcal{C}(6)$ and $\mathcal{C}(4')$, whose reduced bigraphs are presented in the list below where we replace the vertices by the coefficients of the characteristic vector. The coefficient of the center $\omega$ is the encircled number in the lower right corner.






LEMMA. *If $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a critical semigraphical unit form and* $\operatorname{card} I > 2$, *then $\chi$ is actually graphical. Hence the critical semigraphical forms are exactly the forms $\mathcal{C}(1), \ldots, \mathcal{C}(6)$ and $\mathcal{C}(4')$.*

Proof. As $\operatorname{card} I > 2$, clearly $\chi_{\omega i} = -1$ for all $i \neq \omega$. Hence we only have to show $\chi_{ij} \le 1$ for all $i, j$ different from $\omega$. Observing that for $i \neq j$ the vector $e(i) - e(j)$ and the characteristic vector of $\chi$ are linearly independent, we obtain $1 \le \chi(e(i) - e(j)) = 2 - \chi_{ij}$. ∎

**6.3.** LEMMA. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly non-negative semigraphical unit form with center $\omega$ satisfying $\chi_{\omega j} = -1$ for all $j \in I$ different from $\omega$. We fix $i \in I$ different from $\omega$ and put $S_i = \{j \in I : \chi_{ij} > 0\}$. If $x$ is a positive sincere vector, $S$ is a subset of $S_i$ and $x_i - 2(x, e(i)) \ge x_\omega - \sum_{j \in S} x_j$, then $S = S_i$ and $\chi_{ij} = 1$ for all $j \in S_i$.*

Proof. We calculate

$$x_i - 2(x, e(i)) = x_i - \sum_{\substack{j \in S_i \\ j \neq i}} \chi_{ij} x_j - 2x_i + x_\omega = x_\omega - \sum_{j \in S} x_j - \Delta$$

where $\Delta = \sum_{j \in S} (\chi_{ij} - 1) x_j + \sum_{j \in S_i \setminus S} \chi_{ij} x_j \ge 0$. By assumption it follows that $\Delta = 0$, which implies $S_i \setminus S = \emptyset$ and $\chi_{ij} = 1$ for all $j \in S$ since $x$ is sincere. ∎

THEOREM. *Suppose $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a weakly non-negative, finitely sincere, semigraphical form with center $\omega$. Let $x$ be a maximal sincere positive* 1-*root of $\chi$ such that $x_\omega$ is maximal among all those* 1-*roots. If $x$ has only* 1 *exceptional vertex, then $x_\omega \le 7$.*

Proof. Note that $\operatorname{card} I \ge 3$. Let $i$ be the exceptional vertex of $x$, thus $2(e(i), x) = 1$, $x_i = 2$ and $2(e(j), x) = 0$ for all $j \neq i$. Without loss of generality we may assume $x_\omega \ge 7$, which shows $\omega \neq i$. We will carry out the proof by showing several claims.

CLAIM 1. *$\chi_{\omega j} = -1$ for all $j \neq \omega$.*

Assuming that this is false furnishes a point $j$ such that $\chi_{\omega j} = -2$ and consequently $\chi(e(\omega) + e(j)) = 0$. In the case $j \neq i$ we obtain $(e(\omega) + e(j), x)$

$= 0$ and therefore $e(\omega) + e(j) \in \mathrm{Rad}^+ \chi$ by 5.1. By Proposition 3.2 this yields a contradiction. For $j = i$ because of $x - 2e(i) - 2e(\omega) > 0$ we get the contradiction $0 \leq \chi(x - 2e(i) - 2e(\omega)) = 1 + 0 - 4(x, e(i) + e(\omega)) = -1$.

Claim 2. *$\chi_{ij} \leq 1$ for all $j \neq \omega$.*

Assuming $\chi_{ij} \geq 2$ for some $j \neq i, \omega$ would give $\chi(e(i) - e(j)) \leq 0$. Since $x - 2(e(i) - e(j)) \geq 0$, we would arrive at the contradiction $0 \leq \chi(x - 2(e(i) - e(j))) = \chi(x) + 4\chi(e(i) - e(j)) - 4(x, e(i) - e(j)) \leq 1 + 0 - 2 = -1$.

Claim 3. *$\chi_{i_1 j_2} > 0$ for all $j_1, j_2 \neq \omega$ satisfying $\chi_{ij_1} > 0 < \chi_{ij_2}$.*

Of course, for $j_1 = j_2$ nothing has to be proved. Assuming that the claim fails for some $j_1 \neq j_2$, we first deal with the case where $j_1, j_2$ are both different from $i$. Using Claims 2 and 1, we get $y := \sigma_\omega \sigma_{j_1} \sigma_{j_2} \sigma_i(x) = \sigma_\omega(x - e(i) + e(j_1) + e(j_2)) = x - e(i) - e(\omega) + e(j_1) + e(\omega) + e(j_2) + e(\omega) = x + e(\omega) + e(j_1) + e(j_2) - e(i)$. This is a contradiction to the maximal choice of $x$ since $y$ is a sincere positive 1-root of $\chi$ satisfying $y_\omega = x_\omega + 1$. In the case $j_1 = i$ we can use the same vector $y$ for a similar argument.

Claim 4. *For $S_i := \{j : \chi_{ij} > 0\}$, the following assertions hold*:

(i) *$x_j = 1$ for all $j \in S_i$, $j \neq i$.*
(ii) *$\chi_{jk} = 1$ for all $j, k \in S_i$.*
(iii) *If $j \in S_i$ and $k \in I$ such that $\chi_{jk} > 0$, then $k \in S_i$.*
(iv) card $S_i = x_\omega - 2$.

Using Claim 2, we calculate $1 = x_i - 2(e(i), x) = x_i - \sum_{j \in S_i} \chi_{ij} x_j - x_i + x_\omega = x_\omega - \sum_{j \in S_i} x_j$. By Claim 3 we obtain $S_i \subseteq S_j$ for an arbitrary $j \neq i$. Observing that

$$\begin{aligned} 1 \leq x_j &= x_j - 2(e(j), x) \\ &= x_j - \sum_{k \in S_j, k \neq j} \chi_{jk} x_k - 2x_j + x_\omega \leq x_\omega - \sum_{k \in S_j} x_k \leq x_\omega - \sum_{k \in S_i} x_k = 1, \end{aligned}$$

by application of Lemma 6.3 we obtain (i)–(iii). From the equation $1 = x_\omega - 1 - \sum_{k \in S_i} 1$ it follows that card $S_i = x_\omega - 2$, which is (iv).

Proceeding with the proof we see that because of $x_\omega \geq 7$ the form $\chi$ cannot be weakly positive. Thus there exists a critical restriction $\chi|J$ which has to be one of the $\mathcal{C}(i)$, $2 \leq i \leq 6$, or $\mathcal{C}(4')$.

Claim 5. *$S_i \subseteq J$.*

If $\mu$ is the characteristic vector of $\chi|J$ then by 5.1 we get $0 \neq 2(\mu, x) = \sum_{j \in I} \mu_j 2(e(j), x) = \mu_i$ and thus $i \in J$. For all $j \in S_i$, $j \neq i$, we see that $\sigma_j \sigma_i(x) = x - e(i) + e(j)$ is still a positive sincere vector. Again using 5.1 we get $0 \neq 2(\mu, \sigma_j \sigma_i(x)) = \sum_{k \in I} \mu_k 2(e(k), x - e(i) + e(j)) = \mu_j$, where the last equality is obtained by going through all possibilities for $k$ and applying Claim 4.

Using again Claim 4, we see that the bigraph of $\chi|J$ contains as full subbigraph a full graph on $x_\omega - 2 \geq 5$ vertices with single dotted edges. Hence $\chi|J$ can be identified with $\mathcal{C}(6)$. We finish by deriving $x_\omega - 2 = 5$ and hence $x_\omega = 7$. ∎

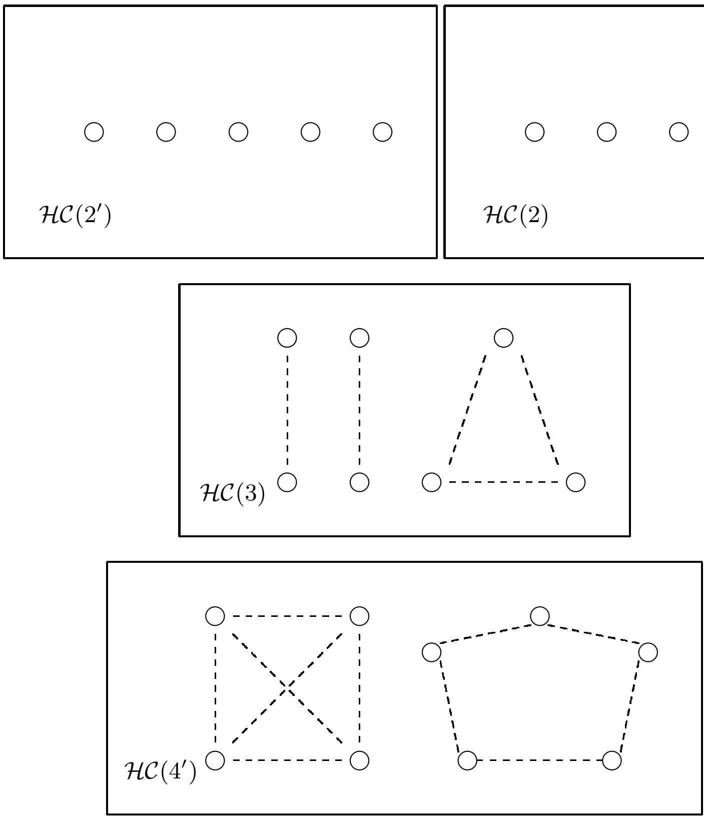
**6.4.** We will need a classification of the hypercritical semigraphical unit forms $\chi$ such that $-1 \leq \chi_{ij}$ for all $i, j$. It is helpful to observe that they actually satisfy $\chi_{ij} \leq 1$ for all $i, j$.

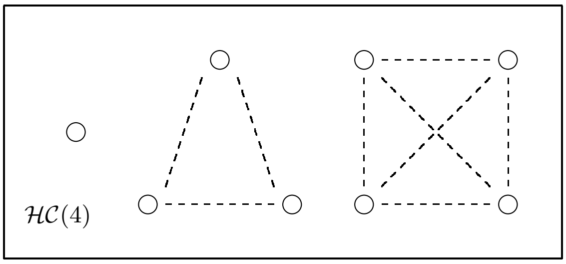
Namely, if $\chi_{ij} \geq 2$ for some $i, j$, then $\chi(e(i) - e(j)) \leq 0$. Since $\chi$ is not weakly non-negative, we find $0 \leq v \in \mathbb{Z}^I$ such that $\chi(v) < 0$. Possibly interchanging $i$ and $j$ we may suppose that $(v, e(i)) \geq 0$. We obtain

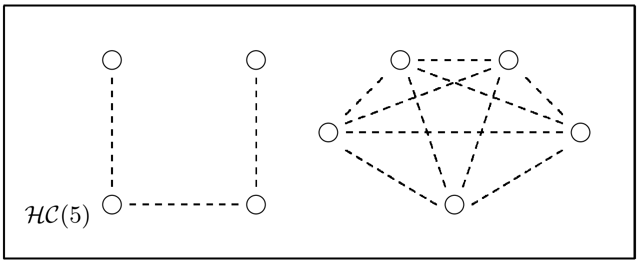
$$\chi(v - v_i(e(i) - e(j))) = \chi(v) + \chi(e(i) - e(j)) - 2v_i(v, e(i) - e(j)) < 0,$$

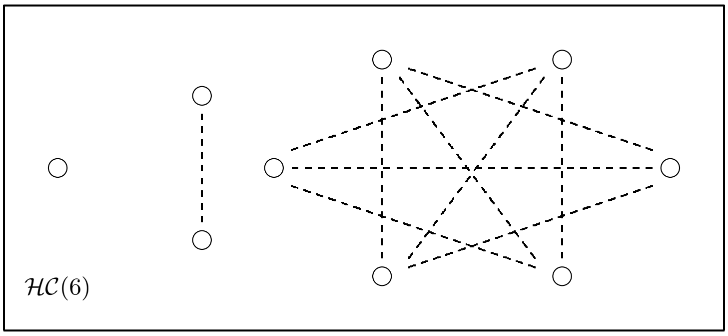
contradicting the weak non-negativity of $\chi|I \setminus \{i\}$.

Lemma. *If $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a hypercritical semigraphical unit form such that $-1 \leq \chi_{ij}$ for all $i, j$, then $\chi$ is one of the graphical forms $\mathcal{HC}(2')$, $\mathcal{HC}(2)$, $\mathcal{HC}(3)$, $\mathcal{HC}(4')$, $\mathcal{HC}(4)$, $\mathcal{HC}(5)$, $\mathcal{HC}(6)$ whose reduced bigraphs are shown in the following list.*






Proof. Since $-1 \le \chi_{ij}$, there has to be a critical restriction $\chi|J$ of type $\mathcal{C}(2)$, $\mathcal{C}(3)$, $\mathcal{C}(4')$, $\mathcal{C}(4)$, $\mathcal{C}(5)$ or $\mathcal{C}(6)$. By [Pe2] the set $I \setminus J$ consists of exactly one element $s$ and for the characteristic vector $\mu$ of $\chi|J$ the inequality $2(\mu, e(s)) < 0$ holds. For all $j \in J$ satisfying $\mu_j = 1$ we obtain $0 \le \chi(\mu - e(j) + e(s)) = 2 + 2(\mu, e(s)) - \chi_{sj}$. Hence $2 \ge -2(\mu, e(s)) = \mu_\omega - \sum_{i \ne \omega} \mu_i \chi_{si} > 0$ and even $-2(\mu, e(s)) = 1$ provided there exists $j \in J$ such that $\chi_{sj} = 1$.

A case by case inspection using these numerical conditions easily shows that the above bigraphs are the only possible candidates. Then one readily checks that these forms are actually hypercritical. ∎

## 7. 0-sincere unit forms

**7.1.** The following basic results on 0-sincere forms can be found in [DP]. For convenience of the reader we include a short sketch of the proof.

Proposition. *Suppose $\chi : \mathbb{Z}^I \to \mathbb{Z}$ is a 0-sincere unit form and let $\{J_1, \ldots, J_m\}$ be the set of all subsets $J$ of $I$ such that $\chi|J$ is critical. For* *all* $i = 1, \dots, m$ *we denote by* $\mu_i$ *the characteristic vector of* $\chi|J_i$. *Then the following assertions hold*:

(a) *If* $\mu \in \mathrm{Rad}^+ \chi$ *is sincere*, *then there exist* $p \in \mathbb{N}$ *and* $q_i \in \mathbb{N}_0$ *such that* $p\mu = \sum_{i=1}^m q_i \mu_i$.

(b) $I = \bigcup_{i=1}^m J_i$.

(c) *For any* $x \in \mathrm{Rad}\,\chi$ *there exists* $r \in \mathbb{Z}$ *such that* $rx \in \sum_{i=1}^m \mathbb{Z}\mu_i$.

(d) $\sum_{i=1}^m \mathbb{Z}\mu_i$ *is a subgroup of* $\mathrm{Rad}\,\chi$ *of finite index. In particular*, $\chi$ *is of corank* $\leq m$.

Proof. Let $\mu \in \mathrm{Rad}^+ \chi$ be sincere. We prove (a) by induction on $n = \mathrm{card}\, I$. Since $\chi$ is 0-sincere, we know $n \geq 0$. For $n = 2$ we obtain $\chi = \mathcal{C}(1)$ where the claim is obvious. For $n > 2$ there is some linear combination $0 \leq w = q\mu - p\mu_n$ with $p, q \in \mathbb{N}$ and $K := \mathrm{supp}\, w$ is a proper subset of $I$. Then by induction (a) holds for the sincere vector $w \in \mathrm{Rad}^+ \chi|K$. Consequently, (a) and also (b) are proved.

For (c) we choose a sincere $\mu \in \mathrm{Rad}^+ \chi$ and $a \in \mathbb{N}$ such that $w = a\mu$ lies in $\mathrm{Rad}^+ \chi$ and is again sincere. Now we apply (a) to $w$ and $\mu$. Clearly, (d) follows from (c). ∎

**7.2.** It does not seem to be obvious that a 0-sincere form of corank $n$ has to have a 0-sincere subform of corank $n-1$. To establish this result, we need the following lemma.

Lemma. *Suppose* $x^{(1)}, \dots, x^{(r)} \in \mathbb{N}_0^n$ *are* $\mathbb{Q}$*-linearly independent vectors satisfying* $\mathrm{supp}(\sum_{i=1}^r x^{(i)}) = \{1, \dots, n\}$. *Then there exist* $\mathbb{Q}$*-linearly independent vectors* $z^{(1)}, \dots, z^{(r-1)} \in \mathbb{N}_0^n$ *in* $\sum_{i=1}^r \mathbb{Z}x^{(i)}$ *such that* $\mathrm{supp} \sum_{i=1}^{r-1} z^{(i)}$ *is a proper subset of* $\{1, \dots, n\}$.

Proof. We apply induction on $r$. For $r = 1$ there is nothing to prove. In the case $r > 1$, if necessary, we replace $x^{(1)}$ by $\sum_{i=1}^r x^{(i)} = \{1, \dots, n\}$ in order to establish that $x^{(1)}$ is sincere. By induction for $x^{(1)}, \dots, x^{(r-1)}$ there are $z^{(1)}, \dots, z^{(r-2)}$ as required. Since $\mathrm{supp} \sum_{i=1}^{r-2} z^{(i)}$ is a proper subset of $\{1, \dots, n\}$, after possibly rearranging indices and vertices, we may suppose that there is some $l \in \mathbb{N}$, $1 \leq l \leq n$, such that for all $i = 1, \dots, r-2$ we have $z_j^{(i)} = 0$ for all $j = 1, \dots, l$, whereas for all $j > l$ there is an index $i$, $1 \leq i \leq r-2$, satisfying $z_j^{(i)} > 0$. As $x^{(1)}$ is sincere, the vectors $x^{(1)}, z^{(1)}, \dots, z^{(r-2)}$ are $\mathbb{Q}$-linearly independent. Because $x^{(1)}, \dots, x^{(r)}$ are $\mathbb{Q}$-linearly independent, without loss of generality we may suppose that even $x^{(r)}, x^{(1)}, z^{(1)}, \dots, z^{(r-2)}$ are $\mathbb{Q}$-linearly independent. Defining $p/q$ as the minimum of all quotients $x_i^{(r)}/x_i^{(1)}$, we find that $z = qx^{(r)} - px^{(1)} \geq 0$ and there is $k$, $1 \leq k \leq n$, such that $z_k = 0$.

Since we may freely replace $x^{(r)}$ by $x^{(r)} + Mz^{(i)}$ where $M$ is any natural number, it is possible to shift the minimum $k$ until $k \leq l$. Therefore without loss of generality we may even suppose $z_1 = 0$. Putting $z^{(r-1)} = z$, we conclude that $x^{(1)}, z^{(1)}, \ldots, z^{(r-1)}$ are still $\mathbb{Q}$-linearly independent and $1 \notin \operatorname{supp} \sum_{i=1}^{r-1} z^{(i)}$. ■

PROPOSITION. *If* $\chi : \mathbb{Z}^I \to \mathbb{Z}$ *is a* 0*-sincere weakly non-negative unit form of corank* $r$, *then there exists a proper subset* $J$ *of* $I$ *such that the restriction* $\chi|J$ *is* 0*-sincere and of corank* $r-1$.

Proof. By 7.1 we can find vectors $x^{(1)}, \ldots, x^{(r)} \in \mathbb{N}_0^n$ which are $\mathbb{Q}$-linearly independent and lie in $\operatorname{Rad}\chi$. We apply the previous lemma and put $J := \operatorname{supp} \sum_{i=1}^{r-1} z^{(i)}$. Obviously, the corank of $\chi|J$ is $\geq r-1$. But, since $\chi$ is non-negative, $\operatorname{Rad}\chi|J$ is a subgroup of $\operatorname{Rad}\chi$. This subgroup has to be proper, as it does not contain any sincere vector. Consequently, the corank of $\chi|J$ is $r-1$. ■

**7.3.** A 0-sincere weakly non-negative unit form $\chi$ is said to be *reduced* provided $\chi_{ij} \leq 1$ for all vertices $i, j$. The reason for calling these forms reduced is the following assertion.

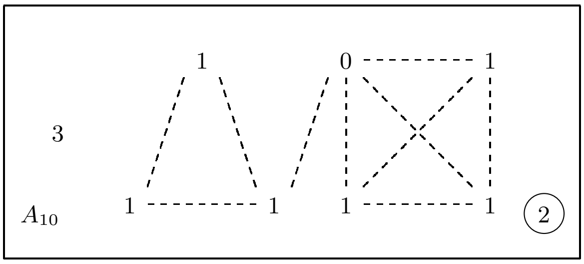
LEMMA. *Suppose* $\chi : \mathbb{Z}^I \to \mathbb{Z}$ *is a* 0*-sincere unit form. Then* $\chi$ *is not reduced if and only if there exists a vertex* $i$ *such that* $\chi$ *is obtained from* $\chi|I \setminus \{i\}$ *by doubling a point.*

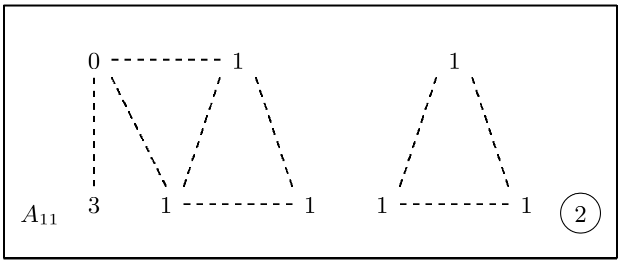
Proof. The sufficiency of the condition is clear. To prove necessity, we fix $i, j$ such that $\chi_{ij} > 1$ and derive from 3.1 that $\chi_{ij} = 2$. Hence $0 = 2 - \chi_{ij} = \chi(e(i) - e(j))$, which by 5.3 and Remark 4.5 implies that $\chi$ is obtained by doubling a point. ■

## 8. Graphical 0-sincere forms of small corank

**8.1.** In [Za] (see also [Si]) the sincere partially ordered sets of polynomial growth were classified. Among others there occurred the sets $A_{10}$, $A_{11}$ and $\Psi_{15}, \ldots, \Psi_{20}$. We denote the corresponding Tits forms by the same symbols. These forms are by construction graphical and are known to be non-negative. In fact, more details about them can be found in [Ri] because the corresponding posets are domestic tubular or tubular.

Let us display the corresponding reduced bigraphs. We replace each vertex by a tuple of numbers which are the coefficients of the characteristic vectors of the critical restrictions. Since we know that these coefficients are integers between 0 and 6, we do not insert any separators between the different numbers. Again we provide the coefficients at the center inside a circle in the lower right corner.


1
0
1
3
A10
1
1
1
1
2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A11
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1
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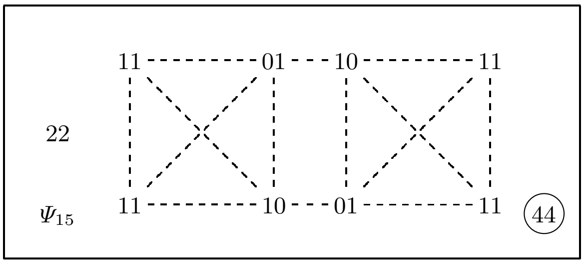
2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22
Ψ15
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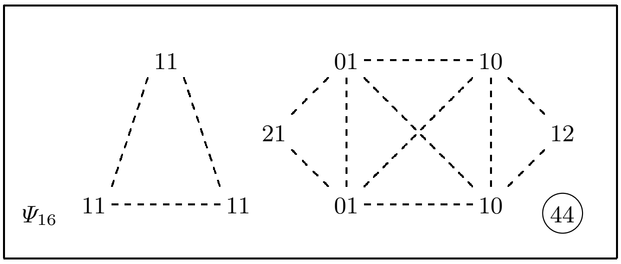
11
44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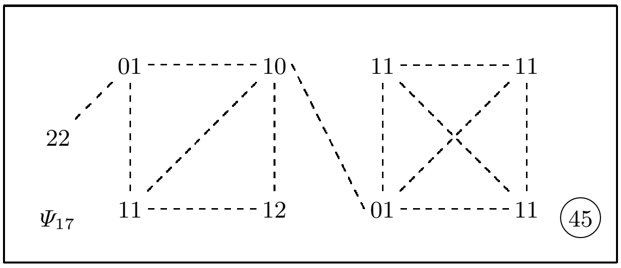
01
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Ψ17
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01
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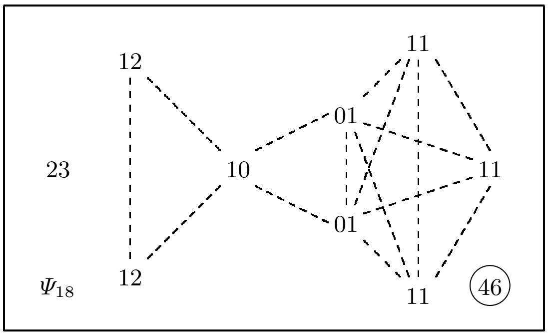
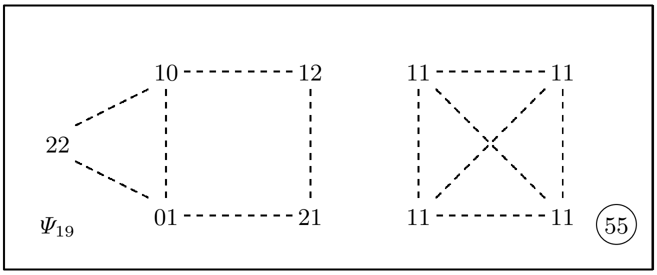
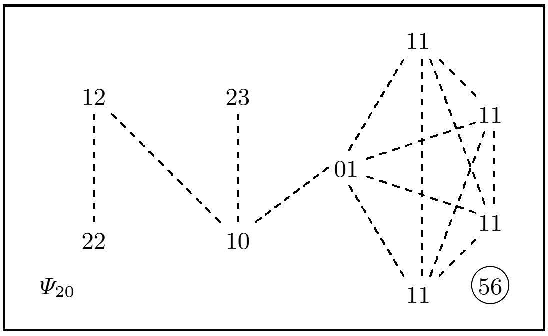
45




Lemma. *Let* $\chi : \mathbb{Z}^K \to \mathbb{Z}$ *be a reduced non-negative semigraphical form such that there is no critical restriction of the form* $\mathcal{C}(1)$, $\mathcal{C}(2)$, $\mathcal{C}(3)$ *or* $\mathcal{C}(4')$. *Suppose that* $J$ *is a subset of* $K$ *such that* $\operatorname{card} J = \operatorname{card} K - 1$.

(a) *If* $\chi|J = \mathcal{C}(4)$, *then* $\chi$ *coincides with one of* $A_{10}$, $A_{11}$ *and is of corank* 1.

(b) *If* $\chi|J = \mathcal{C}(5)$, *then* $\chi$ *coincides with one of* $\Psi_{17}$, $\Psi_{19}$, $\Psi_{20}$ *and is of corank* 2.

(c) *If* $\chi|J = \mathcal{C}(6)$, *then* $\chi$ *coincides with one of* $\Psi_{18}$, $\Psi_{20}$ *and is of corank* 2.

(d) *If* $\chi|J = A_{10}$ *or* $= A_{11}$ *and* $\mu$ *is the characteristic vector of the unique critical restriction* (*of type* $\mathcal{C}(4)$), *then* $\chi$ *is of corank* 2 *and either* $\chi$

*coincides with one of* $\Psi_{15}$, $\Psi_{16}$, $\Psi_{17}$, $\Psi_{18}$ *or* $\operatorname{Rad}\chi$ *has generators* $\mu, v$ *such that* $v_s = 1$ *and* $v_t = -1$ *where* $s, t$ *are the* 2 *vertices not belonging to* $\operatorname{supp}\mu$. *In particular, in this second case the form* $\chi$ *is not* 0*-sincere.*

Proof. The statements about the coranks can be found in [Ri] with the exception of the non-0-sincere forms appearing in (d), which have to be calculated explicitly.

For (a)–(c) denote by $k$ the vertex in $K \setminus J$ and by $\mu$ the characteristic vector of $\chi|J$. Since $\mu \in \operatorname{Rad}(\chi)$, we get $0 = 2(e(k), \mu) = \mu_\omega - \sum_{j\in S}\mu_j$ where $S := \{j \in J : \chi_{kj} = 1\}$. The rest of the proof consists of an inspection of the few cases left possible by this condition.

To show (d) we observe that we can apply (a) to the restriction of $\chi$ to the union of $\mathcal{C}(4)$ and the additional vertex. Hence there are again only a few cases left to examine. ∎

**8.2.** Theorem. *Let* $\chi : \mathbb{Z}^I \to \mathbb{Z}$ *be a reduced* 0*-sincere semigraphical form such that there is no critical restriction of the form* $\mathcal{C}(1)$, $\mathcal{C}(2)$, $\mathcal{C}(3)$ *or* $\mathcal{C}(4')$. *Then the corank of* $\chi$ *is* 2 *if and only if* $\chi$ *is one of the forms* $\Psi_{15}, \ldots, \Psi_{20}$.

Proof. That the forms $\Psi_{15}, \ldots, \Psi_{20}$ have the desired properties follows from [Ri], which was already stated in the proof of the above lemma.

For the converse we choose a critical restriction $\chi|J$. Since $\chi$ is of corank 2, the set $J$ cannot exhaust $I$ and we can apply the above lemma to the union $K$ of $J$ with an arbitrary vertex. In the case when $\chi|J$ coincides with $\mathcal{C}(5)$ or $\mathcal{C}(6)$, parts (b), (c) of the lemma show that $\chi|K = \chi$ since corank 2 is already reached. If $\chi|J$ is of shape $\mathcal{C}(4)$, then part (a) shows that $\chi|K$ is still of corank 1; therefore $K$ is still a proper subset of $I$, and we can add another point to obtain a subset $K'$.

Changing the notation for $K$ to $J$ and for $K'$ to $K$ we are able to apply part (d) of the above lemma. Thus it remains to exclude the possibility of $\chi|K$ being a form of corank 2 which is not 0-sincere. But, since $\chi$ is 0-sincere, the last condition shows that in this case $K$ would still be a proper subset of $I$, leading to the contradiction that $\chi$ would be of corank $\geq 3$. ∎

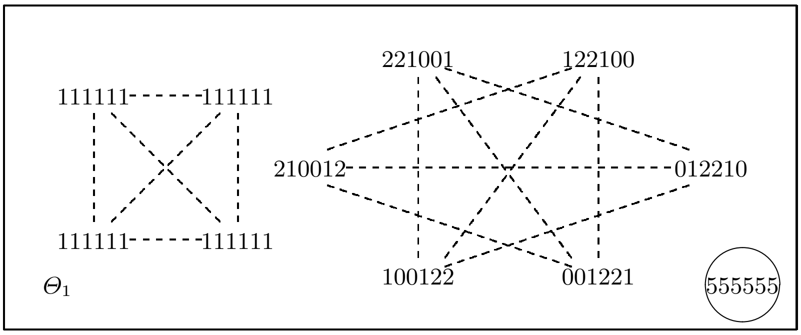
Remark. The forms $\Psi_{15}, \ldots, \Psi_{20}$ appearing in the above list have precisely two critical restrictions. An inspection shows that there are vertices $i, j$ such that the restrictions of the two characteristic vectors $\mu^{(1)}$, $\mu^{(2)}$ corresponding to $\{i, j\}$ are just the canonical base vectors $(1, 0)$, $(0, 1)$. In particular, this means that any sincere $\mu \in \operatorname{Rad}^+\chi$ can be written as $\mu = n_1\mu^{(1)} + n_2\mu^{(2)}$ where $n_1, n_2$ are positive integers.

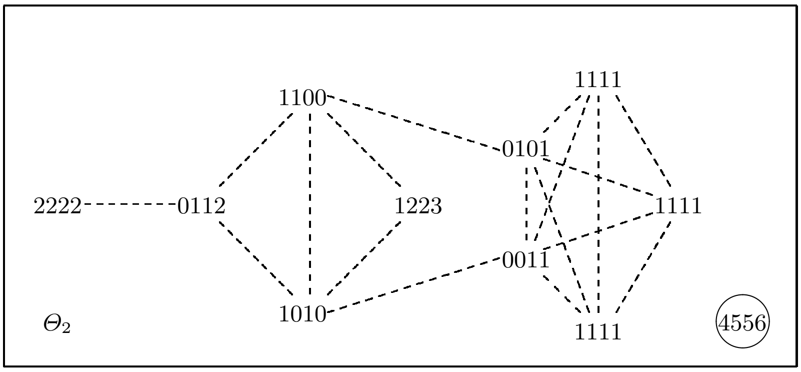
**8.3.** Fortunately we do not need the corresponding complete classification of forms of corank 3 but will only use certain properties of these forms which we will establish in this final part of this section. Let us start with an ad hoc definition. We call a 0-sincere reduced graphical form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ *triangular* provided there are precisely three critical restrictions $I_1, I_2, I_3$ such that for all $i \neq j \in \{1,2,3\}$ the restriction $\chi|I_i \cup I_j$ is a 0-sincere form of corank 2.

LEMMA. *Let $\chi : \mathbb{Z}^K \to \mathbb{Z}$ be a reduced non-negative semigraphical form such that there is no critical restriction of the form $\mathcal{C}(1), \mathcal{C}(2), \mathcal{C}(3)$ or $\mathcal{C}(4')$. Suppose that $J$ is a subset of $K$ such that* $\operatorname{card} J = \operatorname{card} K - 1$. *If $\chi|J$ is* 0*-sincere of corank* 2, *then $\chi$ is a* 0*-sincere form of corank* 3 *which is either triangular or one of the forms $\Theta_1$, $\Theta_2$ whose reduced bigraphs are shown below.*

Proof. Since $\chi|J$ is of corank 2, we can apply Theorem 8.2 to deduce that $\chi|J$ is one of the forms $\Psi_{15}, \ldots, \Psi_{20}$. We denote by $k$ the vertex in $K \setminus J$, choose a critical restriction $J'$ of $J$ and apply Lemma 8.1(a), (b) or (c) to the restriction of $\chi$ to $J' \cup \{k\}$. By inspection of all possibilities how $k$ can be connected to the vertices of $J \setminus J'$ we obtain the result. ∎

Let us present the reduced bigraphs of $\Theta_1$, $\Theta_2$. We again replace the vertices by the coefficients of the characteristic vectors of the critical restrictions.



THEOREM. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a reduced* 0*-sincere semigraphical form such that there is no critical restriction of the form* $\mathcal{C}(1)$, $\mathcal{C}(2)$, $\mathcal{C}(3)$ *or* $\mathcal{C}(4')$. *If the corank of $\chi$ is* 3 *then $\chi$ is either triangular or one of the forms* $\Theta_1$, $\Theta_2$.

Proof. By Proposition 7.2 there is a restriction $\chi|J$ of $\chi$ which is of corank 2. We choose a vertex $k \in I \setminus K$ and apply the above lemma to $K := J \cup \{k\}$. Since the lemma tells us that the corank of $\chi|K$ is 3, we get $K = I$ and the theorem is proved. ■

REMARK. We need a remark corresponding to the above for the corank 2 case. Namely, here an inspection shows that for the triangular forms appearing in the above theorem there are vertices $i, j, k$ such that the restrictions of the characteristic vectors $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$ to $\{i, j, k\}$ are just the canonical base vectors $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$. In particular, this means that any sincere $\mu \in \mathrm{Rad}^+ \chi$ can be written as $\mu = n_1\mu^{(1)} + n_2\mu^{(2)} + n_3\mu^{(3)}$ where $n_1, n_2, n_3$ are positive integers.

A similar property still holds for $\Theta_2$ where there are vertices $i, j, k, l$ such that the restrictions of the characteristic vectors $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$, $\mu^{(4)}$ to $\{i, j, k, l\}$ are the vectors $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(0, 1, 0, 1)$, $(0, 0, 1, 1)$. This shows that after possibly renaming the vertices any sincere $\mu \in \mathrm{Rad}^+ \chi$ can be written as $\mu = n_1\mu^{(1)} + n_2\mu^{(2)} + n_3\mu^{(3)}$ where $n_1, n_2, n_3$ are non-negative integers and $n_1, n_2$ are positive.

## 9. Proof of the main theorem

**9.1.** We assume the existence of a weakly non-negative, finitely sincere semiunit form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ which has a maximal sincere positive 1-root $x$ with a coordinate $x_i > 12$ and try to establish a contradiction.

Applying 4.4 we replace $\chi, x$ by $\chi', x'$ where $(\chi', x', \phi)$ is a full reduction of $(\chi, x)$ with respect to $I \setminus \{i\}$. Obviously, $\mathrm{card}\, I' \geq 2$. If there exists $i \neq j \in I'$ such that $\chi'_{ij} \geq 0$, then $\chi'_{kj} \geq 0$ for all $k \in I'$, which by Remark 3.2 implies $\chi'_{jj} = 0$. This shows that $e(j)$ is a positive 0-root with the property $x' > e(j)$. By Corollary 5.2 the 1-root $x'$ is 2-layer, which by Theorem 5.3 yields the contradiction $x'_i \leq 12$.

Hence in the sequel we only have to deal with the case of $\chi'$ being a semigraphical unit form with center $i$. In order to simplify notation we replace $\chi'$ by $\chi$, $x'$ by $x$ and $i$ by $\omega$. If $\chi$ has a critical restriction $\chi|J$ which is of the shape $\mathcal{C}(1)$, $\mathcal{C}(2)$, $\mathcal{C}(3)$ or $\mathcal{C}(4')$, then the characteristic vector of this restriction is a 0-root $\mu$ satisfying $x > \mu$. Again we obtain a contradiction by Corollary 5.2 and Theorem 5.3.

**9.2.** Let us sum up to what extent we have already reduced our original problem. To produce a contradiction, we now have at our disposal a weakly non-negative, finitely sincere semigraphical form $\chi : \mathbb{Z}^I \to \mathbb{Z}$ with center

$\omega$ having a maximal sincere positive 1-root $x$ with $x_\omega > 12$ and having the property that any critical restriction is of the shape $\mathcal{C}(4)$, $\mathcal{C}(5)$ or $\mathcal{C}(6)$. Since $\mathcal{C}(1)$ does not occur, we know that $\chi_{i\omega} = 1$ for all $i \in I$ different from $\omega$.

Among all forms $\chi$ as above we consider one where the number of vertices $\operatorname{card} I$ is minimal. Section 4.5 shows that this choice implies that $\chi$ is not obtained from any subform by doubling a vertex. By 3.4 the 1-root $x$ has one or two exceptional vertices. If $x$ has only one exceptional vertex, applying Theorem 6.3 we arrive at the contradiction $x_\omega \le 7$.

Thus there are exactly two exceptional vertices $i \neq j$ which have the property that $x_i = x_j = 2(x, e(i)) = 2(x, e(j)) = 1$ and $2(x, e(k)) = 0$ for all $k \neq i, j$. Because of $x_\omega > 12$ we see $\omega \neq i, j$.

**9.3.** We observe that the form $\chi | I \setminus \{i, j\}$ is weakly positive. Indeed, otherwise there exists a critical restriction of this form having a characteristic vector $\mu$, which yields the contradiction $\chi(\mu + x) = \chi(x) = 1$. Using this observation we can prove:

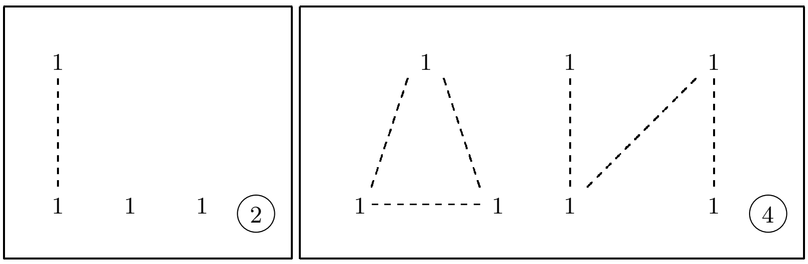
LEMMA. *If* $0 \le z \in \mathbb{Z}^I$ *satisfies* $z_k \le 1$ *for all* $k \in I$, $k \neq \omega$, $z_i = z_j = 1$ *and* $\chi(z) \le 2$, *then* $z_\omega > 6$.

Proof. Assuming $z_\omega \le 6$, we obtain $x - z > 0$ and therefore $0 < \chi(x - z) = \chi(z) - 1 \le 1$. Thus $x - z$ is a positive 1-root of the weakly positive form $\chi | I \setminus \{i, j\}$, which shows $(x - z)_\omega \le 6$ and finally the contradiction $x_\omega \le 12$. ∎

We can apply this lemma immediately to show that $\chi_{ij} \in \{2, 3\}$. By 3.1 we know that $0 \le \chi_{ij} \le 3$. The lemma shows that $\chi_{ij} \in \{0, 1\}$ is impossible by using the vector $z := e(\omega) + e(i) + e(j)$, which satisfies $\chi(z) \le 2$.

**9.4.** Let us consider the unit form $\overline{\chi} : \mathbb{Z}^I \to \mathbb{Z}$ defined by $\overline{\chi}(y) = \chi(y) - y_i y_j$ for all $y \in \mathbb{Z}^I$. Clearly, $x$ is a sincere positive 0-root of $\overline{\chi}$. We want to show that $\overline{\chi}$ is weakly non-negative and consequently a 0-sincere unit form.

If we assume that $\overline{\chi}$ is not weakly non-negative, then there has to be a critical restriction $\overline{\chi}|J$ where $J$ has to contain both $i$ and $j$. If $\chi_{ij} = 3$, we have a contradiction to 6.4. If $\chi_{ij} = 2$, we see from 6.4 that the reduced bigraph of $\overline{\chi}|J$ contains a subbigraph $K$ satisfying $i, j \in K$ of one of the following two shapes:



Defining vectors $z \in \mathbb{Z}^{K\cup\{\omega\}}$ as shown in the pictures above and considering them as elements of $\mathbb{Z}^I$, we obtain $\overline{\chi}(z) = 1$ resp. $\chi(z) = 2$. This is a contradiction to the previous lemma.

**9.5.** As the next step we convince ourselves that $\overline{\chi}_{st} \leq 1$ for all $s, t \in I$ such that $\{s,t\} \neq \{i,j\}$. As $\overline{\chi}$ is 0-sincere, by 3.1 we only have to show that $\chi_{st} = 2$ leads to a contradiction for $s \neq t$ such that $\{s,t\} \neq \{i,j\}$.

If $\{s,t\} \cap \{i,j\} = \emptyset$, then by 7.3 the forms $\overline{\chi}$ and also $\chi$ would be obtained form the restriction to $I \setminus \{t\}$ by doubling the point $s$.

If the intersection of $\{s,t\}$ and $\{i,j\}$ is just $t = j$, then we consider the positive 1-root $y := x - e(i)$ of $\eta = \chi|I \setminus \{i\}$. If $\eta$ were weakly positive, then we would get $x_\omega = y_\omega \leq 6$. Hence $\eta$ has a critical restriction with a characteristic vector $v$. By 7.3 the form $\overline{\chi}$ is obtained from $\chi|I \setminus \{j\}$ by doubling $s$. Hence also $\eta$ is obtained from $\eta|I \setminus \{i,j\}$ by doubling $s$. Consequently, the vector $w := v - v_j e(j) + v_j e(s)$ is a 0-root of $\eta$ and also of $\chi$, which gives the contradiction $\chi(x+w) = \chi(x) + 2(w,x) = \chi(x) + 2w_i(e(i),x) + 2w_j(e(j),x) = \chi(x) = 1$.

**9.6.** Before we continue, let us recall that because of the positive definiteness of $\overline{\chi}$ any 0-root of $\overline{\chi}$ lies in $\operatorname{Rad}\overline{\chi}$. Consider the restriction $\xi$ of $\chi$ to $I' := I \setminus \{i\}$ which, being a restriction of $\overline{\chi}$ as well, is a non-negative reduced semigraphical form. The vector $y = x - e(i) = \sigma_i(x)$ is a positive 1-root of $\chi$ and therefore a sincere positive 1-root of $\xi$. As $y_\omega > 12$, we know that $\xi$ is not weakly positive and consequently the 0-sincere kernel $\xi^+$ of $\xi$ is non-trivial. We want to establish that $\xi^+$ is of corank $\leq 2$.

Assume that the corank of $\xi^+$ is at least 3. Then by 7.3 the form $\xi^+$ has a 0-sincere restriction $\zeta$ whose corank is precisely 3. We will obtain a contradiction using again Corollary 5.2 and Theorem 5.3 by constructing a positive 0-root $\mu$ such that $x > \mu$.

In fact, by Theorem 8.3 the form $\zeta$ is either triangular or coincides with $\Theta_1$ or $\Theta_2$. If $\zeta = \Theta_1$, the positive 0-root $z$ defined by $z_\omega = 5$ and $z_k = 1$ for all other vertices allows us to apply Corollary 5.2 and Theorem 5.3 immediately. If $\zeta$ happens to be triangular, we denote by $\mu_1, \mu_2, \mu_3$ the characteristic vectors of the three critical restrictions $I_1, I_2, I_3$ such that for all $s \neq t$ the union $I_s \cup I_t$ is 0-sincere of corank 2 and consequently, by Theorem 8.2, is one of the forms $\Psi_{15}, \dots, \Psi_{20}$. Looking at these forms, we find that $\{(\mu_s - \mu_t)_k : k \in I,\ k \neq \omega\} = \{-1, 0, 1\}$ and $(\mu_s - \mu_t)_\omega \in \{-2, 0, 2\}$. If $\zeta = \Theta_2$ the same holds for the the first three vectors $\mu_1, \mu_2, \mu_3$ displayed in 8.3 although this form is not triangular.

We claim that at least two of the three numbers $2(e(i), \mu_1)$, $2(e(i), \mu_2)$, $2(e(i), \mu_3)$ are equal. Indeed, otherwise there exist $s, t$ satisfying $2(e(i), \mu_s) - 2(e(i), \mu_t) \geq 2$ and therefore also $2(x, \mu_s - \mu_t) = 2(y + e(i), \mu_s - \mu_t) =$

$2(e(i), \mu_s - \mu_t) \geq 2$. On the other hand, $x - (\mu_s - \mu_t) > 0$, yielding the contradiction $0 \leq \chi(x - (\mu_s - \mu_t)) = \chi(x) + \chi(\mu_s - \mu_t) - 2(x, \mu_s - \mu_t) \leq -1$.

Supposing now that without loss of generality $2(e(i), \mu_1) = 2(e(i), \mu_2)$ and $(\mu_1 - \mu_2)_\omega \geq 0$, we obtain $\mu_1 - \mu_2 \in \operatorname{Rad}\chi$. Putting $d := \min\{x_k : (\mu_1 - \mu_2)_k = -1\}$, it follows that $z := x + d(\mu_1 - \mu_2)$ is a positive 1-root of $\chi$ satisfying $z_\omega > 12$ but $z$ is not sincere. By the subsequent lemma we arrive at a contradiction to the minimal choice of $\operatorname{card} I$.

LEMMA. *Let $\chi : \mathbb{Z}^I \to \mathbb{Z}$ be a weakly non-negative semiunit form. Suppose $x$ is a maximal sincere positive 1-root of $\chi$ and $\mu \in \operatorname{Rad}\chi$. If $x + \mu > 0$ and $J := \operatorname{supp}(x + \mu)$, then $x + \mu$ is a maximal sincere positive 1-root of $\chi|J$.*

Proof. Let $y \geq x + \mu$. Then $y - \mu > x$ and $\chi(y - \mu) = 1$. Hence $y - \mu = x$. ∎

**9.7.** It remains to deal with the case where the corank of $\xi^+$ is 1 or 2. But, before doing so, we also want to establish that $\overline{\chi}$ does not admit a critical restriction of type $\mathcal{C}(1), \mathcal{C}(2), \mathcal{C}(3)$ or $\mathcal{C}(4')$. It is clear that $\mathcal{C}(1)$ cannot occur. If one of the others occurred, then the support of its characteristic vector $z$ would have to contain $i$ and $j$ because otherwise it would even be a critical restriction of $\chi$. Therefore $z$ would be a positive 1-root of $\chi$ to which Lemma 9.3 would apply, furnishing the contradiction $z_\omega > 6$.

**9.8.** Suppose now that $\xi^+ : \mathbb{Z}^K \to \mathbb{Z}$ is of corank 2 and therefore classified in Theorem 8.2. Assume that there exists $k \in I' \setminus K$. Then by Lemma 8.3 the restriction of $\xi$ to $K \cup \{k\}$ is a 0-sincere form of corank 3 and consequently $\xi^+$ is of corank $\geq 3$, a contradiction. Hence $I = K \cup \{i\}$.

If $\overline{\chi}_{ij} = 2$, then $\overline{\chi}$ is obtained from $\xi^+$ by doubling $j$, and $u := x - e(i) + e(j)$ is a sincere 0-root of $\xi^+$ such that $u_j = 2$. If $\mu^{(1)}$ and $\mu^{(2)}$ are the characteristic vectors of the 2 critical restrictions of $\xi^+$, then $u = p\mu^{(1)} + q\mu^{(2)}$ where $p, q \in \mathbb{N}$. Up to symmetry we only need to consider the cases $\mu_j^{(1)} = 1$ and $\mu_j^{(1)} = 0$. In both cases we obtain $x \geq \mu^{(1)}$, which yields the usual contradiction to Theorem 5.3 by Corollary 5.2 since $\mu^{(1)}$ is a positive 0-root of $\chi$.

If $\overline{\chi}_{ij} = 1$, then again by Lemma 8.3 we know that $\overline{\chi}$ is one of the 0-sincere semigraphical forms of corank 3 which are classified in Theorem 8.3. If $\overline{\chi} = \Theta_1$, then the vector $z$ defined by $z_\omega = 5$ and $z_k = 1$ for all other vertices is a 1-root of $\chi$, thus leading to a contradiction via Lemma 9.3. In the other cases, by Remark 8.3 we always find characteristic vectors $\mu^{(1)}$, $\mu^{(2)}$, $\mu^{(3)}$ of critical restrictions generating the radical of $\overline{\chi}$ such that $x = p_1\mu^{(1)} + p_2\mu^{(2} + p_3\mu^{(3)}$, where $p_1, p_2, p_3$ are non-negative integers and at least $p_1, p_2 > 0$. Since $x_i = 1$, we may assume that $\mu_i^{(1)} = 0$. Hence we end up with the same contradiction as above because this $\mu^{(1)}$ is a 0-root of $\chi$ such that $x \geq \mu^{(1)}$.

**9.9.** Finally, we have to examine the case where $\xi^+$ is of corank 1, or in other words, is critical. We denote by $\mu$ the corresponding characteristic vector. This case splits into two subcases: in the first one $\xi^+$ is of the shape $\mathcal{C}(5)$ or $\mathcal{C}(6)$, and in the second one $\xi^+$ is of the shape $\mathcal{C}(4)$.

In the first subcase we see from parts (b) and (c) of Lemma 8.1 that $K = I'$. If $\overline{\chi}_{ij} = 2$ then $\overline{\chi}$ is obtained from $\xi^+$ by doubling $j$ and we consider again the sincere positive 0-root $u = x - e(i) + e(j)$ of $\xi^+$. Because $u_j = 2$ we get $u = p\mu$, where $p \in \{1, 2\}$, yielding the contradiction $x_\omega = u_\omega \leq 2\mu_\omega \leq 12$.

If $\overline{\chi}_{ij} = 1$, then again from Lemma 8.1 we see that $\overline{\chi}$ is one of the forms $\Psi_{17}, \ldots, \Psi_{20}$. Since $x_i = 1$, the existence of a critical restriction avoiding $i$ whose characteristic vector $\mu$ satisfies $x \geq \mu$ is easy to derive.

Let us consider the second subcase. If $I' \setminus K$ happens to be empty, then for $\overline{\chi}_{ij} = 2$ we can argue as in the first subcase and for $\overline{\chi}_{ij} = 1$, by part (a) of Lemma 8.1, the form $\overline{\chi}$ is not 0-sincere, a contradiction.

If $I' \setminus K$ is non-empty, we fix a vertex $k$ in this set. Let us first analyze the situation where $I' \setminus K = \{k\}$. Then either $\overline{\chi}$ is not 0-sincere if $\overline{\chi}_{ij} = 2$ or $\overline{\chi}$ is one of the forms occurring in part (d) of Lemma 8.1. For $\overline{\chi}$ among $\Psi_{15}, \Psi_{16}, \Psi_{17}, \Psi_{18}$ we obtain a contradiction as in the first subcase. Otherwise $\overline{\chi}$ again fails to be 0-sincere.

Supposing now that there are elements in $I' \setminus K$ different from $k$, we observe that for any $k \neq l \in I' \setminus K$, by part (d) of 8.3, using the fact that $\xi^+$ is of corank 1, we find a radical vector $\nu^{(l)}$ such that $\nu^{(l)}_l = 1$ and $\nu^{(l)}_k = -1$. Moreover, it is easy to see that the vectors $\nu^{(l)}$ together with the characteristic vector $\mu$ of $\xi^+$ form a basis of $\operatorname{Rad}\xi$. This shows that $\xi$ is not 0-sincere, which remains true for $\overline{\chi}$ if $\overline{\chi}_{ij} = 2$. To solve the case $\overline{\chi}_{ij} = 1$, we apply part (d) of Lemma 8.1 to $\overline{\chi}|K \cup \{k, i\}$. If this form is not 0-sincere, we obtain another radical vector $\nu^{(i)}$ as above and $\overline{\chi}$ is not 0-sincere.

If $\overline{\chi}$ is among $\Psi_{15}, \Psi_{16}, \Psi_{17}, \Psi_{18}$, the argument is slightly more subtle. Denote by $\mu'$ the second characteristic vector for $\overline{\chi}|K \cup \{k, i\}$ besides $\mu$. One sees that $\mu'_i = \mu'_k = 1$ and the vectors $\nu^{(l)}$ together with $\mu$ and $\mu'$ form a basis of $\operatorname{Rad}\overline{\chi}$. If we write $x$ as a linear combination of these base vectors and observe that $x_1 = 1$, it follows that $x_k \leq 0$, which is the final contradiction finishing the proof.

## 10. Applications to finite-dimensional algebras

**10.1.** For more details concerning the notions used in this section we refer to [Ri]. We suppose that $\Lambda$ is a finite-dimensional basic algebra over an algebraically closed field $k$. We write $A = k[Q]/I$, where $Q$ is the ordinary quiver of $A$ and $I$ is an admissible ideal of the path algebra $k[Q]$. To consider the Tits form $\chi_A$ as defined in the introduction we demand that $Q$ is directed, i.e. does not admit oriented cycles.

We denote by $A$-mod the category of all finite-dimensional left $A$-modules, which we identify with the representations of $Q$ satisfying $I$. We do not distinguish between a module $X$ and its isomorphism class.

On the indecomposable modules $X$ in $A$-mod we consider the relation $\prec$ which is defined as the transitive closure of the relation given by putting $X \prec Y$ if there is a non-zero non-isomorphism $X \to Y$. An indecomposable $A$-module $X$ is said to be *directing* if $X \not\prec X$.

For a given indecomposable module $X$ we define $\Lambda(X)$ to be the subalgebra of $\Lambda$ induced by the support of $X$. In the natural way $X$ will be considered as a $\Lambda(X)$-module, which is directing provided $X$ is a directing $\Lambda$-module. We say that $X$ is *properly directing* if $X$ is a directing $\Lambda$-module but belongs neither to a postprojective nor to a preinjective component of the Auslander–Reiten quiver of $\Lambda(X)$.

PROPOSITION. *Let $\Lambda$ be a tame algebra. If $X$ is an indecomposable properly directing $\Lambda$-module*, *then* $\dim_k X(i) \le 12$ *for all vertices $i \in Q_0$.*

Proof. Without loss of generality we may suppose $\Lambda = \Lambda(X)$. Hence $\Lambda$ is a tame tilted algebra having the sincere directing module $X$. Consequently, $\chi_A$ is a weakly non-negative unit form and $\mathbf{dim}\, X$ is a sincere positive 1-root of $\chi_A$. Thus by our main theorem it remains to show that $\chi_A$ is finitely sincere. By [Pe2, 3.2], $A$ is domestic in at most two 1-parameters.

If $A$ were domestic in less than two 1-parameters, then $A$ would be representation-finite or tame concealed. Thus $X$ would be postprojective or preinjective.

Consequently, $A$ has to be domestic in exactly two 1-parameters. By [Pe3, 1.6] and [Pe3, 2.1] it follows that $\chi_A$ is finitely sincere. ■

COROLLARY. *If $X$ is an indecomposable properly directing module over a tame algebra $\Lambda$ whose ordinary quiver has $n$ vertices*, *then* $\dim_k X \le 2n + 180$.

Proof. Again we may assume $\Lambda = \Lambda(X)$. If $n \ge 20$, then $\Lambda$ belongs to the list presented in [Pe4] and one checks directly that $\dim_k X \le 2n - 2$.

For $n \le 19$ we assume that $\mathbf{dim}\, X$ is a maximal positive 1-root of $\chi_A$, which consequently has an exceptional vertex $i$ satisfying $\dim_k X(i) \le 2$. Therefore $\dim_k X \le 2n - \sum j \ne i |\dim X(j) - 2| \le 2n + 18(12 - 2)$. ■

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