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ON BLOCK RECURSIONS, ASKEY'S SIEVED JACOBI POLYNOMIALS AND TWO RELATED SYSTEMS

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WALEED AL-SALAM, TEACHER AND FRIEND, IN MEMORIAM

Two systems of sieved Jacobi polynomials introduced by R. Askey are considered. Their orthogonality measures are determined via the theory of blocks of recurrence relations, circumventing any resort to properties of the Askey–Wilson polynomials. The connection with polynomial mappings is examined. Some naturally related systems are also dealt with and a simple procedure to compute their orthogonality measures is devised which seems to be applicable in many other instances.

1. Introduction. Since the works of L. J. Rogers ([32]–[36]), the orthogonal systems of q-polynomials, or "basic" orthogonal polynomials, play important roles in diverse branches of mathematics. Just as the Rogers polynomials (also known as the q-Rogers or q-ultraspherical polynomials) have provided the key for the proof of the celebrated Rogers–Ramanujan identities (see [3], [22]), a new system, the q-Wilson polynomials, also known as the Askey–Wilson polynomials ([3], [8], [22]), has been crucial in establishing some surprising identities of R. Baxter, used by this author in his solution of the "Hard Hexagon" and other models in statistical mechanics (see [3]).

Nowadays, q-versions (frequently several of them) of almost all classical systems of polynomials are known (the latter can be recovered from the

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q-versions by appropriately letting $q \to 1$). The relevant literature is extensive, the recent book by G. Gasper and M. Rahman [22] being a basic reference on the subject.

In a remarkable work [2], W. Al-Salam, W. Allaway and R. Askey describe a manner through which the q-Rogers polynomials ([5], [6], [32]–[36]) generate new orthogonal systems: to let q conveniently approach a k-root of unity, where $k \geq 2$ is an integer. According to this process, the recurrence relation of the q-system breaks up for each $n \geq 0$ into blocks of k equations each. Al-Salam, Allaway and Askey call this a *process of sieving*. The outcoming polynomials are the *sieved polynomials*; in their case, the *sieved ultraspherical polynomials*.

According to how $q \to \exp(2\pi i/k)$, a given system of q-polynomials may give rise to several kinds of sieved polynomials. Those known as of the first and second kinds have received much attention ([1], [2], [12]–[15], [24], [25]). Other classes have been barely touched ([1], [14]). In [2], the basic properties of the sieved polynomials, including their orthogonality, are formally deduced from properties of the q-polynomials they originate from. This demands good knowledge of q-polynomials and frequently the extrapolation of established results. To give the whole procedure a sound basis is a delicate matter.

The sieved ultraspherical polynomials are systems of sieved random walk polynomials (see Section 2.10 below). In [11], a theory of systems of sieved polynomials directly built from the recurrence relations was attempted. In order to keep close to the model in [2], the authors only deal with symmetric random walk polynomials. However, J. A. Charris and M. E. H. Ismail [12] extended [2] to the case of the sieved Pollaczek polynomials. The Pollaczek polynomials are not random walk ones. In order to parallel the approach in [2], the authors of [12] resorted to the so-called *q-Pollaczek polynomials*. But, unlike the Rogers polynomials, known properties of the q-Pollaczek polynomials were scarce at the time, and hard to establish ([12], [10]). Thus, it was natural they intended a direct treatment more deeply rooted in the blocks of recurrence relations. This approach evolved to the form exhibited in [13] and [14], which, far from being limited to sieved polynomials, covers systems such as those outcoming from the theory of polynomial mappings in [23] and even more general.

In a panoramic paper [4] on old and new orthogonal systems, R. Askey introduced some new systems of sieved polynomials. He follows the approach in [2] but now starting from the q-Wilson polynomials. He calls these *sieved Jacobi polynomials*. The main purpose of the present paper is *to deal with Askey's sieved Jacobi polynomials and some related systems from the point of view of the theory of block recursions*, *avoiding any reference to properties of the Askey–Wilson polynomials*. We hope this will make clear some of the advantages of the k-block approach. However, we briefly review the manner through which Askey's sieved Jacobi polynomials originate from the Askey–Wilson polynomials.

The *symmetric q-Wilson* (or *Askey–Wilson*) *polynomials* $W_m(x) = W_m(x; a, b|q)$, $m \geq 0$, are given by

$$xW_m(x) = W_{m+1}(x) + C_m W_{m-1}(x), \qquad m \geq 0, \tag{1.1}$$

and $W_{-1}(x) = 0$, $W_0(x) = 1$, where

$$C_m = \frac{(1-q^m)(1+a^2q^{m-1})(1+b^2q^{m-1})(1-a^2b^2q^{m-2})}{4(1-a^2b^2q^{2m-3})(1-a^2b^2q^{2m-1})}, \qquad m \geq 0. \tag{1.2}$$

If a, b are either real or conjugate complex, q is real and $|q|, |a|, |b| < 1$, then the orthogonality measure of $\{W_n(x)\}$ is (see [3], [4], [8] and [22], pp. 143 and 172–179)

$$d\mu(x) = \frac{h(x,1)h(x,-1)h(x,q^{1/2})h(x,-q^{1/2})}{h(x,a)h(x,b)h(x,-a)h(x,-b)}(1-x^2)^{-1/2}\chi(x)dx, \tag{1.3}$$

where $\chi(x)$ is the characteristic function of $[-1, 1]$ and

$$h(x, \alpha) = \prod_{m=0}^{\infty} (1 - 2x(\alpha q^m) + (\alpha q^m)^2).$$

Verification of (1.3) relies on non-trivial properties of the so-called *q-beta integral* of Askey and Wilson, an important q-extension of the beta function (see (2.31) below and [22], Chap. VI).

The Askey–Wilson polynomials contain as special cases many classical systems of q-polynomials. For example, if we let $\gamma = a^2 = b^2/q$ in (1.2) we get

$$C_m = \frac{(1-q^m)(1-\gamma^2 q^{m-1})}{4(1-\gamma q^{m-1})(1-\gamma q^m)}, \qquad m \geq 0, \tag{1.4}$$

and $W_m(x)$ is denoted by $C_m(x; \gamma|q)$. These are the *q-Rogers* or *q-ultraspherical polynomials* mentioned above. The name *q-ultraspherical* given to these polynomials comes from the fact that letting $\gamma = q^\lambda$ and $q \to 1$ in (1.4) yields

$$\lim_{q \to 1} C_m(x; q^\lambda|q) = C_m^\lambda(x) \tag{1.5}$$

where $C_m^\lambda(x)$ is the monic mth ultraspherical polynomial ([31], Chap. 17; [40], Chap. IV, Section 4.7).

Now taking $q=s\exp(\pi i/k)$, $a=s^{(\alpha+1/2)k}$, $b=s^{(\beta+1/2)k+1/2}\exp(\pi i/(2k))$ in (1.2) and letting $s\to 1$ we obtain

$$\lim_{s\to 1} C_m = \begin{cases} \dfrac{n}{2(2n+\alpha+\beta+1)}, & m=2nk,\\ \dfrac{n+\alpha+\beta+1}{2(2n+\alpha+\beta+1)}, & m=2nk+1,\\ \dfrac{n+\beta+1}{2(2n+\alpha+\beta+2)}, & m=(2n+1)k,\\ \dfrac{n+\alpha+1}{2(2n+\alpha+\beta+2)}, & m=(2n+1)k+1,\\ 1/4, & m\neq nk, nk+1. \end{cases} \tag{1.6}$$

Also, if we set

$$q=s\exp(\pi i/k),\quad a=s^{(\alpha+1/2)k+1}\exp(\pi i/k),\quad b=s^{(\beta+1/2)k+1/2}\exp(\pi i/(2k))$$

and let again $s\to 1$ in (1.2), the result is

$$\lim_{s\to 1} C_m = \begin{cases} \dfrac{n+\alpha+\beta+1}{2(2n+\alpha+\beta+1)}, & m=2nk-1,\\ \dfrac{n}{2(2n+\alpha+\beta+1)}, & m=2nk,\\ \dfrac{n+\alpha+1}{2(2n+\alpha+\beta+2)}, & m=(2n+1)k-1,\\ \dfrac{n+\beta+1}{2(2n+\alpha+\beta+2)}, & m=(2n+1)k,\\ 1/4, & m\neq nk, nk-1,\ k>2. \end{cases} \tag{1.7}$$

Relation (1.7) gives rise to the blocks

$$xp_{2nk+j}(x) = p_{2nk+j+1}(x) + a_n^{(j)} p_{2nk+j-1}(x),\qquad n\geq 0, \tag{1.8}$$

where $0\leq j\leq 2k-1$, $p_{-1}(x)=0$, $p_0(x)=1$,

$$\begin{aligned} a_n^{(0)} &= \frac{n}{2(2n+\alpha+\beta+1)}, & a_n^{(1)} &= \frac{n+\alpha+\beta+1}{2(2n+\alpha+\beta+1)},\\ a_n^{(k)} &= \frac{n+\beta+1}{2(2n+\alpha+\beta+2)}, & a_n^{(k+1)} &= \frac{n+\alpha+1}{2(2n+\alpha+\beta+2)}, \end{aligned} \tag{1.9}$$

and, if $k>2$,

$$a_n^{(j)} = 1/4,\qquad 2\leq j\leq 2k-1,\ j\neq k, k+1. \tag{1.10}$$

This is the system Askey calls *sieved Jacobi polynomials of the first kind*.

Also, (1.7) leads to the system (1.8) with

$$\begin{aligned} a_n^{(0)} &= \frac{n}{2(2n+\alpha+\beta+1)}, \qquad a_n^{(k-1)} = \frac{n+\alpha+1}{2(2n+\alpha+\beta+2)}, \\ a_n^{(k)} &= \frac{n+\beta+1}{2(2n+\alpha+\beta+2)}, \qquad a_n^{(2k-1)} = \frac{n+\alpha+\beta+2}{2(2n+\alpha+\beta+3)}, \end{aligned} \tag{1.11}$$

and, when $k > 2$, with

$$a_n^{(j)} = 1/4, \qquad 0 < j < 2k-1,\ j \neq k-1, k. \tag{1.12}$$

This is the system of Askey's *sieved Jacobi polynomials of the second kind*.

The recurrence relations above do not completely fit the scheme of sieved polynomials of a given system in [13], [14] or in Section 2.10 below. In this sense they are not sieved Jacobi polynomials. However, they looked amenable to the general theory of blocks in these papers. Furthermore, though not sieved polynomials of the Jacobi system, they are sieved polynomials of a system first examined by T. S. Chihara in [20], which actually is, in the sense of [13], [14], or of Section 2.10 below, a system of sieved Jacobi polynomials of the first kind (see Section 2.11 below). Thus, this yields an interesting double-sieved system of polynomials.

Remark 1.1. The case $\beta = -1/2,\ \alpha + 1/2 = \lambda$ of (1.9) is that of the sieved ultraspherical polynomials of the first kind in [2]. Relation (1.8) becomes

$$xp_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)} p_{nk+j-1}(x), \qquad n \geq 0, \tag{1.13}$$

$j = 0, 1, 2, \ldots, k-1$, with

$$\begin{aligned} a_n^{(0)} &= \frac{n}{4(n+\lambda)}; \qquad a_n^{(1)} = \frac{n+2\lambda}{4(n+\lambda)}; \\ a_n^{(j)} &= 1/4, \quad j = 2, \ldots, k-1. \end{aligned} \tag{1.14}$$

Also, (1.13) with

$$\begin{aligned} a_n^{(0)} &= \frac{n}{4(n+\lambda)}; \qquad a_n^{(k-1)} = \frac{n+2\lambda+1}{4(n+\lambda+1)}; \\ a_n^{(j)} &= 1/4, \quad j = 1, \ldots, k-2, \end{aligned} \tag{1.15}$$

which is (1.11) with $\beta = -1/2,\ \alpha + 1/2 = \lambda$, is the system of sieved ultrasphericals of the second kind.

The paper is organized as follows. In Section 2 we give definitions and results which are basic for the rest of the work. This allows for handy reference in the appropriate form. A brief survey of the theory of block recursions and of sieved polynomials is included. In Section 2.2 we describe a procedure to determine the orthogonality measures in Sections 5 and 6 which seems suitable in many other circumstances. Askey's sieved Jacobi polynomials are dealt with in Sections 3 and 4: in Section 3, those of the first kind, identified as double-sieved polynomials; in Section 4, those of the second kind, from the point of view of the general theory of block recursions. Some closely related systems, typically given by block recurrence relations, are examined in Sections 5 and 6.

This paper is dedicated to the memory of Professor Waleed Al-Salam. Many ideas on which it is based originated in his deep and important work.

2. Preliminary notions and results

2.1. *Moment functionals and orthogonal polynomials.* A *moment functional* $\mathcal{L}$ is a complex linear map from the space $\mathbb{C}[x]$ of complex polynomials into the field $\mathbb{C}$ of complex numbers. The moment functional $\mathcal{L}$ is *regular* ([19], Chap. I) if there exists a system $\{P_n(x) \mid n \geq 0\}$ of complex polynomials which satisfies the recurrence relation

$$\begin{gathered} xP_n(x) = P_{n+1}(x) + B_nP_n(x) + C_nP_{n-1}(x), \qquad n \geq 0; \\ P_{-1}(x) = 0, \qquad P_0(x) = 1, \end{gathered} \tag{2.1}$$

where $B_n,\ C_n$ are complex numbers with

$$C_{n+1} \neq 0, \qquad n \geq 0, \tag{2.2}$$

and is such that

$$\mathcal{L}(P_n(x)P_m(x)) = \lambda_n\delta_{mn}, \quad m, n \geq 0, \quad \lambda_n \neq 0, \quad \lambda_0 = 1. \tag{2.3}$$

Since $\{P_n(x)\}$ is a basis of $\mathbb{C}[x]$, it follows that $\mathcal{L}$ is uniquely determined by

$$\mathcal{L}(P_0(x)) = 1; \quad \mathcal{L}(P_n(x)) = 0, \quad n \geq 1. \tag{2.4}$$

Moreover,

$$\mathcal{L}(P_n^2(x)) = \lambda_n = C_1 \dots C_n, \qquad n \geq 1. \tag{2.5}$$

The system $\{P_n(x)\}$ is also uniquely determined by $\mathcal{L}$ and is called its *monic orthogonal system*. *Monic* refers to the leading coefficient of $P_n(x)$ being 1; and *orthogonal* to relation (2.3). Also, $\mathcal{L}$ is called the *moment functional* of $\{P_n(x)\}$. If $\{P_n(x)\}$ is given through a recurrence relation (2.1), $\mathcal{L}$ is defined by (2.4) and linear extension, and (2.2) holds, then $\mathcal{L}$ is regular and its monic orthogonal system is $\{P_n(x)\}$.

The moment functional $\mathcal{L}$ is *bounded* if B_n, C_n in (2.1) satisfy

$$|B_n| \leq \frac{M}{3}, \quad |C_{n+1}| \leq \frac{M}{3}, \qquad n \geq 0,\ M \geq 3. \tag{2.6}$$

If $\mathcal{L}$ is regular and bounded (by M), the *continued fraction*

$$\frac{1|}{|z - B_0} - \frac{C_1|}{|z - B_1} - \frac{C_2|}{|z - B_2} - \dots \tag{2.7}$$

([19], Chap. III; see also [41] for a more detailed study of continued fractions) is uniformly convergent on $|z| > M'$, for all $M' > M$, to a limit $X(z)$, which is an analytic function on $|z| > M$. Then

$$\mathcal{L}(P(x)) = \frac{1}{2\pi i} \int_C P(z)X(z)\,dz, \tag{2.8}$$

where C is any positively oriented contour of $|z| > M$ with $z = 0$ in its interior. Proofs of this and related results can be found in [7], [15], [16] and [27].

If $\mathcal{L}$ *is positive*, which means that B_n, C_n in (2.1) are real numbers and

$$C_{n+1} > 0, \qquad n \geq 0, \tag{2.9}$$

then $\mathcal{L}$ has representations of the form ([19], Chap. II, Theorem 3.1)

$$\mathcal{L}(P(x)) = \int_{-\infty}^{\infty} P(x)\,d\mu(x), \tag{2.10}$$

where μ is a positive measure supported by the real line. This is known as *Favard's theorem*. Clearly, (2.8) and (2.10) hold if and only if they hold for any polynomial $P(x)$ with real coefficients. In general, μ is not unique. When μ is unique, we say that the *moment problem for* $\mathcal{L}$ *is determined* and μ is called the *orthogonality measure* of $\{P_n(x)\}$. This is so, for example, when (2.6) holds, in which case $\operatorname{supp}\mu \subseteq [-M, M]$. When $\mathcal{L}$ is positive and bounded by M, (2.7) converges to $X(z)$ uniformly on compact subsets of $\mathbb{C} - [-M, M]$ and $X(\overline{z}) = \overline{X(z)}$ for $z \notin [-M, M]$. In particular, $X(x)$ is real for $x \in \mathbb{R} - [-M, M]$. Furthermore,

$$\mathcal{L}(P(x)) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{Im} X(t - i\varepsilon) P(t)\,dt. \tag{2.11}$$

This is Stieltjes' inversion formula ([19], Chap. II). It frequently allows one to determine μ. In some instances the limit and the integral in (2.11) can be interchanged, i.e.,

$$\mathcal{L}(P(x)) = \int_{-\infty}^{\infty} P(t)w(t)\,dt \tag{2.12}$$

with

$$w(t) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \operatorname{Im} X(t - i\varepsilon) \tag{2.13}$$

wherever the limit exists. Observe that $w(t) = 0$ for $t \in \mathbb{R} - [-M, M]$. This is obviously the case if the limit in (2.13) holds almost everywhere and there is an integrable function g on $\mathbb{R}$ such that $|\operatorname{Im} X(t - i\varepsilon)| \leq g(t)$ for almost all $t \in [-M, M]$ and all sufficiently small ε (Lebesgue's dominated convergence theorem, [28], p. 141). It is also the case if

$$F(z) = \begin{cases} \frac{1}{\pi}\operatorname{Im} X(z), & \operatorname{Im}(z) < 0, \\ w(z), & \operatorname{Im}(z) = 0, \end{cases} \tag{2.14}$$

is continuous, as a uniform continuity argument on $[-M-1, M+1] \times [-1, 0]$ readily shows.

REMARK 2.1. More frequently $\lim_{\zeta\to x} X(\zeta)$ exists for all $x \in \mathbb{R}$ except at most finitely many points $-M = \zeta_0 < \zeta_1 < \ldots < \zeta_k = M$; also, the function $\widehat{X}(z)$ given by

$$\widehat{X}(z) = \begin{cases} X(z), & \operatorname{Im}(z) < 0, \\ \lim_{\zeta\to z} X(\zeta), & \operatorname{Im}(\zeta) < 0,\ z \in \mathbb{R}, \end{cases} \tag{2.15}$$

is continuous except perhaps at the points $z = \zeta_0, \ldots, \zeta_k$, at which we nevertheless see that $\lim_{z\to\zeta_j}(z-\zeta_j)\widehat{X}(z)$ exists and is finite. We observe that $F(z) = \frac{1}{\pi}\operatorname{Im}\widehat{X}(z)$ wherever $\widehat{X}(z)$ is defined. In particular, $w(x) = \frac{1}{\pi}\operatorname{Im}\widehat{X}(x)$ for $x \in \mathbb{R}$, $x \neq \zeta_j$, $j = 0, 1, \ldots, k$.

Now let C be a positively oriented contour enclosing $[-M, M]$ and let $r, \varepsilon > 0$ be such that $0 < r < \frac{1}{2}|\zeta_{i+1} - \zeta_i|$, $i = 0, 1, \ldots, k-1$, $0 < \varepsilon < r$. Let $\Gamma_{r,\varepsilon}$ be the positively oriented boundary of the set of points which are either inside one of the circles of center ζ_i, $i = 1, \ldots, k$, and radius r or lie between the line segments parallel to $[-M, M]$ and respectively joining the points $(-M, \varepsilon)$ and (M, ε) or $(-M, -\varepsilon)$ and $(M, -\varepsilon)$. Assume r is small enough for $\Gamma_{r,\varepsilon}$ to be in the interior of C. Then, for $P(x)$ a real polynomial,

$$\int_C X(z)P(z)\,dz = \lim_{r\to0}\lim_{\varepsilon\to0}\int_{\Gamma_{r,\varepsilon}} X(z)P(z)\,dz,$$

and taking into account that $X(\overline{z}) = \overline{X(z)}$, $\operatorname{Im}(z) \neq 0$, it follows that

$$\begin{aligned} \int_{\Gamma_{r,\varepsilon}} X(z)P(z)\,dz = {} & \sum_{j=0}^{k} 2i \int_{-\pi}^{0} \operatorname{Re}(re^{i\theta}\widehat{X}(\zeta_j + re^{i\theta})P(\zeta_j + re^{i\theta}))\,d\theta \\ & + \sum_{j=0}^{k-1} 2i \int_{\zeta_j + r}^{\zeta_{j+1} - r} \operatorname{Im}(\widehat{X}(t - i\varepsilon)P(t - i\varepsilon))\,dt. \end{aligned}$$

Thus, since

$$\begin{aligned} \operatorname{Re}((z-\zeta_j)\widehat{X}(z)P(z)) = {} & \operatorname{Re}((z-\zeta_j)\widehat{X}(z))\operatorname{Re}(P(z)) \\ & - \operatorname{Im}((z-\zeta_j)\widehat{X}(z))\operatorname{Im}(P(z)), \\ \operatorname{Im}(\widehat{X}(z)P(z)) = {} & \operatorname{Im}(\widehat{X}(z))\operatorname{Re}(P(z)) \\ & + \operatorname{Re}(\widehat{X}(z))\operatorname{Im}(P(z)), \end{aligned}$$

$\mathrm{Re}(\widehat{X}(z))$ and $\mathrm{Im}(\widehat{X}(z))$ are both continuous on $[\zeta_j + r, \zeta_{j+1} - r] \times [-1, 0]$, $j = 0, 1, \dots, k-1$, and $\mathrm{Re}(P(t - i\varepsilon)) \to P(t)$, $\mathrm{Im}(P(t - i\varepsilon)) \to 0$ as $\varepsilon \to 0$, it follows that

$$\mathcal{L}(P(x)) = \sum_{j=0}^{k} A_j P(\zeta_j) + \mathrm{P.V.} \int_{-\infty}^{\infty} P(t) w(t)\, dt \tag{2.16}$$

where

$$A_j = \lim_{z \to \zeta_j} \mathrm{Re}((z - \zeta_j)\widehat{X}(z)) \tag{2.17}$$

and

$$\mathrm{P.V.} \int_{-\infty}^{\infty} P(t) w(t)\, dt := \lim_{r \to 0} \sum_{j=0}^{k-1} \int_{\zeta_j + r}^{\zeta_{j+1} - r} P(t) w(t)\, dt$$

is the Cauchy Principal Value.

A new application of Lebesgue's dominated convergence theorem then yields, under the circumstances of Remark 2.1, the following:

THEOREM 2.1. *If w, given by* (2.13), *is integrable on $\mathbb{R}$, then*

$$\mathcal{L}(P(x)) = \sum_{j=0}^{k} A_j P(\zeta_j) + \int_{-\infty}^{\infty} P(t) w(t)\, dt, \tag{2.18}$$

where A_j is given by (2.17). *Furthermore, if $A_j \neq 0$, then ζ_j is a mass point of $\mathcal{L}$ in $[-M, M]$. Thus, the orthogonality measure of $\{P_n(x)\}$ is*

$$d\mu(x) = \sum_{j=0}^{k} A_j \delta_j + w(x) dx \tag{2.19}$$

where δ_j is the Dirac measure at ζ_j.

The above procedure has been devised for use in Sections 5 and 6.

2.2. *Numerator polynomials.* If $\{P_n(x)\}$ satisfies (2.1), and (2.2) and (2.6) hold, then $X(z)$ in (2.8) is given by

$$X(z) = \lim_{n \to \infty} \frac{P_{n-1}^{(1)}(z)}{P_n(z)}, \qquad |z| > M, \tag{2.20}$$

where $\{P_n^{(i)}(x)\}$ is determined, for $i = 0, 1, \dots,$ by

$$x P_n^{(i)}(x) = P_{n+1}^{(i)}(x) + B_{n+i} P_n^{(i)}(x) + C_{n+i} P_{n-1}^{(i)}(x), \qquad n \geq 0, \tag{2.21}$$

with $P_{-1}^{(i)}(x) = 0$, $P_0^{(i)}(x) = 1$ ([19], Chap. III, p. 87). Convergence in (2.20) is uniform on $|z| \geq M'$ for $M' > M$. In view of (2.20), $\{P_n^{(1)}(x)\}$ is known as the *system of numerator polynomials* of $\{P_n(x)\}$. Darboux's asymptotic method ([21]; [30], Chap. 8) is frequently helpful in establishing

(2.20). See [7] for examples of how a combination of Darboux's method and Stieltjes' inversion formula allows one to determine $X(z)$ and μ from the recurrence relation (2.1). The system $\{P_n^{(i)}(x)\}$ is called the set of *i-associated polynomials* of $\{P_n(x)\}$. Clearly, $\{P_n^{(i+1)}(x)\}$ is the set of numerator polynomials of $\{P_n^{(i)}(x)\}$. If $\{\widetilde{P}_n(x)\}$ satisfies (2.1) for $n \geq 1$ and $\widetilde{P}_0(x) = 1$, $\widetilde{P}_1(x) = P_1(x) + Q(x)$, where $Q(x)$ is a polynomial, then (2.21) for $i = 0, 1$ ensures that $\widetilde{P}_n(x) = P_n(x) + Q(x)P_{n-1}^{(1)}(x)$ for all $n \geq 0$. We say that $\{P_n(x)\}$ and $\{\widetilde{P}_n(x)\}$ are *co-recursive*. See [18], [29], [37] for details about co-recursive polynomials and their applications.

2.3. *The Chebyshev polynomials.* The *Chebyshev polynomials of the first and second kinds* $\{T_n(x)\}$ and $\{U_n(x)\}$ are both defined ([31], Chap. 18, pp. 301–302) by the recurrence relation

$$2xy_n(x) = y_{n+1}(x) + y_{n-1}(x), \qquad n \geq 1.$$

The respective initial conditions are $T_0(x) = 1$, $T_1(x) = x$ and $U_0(x) = 1$, $U_1(x) = 2x$. We assume $T_{-1}(x) = U_{-1}(x) = 0$. For $x = \cos\theta$, $0 < \theta < \pi$,

$$T_n(x) = \cos n\theta, \quad U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \qquad n \geq 0. \tag{2.22}$$

The relations

$$\begin{gathered} 2T_n(x) = U_n(x) - U_{n-2}(x), \\ U_{n-1}^2(x) - U_n(x)U_{n-2}(x) = 1, \qquad n \geq 1, \end{gathered} \tag{2.23}$$

and

$$\begin{aligned} 1 - T_n^2(x) &= (1 - x^2)U_{n-1}^2(x), & 1 - T_{2n}(x) &= 2(1 - x^2)U_{n-1}^2(x) \\ U_{2n-1}(x) &= 2U_{n-1}(x)T_n(x), & 1 + T_{2n}(x) &= 2T_n^2(x), \qquad n \geq 0, \end{aligned} \tag{2.24}$$

will be needed. They follow at once from (2.22) for $-1 < x < 1$ and hold for all x by analytic continuation. The corresponding monic Chebyshev polynomials are

$$\widetilde{T}_n(x) = 2^{-n+1}T_n(x), \quad \widetilde{U}_n(x) = 2^{-n}U_n(x), \qquad n \geq 1. \tag{2.25}$$

We also let $\widetilde{T}_{-1}(x) = \widetilde{U}_{-1}(x) = 0$, $\widetilde{T}_0(x) = \widetilde{U}_0(x) = 1$.

2.4. *The gamma function.* The *gamma function* ([31], Chap. 2) is

$$\Gamma(z) := \lim_{n\to\infty} \frac{(n-1)!n^z}{(z)_n}, \qquad z \neq 0, -1, -2, \ldots, \tag{2.26}$$

where $(z)_n$ is the *Pochhammer symbol*:

$$(z)_0 = 1; \quad (z)_1 = z; \quad (z)_n = z(z+1)\ldots(z+n-1), \quad n \geq 2.$$

From (2.26) it follows that

$$\Gamma(z) = \frac{\Gamma(z+n)}{(z)_n}, \qquad n \geq 0,\ z \neq 0, -1, -2, \dots \tag{2.27}$$

If $\mathrm{Re}(z) > 0$, then ([31], Chap. 2)

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \tag{2.28}$$

so that $\Gamma(z)$ is analytic for $\mathrm{Re}(z) > 0$. This and (2.27) ensure that Γ is analytic on $\mathbb{C} - \{0, -1, \dots\}$ with simple poles at $0, -1, -2, \dots$

2.5. *The hypergeometric function.* The analytic continuation to $\mathbb{C} - [1, \infty)$ of the hypergeometric series

$$\sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad |z| < 1,\ c \neq 0, -1, -2, \dots, \tag{2.29}$$

is denoted by $F(a, b; c; z)$ and called the *hypergeometric function*. The analytic continuation can be carried out by means of contour integral representations of the Barnes type ([31], Chap. 5). If $a \in \mathbb{C}$ and $t^a = e^{a \log(t)}$, $t \neq 0$, and $(1-t)^a = e^{a \log(1-t)}$, $t \neq 1$, where $\log(z)$ is the branch of the logarithm in $\mathbb{C} - \{0\}$ with imaginary part in $(-\pi, \pi]$ (the so-called principal branch), then it can be shown ([31], Chap. 4) that, provided $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$,

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-a}\, dt \tag{2.30}$$

holds for $z \notin [1, \infty)$. Formula (2.30), due to Euler, is extremely useful and usually all that is needed in applications. It provides an integral representation of $F(a, b; c; z)$ which is simpler than Barnes'.

The *beta function* or *beta integral* ([31], Chap. 2) is

$$B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1}\, dt, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0. \tag{2.31}$$

Since $F(0, x; x+y; 0) = 1$, (2.30) yields

$$B(x, y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \qquad \mathrm{Re}(x) > 0,\ \mathrm{Re}(y) > 0. \tag{2.32}$$

Also,

$$(1-z)^{-a} = F(a, 1; 1; z), \qquad z \notin [1, \infty), \tag{2.33}$$

as can be easily checked for $|z| < 1$ (from the Taylor development of the left hand side around 0) and follows therefrom by analytic continuation.

Relation (2.33) is known as *Newton's binomial formula*. We will also denote $F(a, b; c; z)$ by

$$F\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right).$$

2.6. *Two contiguous function relations.* Let $F = F(a, b; c; z)$. The functions $F(a+) = F(a+1, b; c; z)$, $F(a-) = F(a-1, b; c; z)$ and similarly $F(b+)$, $F(b-)$, $F(c+)$ and $F(c-)$ are called the *contiguous functions* of F. The contiguous function relations

$$(1-z)F = F(a-) - \frac{c-b}{c} zF(c+) \tag{2.34}$$

and

$$(c-a-b)F = (c-a)F(a-) - b(1-z)F(b+), \tag{2.35}$$

two among fifteen ([31], Chap. 4), will be needed in what follows.

2.7. *The Jacobi polynomials.* The *monic Jacobi polynomials* $\{P_n^{(\alpha,\beta)}(x)\}$ are given ([19], Chap. V; [31], Chap. 16; [40], Chap. IV) by

$$\begin{aligned} &\left(x - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right) P_n^{(\alpha,\beta)}(x) \\ &= P_{n+1}^{(\alpha,\beta)}(x) + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta)^2(2n+\alpha+\beta-1)(2n+\alpha+\beta+1)} P_{n-1}^{(\alpha,\beta)}(x) \end{aligned} \tag{2.36}$$

for $n \geq 0$ and $P_{-1}^{(\alpha,\beta)}(x) = 0$, $P_0^{(\alpha,\beta)}(x) = 1$. When $\alpha = -\beta$, the coefficient of $P_0^{(\alpha,\beta)}(x)$ in (2.36) reduces to $x - \beta$. The regularity condition (2.2) becomes α, β *and* $\alpha+\beta$ *are not integers* < 0, and the positivity condition (2.9) is α, β *are real numbers and* $\alpha > -1$, $\beta > -1$. In the latter case, the moment functional $\mathcal{L}_{\alpha,\beta}$ is represented ([19], Chap. V, p. 148; [31], Chap. 16, p. 258; [40], Chap. IV, p. 68) by the positive measure

$$d\mu_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta+2)}{2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x)^\alpha(1+x)^\beta \chi(x)\, dx, \tag{2.37}$$

$\chi(x)$ being the characteristic function of $(-1, 1)$, which relates to the beta integral (2.31) in a manner analogous to that through which (1.3) relates to the Askey–Wilson q-beta integral. The support of $\mu_{\alpha,\beta}$ is $[-1, 1]$. Under the regularity assumption, the continued fraction of $\{P_n^{(\alpha,\beta)}(x)\}$ converges uniformly to ([27]; [40], p. 75)

$$X_{\alpha,\beta}(z) = \frac{1}{z-1} F\left(\begin{matrix} 1, \alpha+1 \\ \alpha+\beta+2 \end{matrix} \,\middle|\, \frac{2}{1-z}\right) \tag{2.38}$$

in $|z-1| > 2 + \varepsilon$ for all $\varepsilon > 0$. Actually, (2.38) holds for $z \notin [-1, 1]$ and for $\alpha > -1$, $\beta > -1$, convergence is uniform on compact subsets of $\mathbb{C} - [-1, 1]$.

From (2.34) and (2.35) it follows that

$$X_{\alpha+1,\beta+1}(z) = \frac{(\alpha+\beta+2)(\alpha+\beta+3)}{4(\alpha+1)(\beta+1)}(1-z)[(1+z)X_{\alpha,\beta}(z)-1] + \frac{(\alpha+\beta+3)}{2(\alpha+1)}. \tag{2.39}$$

2.8. *Blocks of recurrence relations.* If $k \geq 2$ is an integer, then recurrence relations given for each $n \geq 0$ in the form of blocks of k equations each,

$$(x - b_n^{(j)})p_{nk+j}(x) = p_{nk+j+1}(x) + a_n^{(j)} p_{nk+j-1}(x), \tag{2.40}$$

where $0 \leq j \leq k-1$, arise in many instances. System (2.40) can be written as

$$A_n \begin{bmatrix} p_{nk+1}(x) \\ p_{nk+2}(x) \\ p_{nk+3}(x) \\ \vdots \\ p_{nk+k-1}(x) \\ p_{nk-1}(x) \end{bmatrix} = \begin{bmatrix} (x-b_n^{(0)})p_{nk}(x) \\ a_n^{(1)} p_{nk}(x) \\ 0 \\ \vdots \\ 0 \\ p_{nk+k}(x) \end{bmatrix} \tag{2.41}$$

where $A_n = [a_{n,i,j}]$ is the $k \times k$ matrix given by $a_{n,1,j} = \delta_{1j} + a_n^{(0)} \delta_{1j-k+1}$ and $a_{n,k,j} = -a_n^{(k-1)} \delta_{ij+2} + (x - b_n^{(k-1)})\delta_{kj+1}$, $1 \leq j \leq k$, and by $a_{n,i,j} = -a_n^{(i-1)} \delta_{ij+2} + (x - b_n^{(i-1)})\delta_{ij+1} - \delta_{i,j}$, $1 \leq j \leq k$, for $i = 2, \ldots, k-1$, which allows solving for $p_{nk+j}(x)$, $j = -1, 1, 2, \ldots, k-1$, in terms of $p_{nk}(x)$ and $p_{nk+k}(x)$ (by Cramer's rule, for example). Since $p_{nk-1}(x) = p_{(n-1)k+k-1}(x)$, two representations of $p_{nk-1}(x)$ arise. Eliminating $p_{nk-1}(x)$ from them we obtain the following theorem, with

$$\Delta_n(i,j) = \begin{cases} 0, & j < i-2, \\ 1, & j = i-2, \end{cases} \tag{2.42}$$

and for $j \geq i-1$,

$$\Delta_n(i,j) = \begin{vmatrix} x - b_n^{(i-1)} & -1 & 0 & 0 & \ldots & 0 & 0 \\ -a_n^{(i)} & x - b_n^{(i)} & -1 & 0 & \ldots & 0 & 0 \\ 0 & -a_n^{(i+1)} & x - b_n^{(i+1)} & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & -a_n^{(j)} & x - b_n^{(j)} \end{vmatrix}. \tag{2.43}$$

THEOREM 2.2. *The polynomials $P_n(x) = p_{nk}(x)$, $n \geq 0$, $P_{-1}(x) = 0$, satisfy the recurrence relation*

$$
\begin{aligned}
(2.44)\qquad \{(x-b_n^{(0)})\Delta_n(2,k-1)\Delta_{n-1}(2,k-1)&\\
-a_n^{(1)}\Delta_n(3,k-1)\Delta_{n-1}(2,k-1)&\\
-a_n^{(0)}\Delta_n(2,k-1)\Delta_{n-1}(2,k-2)\}P_n(x)&\\
=\Delta_{n-1}(2,k-1)P_{n+1}(x)&\\
+a_n^{(0)}a_{n-1}^{(1)}\dots a_{n-1}^{(k-1)}\Delta_n(2,k-1)P_{n-1}(x)&
\end{aligned}
$$

for $n\ge 0$, *where by convention* $\Delta_{-1}(2,k-1)=1,\ \Delta_{-1}(2,k-2)=0.$

Corollary 2.1. *If* $\Delta_n(2,k-1)$ *is independent of* n, *i.e., if*

$$\Delta_n(2,k-1)=\Delta_0(2,k-1),\qquad n\ge 0, \tag{2.45}$$

then (2.44) *becomes*

$$
\begin{aligned}
(2.46)\qquad [(x-b_n^{(0)})\Delta_n(2,k-1)&\\
-a_n^{(1)}\Delta_n(3,k-1)-a_n^{(0)}\Delta_{n-1}(2,k-2)]P_n(x)&\\
=P_{n+1}(x)+a_n^{(0)}a_{n-1}^{(1)}\dots a_{n-1}^{(k-1)}P_{n-1}(x),\qquad n\ge 0.&
\end{aligned}
$$

Relation (2.46) is easier to handle than (2.45) and still covers the most important cases. In fact, sieved polynomials and polynomials arising from polynomial mappings (see [23]) can be dealt with through (2.46). The polynomials $P_n(x)=p_{nk}(x)$, $n\ge 0$, are called the *link polynomials* of the blocks (2.40) defining $\{p_n(x)\}$. That (2.45) holds does not imply that $\Delta_n(3,k-1)$ or $\Delta_n(2,k-2)$ are independent of n. We observe (see [13], [14]) that $p_j(x)=\Delta_0(1,j-1)$, $j=0,1,\dots,k$.

For an integer $l\ge 0$, the *l-associated polynomials* $\{P_n^{(l)}(x)\}$ of $\{P_n(x)\}$ are defined through

$$
\begin{aligned}
(2.47)\qquad [(x-b_{n+l}^{(0)})\Delta_{n+l}(2,k-1)&\\
-a_{n+l}^{(1)}\Delta_{n+l}(3,k-1)-a_{n+l}^{(0)}\Delta_{n-1+l}(2,k-2)]P_n^{(l)}(x)&\\
=P_{n+1}^{(l)}(x)+a_{n+l}^{(0)}a_{n-1+l}^{(1)}\dots a_{n-1+l}^{(k-1)}P_{n-1}^{(l)}(x),\qquad n\ge 0,&
\end{aligned}
$$

and $P_{-1}^{(l)}(x)=0,\ P_0^{(l)}(x)=1$. Then (see [13], [14]) we have

Theorem 2.3. *If* $\{p_n^{(1)}(x)\}$ *and* $\{p_n^{(2)}(x)\}$ *are respectively the systems of first and second order associated polynomials of* $\{p_n(x)\}$ *then*

$$
\begin{aligned}
&p_{(n+1)k-1}^{(1)}(x)=\Delta_0(2,k-1)P_n^{(1)}(x),\\
&p_{(n+1)k-2}^{(2)}(x)=\Delta_0(3,k-1)P_n^{(1)}(x)+a_1^{(0)}a_0^{(2)}\dots a_0^{(k-1)}P_{n-1}^{(2)}(x)
\end{aligned}
\tag{2.48}
$$

for $n \ge 0$ (*with* $a_0^{(2)} \dots a_0^{(k-1)} = 1$ *if* $k = 2$). *If*, *furthermore*,

$$|b_n^{(j)}| \le \frac{M}{3}, \quad |a_n^{(j)}| \le \frac{M}{3}, \qquad n \ge 0,\ j = 0, 1, \dots, k-1,\ M \ge 3, \tag{2.49}$$

then the limit of the continued fraction of $\{p_n(x)\}$ *is given for* $|z| > M$ *by*

$$\lim_{n\to\infty} \frac{p_{n-1}^{(1)}(z)}{p_n(z)} = \lim_{n\to\infty} \frac{p_{nk-1}^{(1)}(z)}{p_{nk}(z)} = \Delta_0(2, k-1)Y(z), \tag{2.50}$$

where

$$Y(z) = \lim_{n\to\infty} \frac{P_{n-1}^{(1)}(z)}{P_n(z)}, \qquad |z| > M, \tag{2.51}$$

and that of $\{p_n^{(1)}(x)\}$ *is given by*

$$\lim_{n\to\infty} \frac{p_{n-1}^{(2)}(z)}{p_n^{(1)}(z)} = \frac{1}{\Delta_0(2, k-1)} [\Delta_0(3, k-1) + a_1^{(0)} a_0^{(2)} \dots a_0^{(k-1)} Y^{(1)}(z)] \tag{2.52}$$

(*provided* M *is large enough for all the roots of* $\Delta_0(2, k-1)$ *to be in* $|z| < M$), *where*

$$Y^{(1)}(z) = \lim_{n\to\infty} \frac{P_{n-1}^{(2)}(z)}{P_n^{(1)}(z)}, \qquad |z| > M. \tag{2.53}$$

REMARK 2.2. In general, $\{P_n^{(1)}(x)\}$ is not the system of link polynomials of $\{p_n^{(1)}(x)\}$. If $\Delta_n(2, k-1)$ for $\{p_n(x)\}$ is independent of n, it may be false that this also holds for $\Delta_n(2, k-1)$ of $\{p_n^{(1)}(x)\}$.

2.9. *Sieved polynomials.* If in (2.40), we take $k = 2$ and $b_n^{(1)} = 0$, $n \ge 0$, or $k \ge 3$ and $a_n^{(j)} = 1/4$, $j = 2, \dots, k-1$, $b_n^{(j)} = 0$, $j = 1, \dots, k-1$, then $\{p_n(x)\}$ is called a *system of sieved polynomials of the first kind* ([13], [14]). In this case it follows from (2.43) that $\Delta_n(2, k-1) = \widetilde{U}_{k-1}(x)$, $n \ge 0$, where $\{\widetilde{U}_n(x)\}$ is the system of monic Chebyshev polynomials of the second kind as in (2.25). More generally,

$$\Delta_n(i, j) = \begin{cases} 0, & j < i-2, \\ \widetilde{U}_{j-i+2}(x), & j \ge i-2,\ i \ge 2. \end{cases} \tag{2.54}$$

Provided (2.49) holds, (2.50) yields

$$\lim_{n\to\infty} \frac{p_{n-1}^{(1)}(z)}{p_n(z)} = \widetilde{U}_{k-1}(z)Y(z), \qquad |z| > M, \tag{2.55}$$

with $Y(z)$ as in (2.51).

If $\{p_n(x)\}$ is a system of sieved polynomials of the first kind, $\{p_n^{(1)}(x)\}$ is called a *system of sieved polynomials of the second kind*. If $\{q_n(x)\}$ is a system of sieved polynomials of the second kind and

$$(x - d_n^{(j)})q_{nk+j}(x) = q_{nk+j+1}(x) + c_n^{(j)} q_{nk+j-1}(x), \qquad j = 0, 1, \dots, k-1,\ n \geq 0, \tag{2.56}$$

then $k = 2$ and $d_n^{(0)} = 0$, or $k > 2$, $d_n^{(j)} = 0$, $j = 0, 1, \dots, k-2$, and $c_n^{(j)} = 1/4$, $j = 1, \dots, k-2$. Let $q_n(x) = p_n^{(1)}(x)$, $n \geq 0$, be a system of sieved polynomials of the second kind. If $\{P_n(x)\}$ is the system of link polynomials of $\{p_n(x)\}$ then (2.48) becomes

$$\begin{aligned} q_{(n+1)k-1}(x) &= \widetilde{U}_{k-1}(x) P_n^{(1)}(x), \\ q^{(1)}_{(n+1)k-2}(x) &= \widetilde{U}_{k-2}(x) P_n^{(1)}(x) + \frac{1}{4^{(k-2)}} c_0^{(k-1)} P_{n-1}^{(2)}(x), \end{aligned} \tag{2.57}$$

for $n \geq 0$. When (2.49) holds, then (2.52) is

$$\lim_{n\to\infty} \frac{q_{n-1}^{(1)}(z)}{q_n(z)} = \frac{1}{\widetilde{U}_{k-1}(x)} \left[\widetilde{U}_{k-2}(x) + \frac{1}{4^{(k-2)}} c_0^{(k-1)} Y^{(1)}(z) \right], \qquad |z| > M, \tag{2.58}$$

with $Y^{(1)}(z)$ as in (2.53).

REMARK 2.3. If $\{p_n(x)\}$ is a sieved system of the first kind, then $\{p_n^{(i)}(x)\}$, $i \geq 0$, is called a *sieved system of the* $(i+1)$*th kind*. Not much attention has been paid to $\{p_n^{(i)}(x)\}$ for $i > 1$ (see [1], [14]).

2.10. *Sieved polynomials of an orthogonal system.* Let $\{P_n(x)\}$ be given by (2.1) and assume that (2.2) holds. If $\{p_n(x)\}$ as in (2.40) is a system of sieved polynomials of the first kind and

$$4a_n^{(0)} a_{n-1}^{(1)} = C_n, \qquad n \geq 1, \tag{2.59}$$

then $\{p_n(x)\}$ is called a *system of sieved polynomials of the first kind of* $\{P_n(x)\}$, or a *system of sieved* $\{P_n(x)\}$ *polynomials of the first kind.*

Also, if $\{q_n(x)\}$ is a system of sieved polynomials of the second kind as in (2.56), and the coefficients $c_n^{(j)}$ are related to those of $\{P_n(x)\}$ by

$$4c_n^{(0)} c_n^{(k-1)} = C_n, \qquad n \geq 1, \tag{2.60}$$

then $\{q_n(x)\}$ is called a *system of sieved polynomials of the second kind of* $\{P_n(x)\}$, or a *system of sieved* $\{P_n(x)\}$ *polynomials of the second kind.*

The system $\{P_n(x)\}$ itself is usually considered to be a special case of $\{p_n(x)\}$ (or of $\{q_n(x)\}$) corresponding to $k = 1$.

REMARK 2.4. Assume (C_n) in (2.1) satisfies $C_{n+1} = g_{n+1}(1-g_n)$, $n \geq 0$, where $0 \leq g_0 < 1$ and $0 < g_n < 1$ for $n \geq 1$. Then (C_n) is called a *chain sequence* and (g_n) is said to be a *sequence of parameters* for (C_n) ([19], Chap. VI). If this is the case then $\{P_n(x)\}$ given by (2.1) is called a system of *random walk polynomials*. If $\{p_n(x)\}$ is a system of sieved polynomials of the first kind such that $a_n^{(0)} = \frac{1}{2}g_n$ and $a_n^{(1)} = \frac{1}{2}(1-g_n)$, $n \geq 0$, then $\{p_n(x)\}$ is a system of sieved $\{P_n(x)\}$ polynomials of the first kind. Because of this, systems of sieved polynomials of the first kind (resp. of the second kind) such that $a_n^{(0)} + a_n^{(1)} = 1/2$, $n \geq 0$, (resp. $a_n^{(0)} + a_{n-1}^{(k-1)} = 1/2$, $n \geq 1$) are sometimes called *random walk systems of sieved polynomials of the first kind* (resp. *of the second kind*). For a random walk system of sieved polynomials of the first kind, the system $\{P_n(x)\}$ given by (2.1) with $C_n = 4a_n^{(0)}a_{n-1}^{(1)}$ and, say, $B_n = 0$ for $n \geq 0$, is a system of random walk polynomials; in fact, $(2a_n^{(0)})$ is a sequence of parameters for (C_n). For random walk sieved systems of the second kind, $(2a_n^{(k-1)})$ is a sequence of parameters for (C_n), where $C_n = 4a_n^{(0)}a_n^{(k-1)}, n \geq 1$. Thus, random walk systems of sieved polynomials actually are systems of sieved random walk polynomials.

REMARK 2.5. We warn the reader about the fact that a system of random walk polynomials presented by means of blocks of recurrence relations may fail to be a system of sieved random walk polynomials. This is the case of the little Jacobi polynomials in the next section.

REMARK 2.6. It can be shown (see [23]) that if $\{P_n(x)\}$ is a system of symmetric, monic orthogonal polynomials (i.e., $B_n = 0$ in (2.1)) whose orthogonality measure is supported by $[-1, 1]$ (which is the case of symmetric random walk polynomials) and $k \geq 2$ is an integer, then the system $\{p_n(x)\}$ given by $p_0(x) = 1$ and

$$p_{nk+j}(x) = \frac{2^{-n(k-1)-j+1}}{U_{k-1}(x)}[U_{j-1}(x)P_{n+1}(T_k(x)) + 2a_n^{(1)}U_{k-j-1}(x)P_n(T_k(x))]$$

for $n \geq 0$, $j = 1, \dots, k$, is a system of symmetric sieved polynomials of the first kind with link polynomials $2^{-n(k-1)}P_n(T_k(x))$, $n \geq 0$. Systems of the second kind can be analogously defined (see [23]). We say that $\{p_n(x)\}$ is *obtained from* $\{P_n(x)\}$ *by means of the polynomial mapping* $x \to T_k(x)$.

For non-symmetric systems, matters are rather more delicate. As we show in Sections 3 and 4, Askey's sieved Jacobi polynomials can be obtained through polynomial mappings of the form $x \to T_k(x)$. The theory of block recursions frequently allows one to identify a system as given through a polynomial mapping (see [12], [13]). We observe that not every system of sieved polynomials (even if symmetric) can be obtained by means of a polynomial mapping: this is the case of the polynomials in Section 5 (see [12]).

The above notion of sieved polynomials of a given system originates in [2], [3], [11], [12], [24], [25]. Proofs of all the results in Sections 2.8, 2.9 and 2.10 can be found in [13], [14].

2.11. *The little Jacobi polynomials.* The system $\{p_n^{(\alpha,\beta)}(x)\}$ given by (2.40) with $k=2$, $b_n^{(0)}=b_n^{(1)}=0$ and

$$\begin{aligned} a_n^{(0)} &= \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)},\\ a_n^{(1)} &= \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \end{aligned} \tag{2.61}$$

$n\geq 0$, is a system of sieved polynomials of the first kind of the Jacobi polynomials. $\{p_n^{(\alpha,\beta)}(x)\}$ was studied in detail in Chihara [20]. The case $\alpha=\beta=0$ had been previously examined in Szegő [39] and in Stroud and Secrest [38]. Chihara [20] devises an ingenious procedure which makes use of kernel polynomials to obtain the orthogonality measure of this and other related systems. He gives for $\{p_n^{(\alpha,\beta)}(x)\}$ the orthogonality measure

$$dv^{\alpha,\beta}(x)=\frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}|x|^{2\beta+1}(1-x^2)^{\alpha}\chi(x)dx \tag{2.62}$$

where $\chi(x)$ is the characteristic function of $(-1,1)$. Moreover, its system of link polynomials is $P_n(x)=2^{-n}P_n^{(\alpha,\beta)}(T_2(x))$, $n\geq 0$, and $dv^{\alpha,\beta}(x)$ can also be obtained via the theory of polynomial mappings in [23] or directly from (2.8) and (2.55) by the procedure we will explain in Section 3. We call $\{p_n^{(\alpha,\beta)}(x)\}$ the *little Jacobi polynomials of the first kind*, or, simply, the *little Jacobi polynomials.*

REMARK 2.7. Letting $\alpha=\beta=0$, (2.62) gives for the system

$$\begin{aligned} xp_{2n}(x) &= p_{2n+1}(x)+\frac{n}{4n+2}p_{2n-1}(x),\\ xp_{2n+1}(x) &= p_{2n+2}(x)+\frac{n+1}{4n+2}p_{2n}(x), \end{aligned} \tag{2.63}$$

$n\geq 0$, $p_0(x)=1$, $p_{-1}(x)=0$, the orthogonality measure $dv(x)=|x|\chi(x)dx$.

REMARK 2.8. It is easily verified that $\{p_n^{(\alpha+1,\beta)}(x)\}$ is a system of sieved polynomials of the second kind of $\{P_n^{(\alpha+1,\beta+1)}(x)\}$. The system $\{p_n^{(\alpha,\beta)}(x)\}$ is not, in general, a system of sieved random walk polynomials. However, if $\alpha=\beta=\lambda-1/2$, then $\{p_n^{(\alpha,\beta)}(x)\}$ and $\{p_n^{(\alpha+1,\beta)}(x)\}$ are respectively the systems of sieved ultraspherical polynomials of the first and second kinds (with $k=2$), which are simultaneously random walk polynomials and systems of sieved random walk polynomials. As noted above, $\{p_n^{(\alpha,\beta)}(x)\}$ can be obtained from $\{P_n^{(\alpha,\beta)}(x)\}$ through the polynomial mapping $x\to T_2(x)$.

3. Sieved Jacobi polynomials of the first kind. This is the system $\{p_n(x)\}$ given by the blocks of $2k$ recurrence relations (1.8) with the $a_n^{(j)}$ given by (1.9) and (1.10). It is assumed throughout that the positivity condition $\alpha, \beta > -1$ holds. Viewed as defined by blocks of $2k$ recurrence relations, $\{p_n(x)\}$ is not a system of sieved polynomials in the sense of Section 2.9. As blocks of k equations each, they are sieved polynomials of the first kind as in Section 2.9, but, according to Section 2.10, not of the Jacobi polynomials. They are, however, sieved polynomials of the first kind of the little Jacobi polynomials, as we will see next. Thus, they provide a significant example of a double-sieved system. As defined by blocks of $2k$ equations each, they could be dealt with according to the general theory of blocks (with no need of looking at them as sieved polynomials), thus also providing a meaningful example for this theory.

In this section we deal with $\{p_n(x)\}$ as given by blocks of k recurrence relations, i.e., as sieved polynomials of the first kind, reserving the $2k$-block approach for the polynomials of the second kind and for those in Sections 5 and 6. Thus, we separate the nth block (1.8) of recurrence relations into two blocks respectively embracing the first k (which then corresponds not to n but to $2n$) and the last k equations (which corresponds to $2n+1$). For the k blocks we have

$$\begin{aligned} \Delta_{-1}(2, k-2) = 0; \quad \Delta_n(2, k-2) = \widetilde{U}_{k-2}(x), \quad n \geq 0; \\ \Delta_n(2, k-1) = \widetilde{U}_{k-1}(x); \quad \Delta_n(3, k-1) = \widetilde{U}_{k-2}(x), \quad n \geq 0. \end{aligned} \tag{2.60}$$

Observing that $a_n^{(0)} + a_n^{(1)} = 1/2$ for $n \geq 1$ and $a_0^{(1)} = 1/2$, the system $\{P_n(x)\}$ of link polynomials will satisfy, according to (2.46),

$$\begin{aligned} &\widetilde{T}_k(x) P_{2n}(x) \\ &\quad = P_{2n+1}(x) + \frac{n(n+\alpha)}{4^{k-1}(2n+\alpha+\beta)(2n+\alpha+\beta+1)} P_{2n-1}(x), \qquad n \geq 0, \end{aligned} \tag{3.2}$$

with $P_{-1}(x) = 0$, $P_0(x) = 1$, where $\widetilde{T}_k(x) = 2^{-k+1} T_k(x)$ is as in (2.25). Also, as $a_n^{(k)} + a_n^{(k+1)} = 1/2$ for $n \geq 0$, we have

$$\begin{aligned} &\widetilde{T}_k(x) P_{2n+1}(x) \\ &\quad = P_{2n+2}(x) + \frac{(n+\beta+1)(n+\alpha+\beta+1)}{4^{k-1}(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)} P_{2n}(x), \end{aligned} \tag{3.3}$$

with $P_0(x) = 1$, $P_1(x) = p_k(x) = \Delta_0(1, k-1) = \widetilde{T}_k(x)$.

From (3.2) and (3.3) it then follows that:

THEOREM 3.1. *The system $\{p_n(x)\}$ of sieved Jacobi polynomials of the first kind satisfies*

$$p_{nk}(x) = 2^{n(1-k)} p_n^{(\alpha,\beta)}(T_k(x)), \qquad n \geq 0, \tag{3.4}$$

where $\{p_n^{(\alpha,\beta)}(x)\}$ *is the system of little Jacobi polynomials. Furthermore*,

$$4a_n^{(0)}a_{n-1}^{(k+1)} = \frac{n(n+\alpha)}{(2n+\alpha+\beta)(2n+\alpha+\beta+1)} \tag{3.5}$$

and

$$4a_n^{(k)}a_n^{(1)} = \frac{(n+\beta+1)(n+\alpha+\beta+1)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)}, \qquad n \geq 0. \tag{3.6}$$

Thus, $\{p_n(x)\}$ *actually is*, *in the sense of Section* 2.10, *a system of sieved polynomials of the first kind of* $\{p_n^{(\alpha,\beta)}(x)\}$.

Because of (3.4) we could resort at this point to the theory of polynomial mappings [23]. We follow, however, a more direct approach based on formula (2.8). In fact, from (2.55) and (3.4) it follows that the limit of the continued fraction of $\{p_n(x)\}$ is

$$X(z) = U_{k-1}(z)X^{\alpha,\beta}(T_k(z)) \tag{3.7}$$

where $X^{\alpha,\beta}(z)$ denotes the limit of the continued fraction of $\{p_n^{(\alpha,\beta)}(z)\}$. Now let $\cos_i$, $i = 0, 1, \ldots, k-1$, be the restriction of cos to $(i\pi, (i+1)\pi) \times \mathbb{R}$, where $\mathbb{R}$ denotes the real numbers. If $\Omega = \mathbb{C} - \{x \in \mathbb{R} \mid |x| \geq 1\}$, then $\cos_i$ applies $(i\pi, (i+1)\pi) \times \mathbb{R}$ conformally onto Ω. If $\cos_i^{-1} : \Omega \to (i\pi, (i+1)\pi) \times \mathbb{R}$ is the inverse map of $\cos_i$, then

$$T_k(z) = \cos(k \cos_0^{-1}(z)), \tag{3.8}$$

and if

$$L_i(z) = \cos\left(\frac{1}{k}\cos_i^{-1}(z)\right), \qquad z \in \Omega,\ i = 0, 1, \ldots, k-1, \tag{3.9}$$

then $T_k(L_i(z)) = z$, $z \in \Omega$, so that each L_i is one of the k branches of $T_k^{-1}(z)$. Let $\gamma(\theta) = 1 + Re^{i\theta}$, $0 \leq \theta \leq 2\pi$, $R > 2$. Then $[-1, 1]$ is contained in the interior of γ, and if $\gamma_i = L_i \circ \gamma$ for $i = 0, 1, \ldots, k-1$, then the γ_i piece together on a positively oriented contour $\widetilde{\gamma}$ enclosing $[-1, 1] = T_k^{-1}([-1, 1])$. The contour $\widetilde{\gamma}$ is called the *lifting of* γ *through* T_k. Since $T_k'(L_i(z))L_i'(z) = 1$ and $T_k'(z) = kU_{k-1}(z)$, so that $L_i'(z) = 1/(kU_{k-1}(L_i(z)))$, a change of variables gives, for f continuous on $\widetilde{\gamma}$,

$$\begin{aligned} \int_{\gamma_i} f(z)\,dz &= \frac{1}{k}\int_{\gamma} \frac{f(L_i(z))}{U_{k-1}(L_i(z))}\,dz, \\ \int_{\widetilde{\gamma}} f(z)\,dz &= \sum_{i=0}^{k-1}\int_{\gamma_i} \frac{f(L_i(z))}{U_{k-1}(L_i(z))}\,dz, \end{aligned} \tag{3.10}$$

and for g continuous on γ,

$$\int_{\widetilde{\gamma}} g(T_k(z))U_{k-1}(z)\,dz = \int_{\gamma} g(z)\,dz. \tag{3.11}$$

On the other hand, (2.8) and (3.7) imply, with γ, $\widetilde{\gamma}$ as above, that the moment functional $\mathcal{L}$ of $\{p_n(x)\}$ has the representation

$$\mathcal{L}(P(x)) = \frac{1}{2\pi i}\int_{\widetilde{\gamma}} P(z)U_{k-1}(z)X^{\alpha,\beta}(T_k(z))\,dz, \tag{3.12}$$

so that if $\xi_0 = -1 < \xi_1 < \ldots < \xi_{k-1} < \xi_k = 1$ are the roots of $T_k^2(x) - 1$ (ξ_0 and ξ_k are simple and $\xi_1, \ldots, \xi_{k-1}$ are double; observe that $\xi_1, \ldots, \xi_{k-1}$ are also the roots of $U_{k-1}(x)$), then from (3.10), (3.11) and (3.12), writing $\Gamma(\alpha,\beta) = 1/B(\alpha+1,\beta+1)$ and using (2.62), we have

$$\begin{aligned}
\mathcal{L}(P(x)) &= \frac{1}{2\pi i}\sum_{i=0}^{k-1}\int_{\gamma_i} P(z)U_{k-1}(z)X^{\alpha,\beta}(T_k(z))\,dz \\
&= \frac{1}{2\pi i}\sum_{i=0}^{k-1}\int_{\gamma} P(L_i(z))U_{k-1}(z)X^{\alpha,\beta}(z)\,dz \\
&= \frac{\Gamma(\alpha,\beta)}{k}\sum_{i=0}^{k-1}\int_{-1}^{1} P(L_i(x))|x|^{2\beta+1}(1-x^2)^{\alpha}\,dx \\
&= \Gamma(\alpha,\beta)\sum_{i=0}^{k-1}\int_{\xi_i}^{\xi_{i+1}} P(x)|T_k(x)|^{2\beta+1}(1-T_k^2(x))^{\alpha}|U_{k-1}(x)|\,dx \\
&= \Gamma(\alpha,\beta)\int_{-1}^{1} P(x)(1-x^2)^{\alpha}|T_k(x)|^{2\beta+1}|U_{k-1}(x)|^{2\alpha+1}\,dx.
\end{aligned}$$

Thus:

THEOREM 3.2. *The orthogonality measure of the system $\{p_n(x)\}$ of sieved Jacobi polynomials of the first kind is*

$$d\mu(x) = \frac{\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x^2)^{\alpha}|T_k(x)|^{2\beta+1}|U_{k-1}(x)|^{2\alpha+1}\chi(x)\,dx, \tag{3.13}$$

where $\chi(x)$ denotes the characteristic function of $(-1,1)$.

REMARK 3.1. Representation (2.8) thus allows one to determine $d\mu$ without resorting to Stieltjes' inversion formula (2.11), which is the usual procedure for this type of problems (see [1], [7], [11], [12], [24], [25], etc.). As a matter of fact, to prove that

$$d\mu(x) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+}\operatorname{Im}(X(x-i\varepsilon))\,dx, \tag{3.14}$$

and on this basis to show that (2.38) (with $T_k(z)$ in the place of z) implies

(3.13), is rather sticky. The procedure used above was devised for a slightly different purpose in [16]. See also [15], [17].

REMARK 3.2. When $\beta = -1/2$ and $\alpha + 1/2 = \lambda$, (3.13) becomes

$$d\mu(x) = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1/2)\sqrt{\pi}}(1-x^2)^{\lambda-1/2}|U_{k-1}(x)|^{2\lambda}\chi(x)d(x), \tag{3.15}$$

which is the orthogonality measure of the sieved ultraspherical polynomials of the first kind in [2].

REMARK 3.3. It follows easily from (3.4) and the properties of $\{p_n^{(\alpha,\beta)}(x)\}$ in Section 2.11 that $p_{2nk}(x) = 2^{(1-2k)n}P_n^{(\alpha,\beta)}(T_{2k}(x))$, $n \geq 0$, and $\{p_n(x)\}$ can thus be obtained from the Jacobi polynomials $\{P_n^{(\alpha,\beta)}(x)\}$ by means of the polynomial mapping $x \to T_{2k}(x)$.

4. Sieved Jacobi polynomials of the second kind. The system $\{p_n(x)\}$ of sieved Jacobi polynomials of the second kind is given by (1.8) with the $a_n^{(j)}$ as in (1.11) and (1.12).

For this system $\Delta_n(2, 2k-1)$ depends on n and the direct treatment of $\{p_n(x)\}$ would require using (2.44) instead of (2.46), which is rather cumbersome. This can be circumvented by observing that $\{p_n(x)\}$ is the set of numerator polynomials of the system $\{q_n(x)\}$ given by

$$xq_{2nk+j}(x) = q_{2nk+j+1}(x) + b_n^{(j)}q_{2nk+j-1}(x), \tag{4.1}$$

$k \geq 2,\ 0 \leq j \leq 2k-1$, and $q_{-1}(x) = 0,\ q_0(x) = 1$, where

$$\begin{gathered} b_n^{(0)} = a_{n-1}^{(2k-1)}, \quad b_n^{(1)} = a_n^{(0)}, \quad b_n^{(k)} = a_n^{(k-1)}, \\ b_n^{(k+1)} = a_n^{(k)}, \qquad n \geq 0, \end{gathered} \tag{4.2}$$

and, when $k > 2$,

$$b_n^{(j)} = 1/4, \qquad j = 2, \dots, 2k-1,\ j \neq k, k+1,\ n \geq 0. \tag{4.3}$$

We adopt the convention that $b_0^{(1)} = 0$. As an easy calculation shows,

$$\Delta_n(2, 2k-1) = \widetilde{U}_{2k-1}(x), \qquad n \geq 0, \tag{4.4}$$

for the system of the first kind $\{q_n(x)\}$ so defined. Also,

$$\begin{aligned} \Delta_n(2, 2k-2) = x&\widetilde{U}_{k-1}(x)\widetilde{U}_{k-2}(x) \\ &- a_n^{(k)}\widetilde{U}_{k-1}(x)\widetilde{U}_{k-3}(x) - a_n^{(k-1)}\widetilde{U}_{k-2}^2(x) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} \Delta_n(3, 2k-1) = x&\widetilde{U}_{k-1}(x)\widetilde{U}_{k-2}(x) \\ &- a_n^{(k-1)}\widetilde{U}_{k-1}(x)\widetilde{U}_{k-3}(x) - a_n^{(k)}\widetilde{U}_{k-2}^2(x). \end{aligned} \tag{4.6}$$

Let $\{Q_n(x)\}$ be the link polynomials of $\{q_n(x)\}$. Then $\{Q_n^{(1)}(x)\}$ satisfies

$$(4.7)\quad \begin{aligned}[][x\Delta_{n+1}(2,2k-1) &- a_{n+1}^{(0)}\Delta_{n+1}(3,2k-1) \\ &- a_n^{(2k-1)}\Delta_n(2,2k-2)]Q_n^{(1)}(x) \\ &= Q_{n+1}^{(1)}(x) + (1/4)^{2k-4}a_n^{(0)}a_n^{(k-1)}a_n^{(k)}a_n^{(2k-1)}Q_{n-1}^{(1)}(x)\end{aligned}$$

with $Q_{-1}^{(1)}(x)=0$, $Q_0^{(1)}(x)=1$, where (2.47), (4.2) and (4.3) have been taken into account. Also, because of (4.4)–(4.6), relation (4.7) can be written as

$$(4.8)\quad \begin{aligned}[][xU_{2k-1}(x) &- 4(a_{n+1}^{(0)}+a_n^{(2k-1)})xU_{k-1}(x)U_{k-2}(x) \\ &+ 8(a_{n+1}^{(0)}a_{n+1}^{(k-1)}+a_n^{(2k-1)}a_n^{(k)})U_{k-1}(x)U_{k-3}(x) \\ &+ 8(a_{n+1}^{(0)}a_{n+1}^{(k)}+a_n^{(2k-1)}a_n^{(k-1)})U_{k-2}^2(x)]\widetilde{Q}_n(x) \\ &= \widetilde{Q}_{n+1}(x) + 64a_n^{(0)}a_n^{(k-1)}a_n^{(k)}a_n^{(2k-1)}\widetilde{Q}_{n-1}(x)\end{aligned}$$

where

$$(4.9)\qquad \widetilde{Q}_n(x) = 2^{(2k-1)n}Q_n^{(1)}(x), \qquad n\ge 0.$$

Now, from $a_{n+1}^{(0)}+a_n^{(2k-1)} = a_n^{(k)}+a_n^{(k-1)} = 1/2$ it follows that

$$a_{n+1}^{(0)}a_{n+1}^{(k-1)}+a_n^{(2k-1)}a_n^{(k)} = 1/4-(a_{n+1}^{(0)}a_{n+1}^{(k)}+a_n^{(2k-1)}a_n^{(k-1)}),$$

and from (2.23)–(2.25) we have

$$(4.10)\quad \begin{aligned}[][T_{2k}(x)+1-8(a_{n+1}^{(0)}a_{n+1}^{(k-1)}+a_n^{(2k-1)}a_n^{(k)})]\widetilde{Q}_n(x) \\ = \widetilde{Q}_{n+1}(x) + 64a_n^{(0)}a_n^{(k-1)}a_n^{(k)}a_n^{(2k-1)}\widetilde{Q}_{n-1}(x)\end{aligned}$$

for $n\ge 0$. This translates into

$$(4.11)\quad \left[T_{2k}(x) - \frac{(\beta+1)^2-(\alpha+1)^2}{(2n+\alpha+\beta+2)(2n+\alpha+\beta+4)}\right]\widetilde{Q}_n(x)$$
$$= \widetilde{Q}_{n+1}(x) + \frac{4n(n+\alpha+1)(n+\beta+1)(n+\alpha+\beta+2)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta+2)^2(2n+\alpha+\beta+3)}\widetilde{Q}_{n-1}(x)$$

for $n\ge 0$. Then (2.36) yields $\widetilde{Q}_n(x) = P_n^{(\alpha+1,\beta+1)}(T_{2k}(x))$, and from (2.48) and (2.52) we finally obtain:

THEOREM 4.1. *The system $\{p_n(x)\}$ of sieved Jacobi polynomials of the second kind satisfies*

$$(4.12)\quad \begin{aligned}&p_{2(n+1)k-1}(x) \\ &\quad = 2^{-(2k-1)(n+1)}U_{2k-1}(x)P_n^{(\alpha+1,\beta+1)}(T_{2k}(x)), \qquad n\ge 0.\end{aligned}$$

and

$$p^{(1)}_{2(n+1)k-2}(x) = 2^{-(2k-1)n}\Delta_0(3,2k-1)P_n^{(\alpha+1,\beta+1)}(T_{2k}(x)) + \frac{2^6 a_0^{(k-1)}a_0^{(k)}a_0^{(2k-1)}}{2^{(2k-1)(n+1)}}(P_{n-1}^{(\alpha+1,\beta+1)})^{(1)}(T_{2k}(x)). \tag{4.13}$$

Therefore, the limit $Y(z)$ *of the continued fraction of* $\{p_n(x)\}$ *is*

$$Y(z) = \frac{2^{2k-1}\Delta_0(3,2k-1)}{U_{2k-1}(z)} + 64\frac{a_0^{(k-1)}a_0^{(k)}a_0^{(2k-1)}}{U_{2k-1}(z)}X_{\alpha+1,\beta+1}(T_{2k}(z)). \tag{4.14}$$

The appearance of $U_{2k-1}(z)$ in the denominators of the right hand side terms of (4.14) suggests the presence of masses in $[-1,1]$ (at the roots of $U_{2k-1}(x)$), which may actually be the case for some systems ([13], pp. 89–90, and Section 6 below). However,

$$a_0^{(k-1)}a_0^{(k)}a_0^{(2k-1)}\frac{(\alpha+\beta+2)(\alpha+\beta+3)}{4(\alpha+1)(\beta+1)} = \frac{1}{32}, \tag{4.15}$$

and from $(2,39)$, with $T_{2k}(z)$ in place of z, we see that

$$Y(z) = \frac{2^{2k-1}\Delta_0(3,2k-1) - 2(1-T_{2k}(z)) + 8a_0^{(k)}}{U_{2k-1}(z)} + 2(1-z^2)U_{2k-1}(z)X_{\alpha,\beta}(T_{2k}(z)). \tag{4.16}$$

On the other hand, a calculation based on (2.23)–(2.25) and (4.6) yields $2^{2k-1}\Delta_0(3,2k-1) = 4[U_{k-1}(z)T_{k-1}(z) - 2a_0^{(k)}]$, and it follows that the numerator of the first term on the right hand side of (4.16) reduces to $4[U_{k-1}(z)T_{k-1}(z) + T_k^2(z) - 1] = 2zU_{2k-1}(z)$. Thus

$$Y(z) = 2z + 2(1-z^2)U_{2k-1}(z)X_{\alpha,\beta}(T_{2k}(z)), \tag{4.17}$$

and the moment functional $\mathcal{L}^1$ of $\{p_n(x)\}$ is

$$\mathcal{L}^1(P(x)) = \frac{1}{\pi i}\int_C P(z)(1-z^2)U_{2k-1}(z)X_{\alpha,\beta}(T_{2k}(z))\,dz \tag{4.18}$$

where C is a positively oriented contour around $[-1,1]$. Hence we have $\mathcal{L}^1 = 2(1-x^2)\mathcal{L}$ with $\mathcal{L}$ as in (3.12), i.e., $\mathcal{L}^1$ is obtained from $\mathcal{L}$ by left multiplication by the polynomial $2(1-x^2)$ (see [9] for details about this operation), and thus:

THEOREM 4.2. *The orthogonality measure of the system* $\{p_n(x)\}$ *of sieved Jacobi polynomials of the second kind is*

$$d\vartheta(x) = \frac{2\Gamma(\alpha+\beta+2)}{\Gamma(\alpha+1)\Gamma(\beta+1)}(1-x^2)^{\alpha+1}|T_k(x)|^{2\beta+1}|U_{k-1}(x)|^{2\alpha+1}\chi(x)\,dx \tag{4.19}$$

where $\chi(x)$ *is the characteristic function of* $[-1,1]$. *Moreover*, $2(1-x^2)d\mu(x) = d\vartheta(x)$, *with* μ *as in* (3.13).

REMARK 4.1. If $\beta=-1/2$ and $\alpha+1/2=\lambda$, then $\{p_n(x)\}$ is the system of sieved ultraspherical polynomials of the second kind in [2]. From (4.19), its orthogonality measure is

$$d\vartheta(x) = \frac{2\Gamma(\lambda+1)}{\Gamma(\lambda+1/2)\sqrt{\pi}}(1-x^2)^{\lambda+1/2}|U_{k-1}(x)|^{2\lambda}\chi(x)\,dx. \tag{4.20}$$

REMARK 4.2. The system $\{p_n(x)\}$ of sieved Jacobi polynomials of the second kind is a system of sieved polynomials of the second kind of the little Jacobi polynomials $\{p_n^{(\alpha+1,\beta)}(x)\}$. This follows at once from (2.60), (1.11) and (1.12).

5. A related system of the first kind. The system $\{p_n(x)\}$ in this section is closely related to those in Section 3. It is likewise a system of sieved polynomials of the first kind of $\{p_n^{(\alpha,\beta)}(x)\}$, and a system of sieved random walk polynomials of the first kind as well. It cannot be obtained via a polynomial mapping $x\to T_k(x)$ $(x\to T_{2k}(x))$ except in special cases. $\{p_n(x)\}$ is given by the blocks of $2k$ equations (1.8) with

$$\begin{aligned} a_n^{(0)} &= \frac{(n+\alpha)}{2(2n+\alpha+\beta+1)}, & a_n^{(1)} &= \frac{(n+\beta+1)}{2(2n+\alpha+\beta+1)},\\ a_n^{(k)} &= \frac{(n+\alpha+\beta+1)}{2(2n+\alpha+\beta+2)}, & a_n^{(k+1)} &= \frac{(n+1)}{2(2n+\alpha+\beta+2)} \end{aligned} \tag{5.1}$$

for $n\ge 0$ and, if $k>2$, with

$$a_n^{(j)} = 1/4, \qquad j=2,\dots,2k-1,\ j\ne k,k+1,\ n\ge 0. \tag{5.2}$$

We assume $\alpha>-1,\ \beta>-1,\ \alpha+\beta>-1$ and $k\ge 2$. Clearly, (3.5) and (3.6) hold for $\{p_n(x)\}$, and also $a_n^{(0)}+a_n^{(1)} = a_n^{(k)}+a_n^{(k+1)} = 1/2,\ n\ge 0$. We deal with $\{p_n(x)\}$ by the $2k$-block approach. Thus, for the link polynomials $P_n(x) := p_{2nk}(x),\ n\ge 0$, we have

$$\begin{aligned} &[x\Delta_n(2,2k-1) - a_n^{(1)}\Delta_n(3,2k-1) - a_n^{(0)}\Delta_{n-1}(2,2k-2)]P_n(x)\\ &\qquad = P_{n+1}(x) + 4^{2(2-k)}a_n^{(0)}a_{n-1}^{(1)}a_{n-1}^{(k-1)}a_{n-1}^{(k)}P_{n-1}(x), \qquad n\ge 1, \end{aligned} \tag{5.3}$$

with $\Delta_{-1}(2,2k-2)=0$ and

$$\begin{aligned}
\Delta_n(2,2k-1) &= \widetilde{U}_{2k-1}(x),\\
\Delta_n(2,2k-2) &= x\widetilde{U}_{k-1}(x)\widetilde{U}_{k-2}(x)\\
&\quad - a_n^{(k+1)}\widetilde{U}_{k-1}(x)\widetilde{U}_{k-3}(x) - a_n^{(k)}\widetilde{U}_{k-2}^2(x),\\
\Delta_n(3,2k-1) &= x\widetilde{U}_{k-1}(x)\widetilde{U}_{k-2}(x)\\
&\quad - a_n^{(k)}\widetilde{U}_{k-1}(x)\widetilde{U}_{k-3}(x) - a_n^{(k+1)}\widetilde{U}_{k-2}^2(x)
\end{aligned} \tag{5.4}$$

for $n \geq 0$, where $\{\widetilde{U}_n(x)\}$ are the monic Chebyshev polynomials of the second kind in (2.25). Also

$$\begin{gathered}
P_{-1}(x)=0, \qquad P_0(x)=1,\\
P_1(x) = x\Delta_0(2,2k-1) - a_0^{(1)}\Delta_0(3,2k-1).
\end{gathered} \tag{5.5}$$

Letting

$$\widetilde{P}_n(x) = 2^{(2k-1)n}P_n(x), \qquad n \geq 0, \tag{5.6}$$

and taking into account that

$$a_n^{(1)}a_{n-1}^{(k+1)} + a_n^{(0)}a_{n-1}^{(k+1)} = 1/4 - (a_n^{(1)}a_n^{(k)} + a_n^{(0)}a_{n-1}^{(k+1)})$$

for $n \geq 1$, and relations (2.23)–(2.25), we obtain from (5.1)–(5.4),

$$\begin{aligned}
&\left[T_{2k}(x) - \frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}\right]\widetilde{P}_n(x)\\
&\quad = \widetilde{P}_{n+1}(x) + \frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2(2n+\alpha+\beta+1)}\widetilde{P}_{n-1}(x),\\
&\hspace{28em} n \geq 1,
\end{aligned} \tag{5.7}$$

with $\widetilde{P}_0(x)=1$ and, from (5.5),

$$\widetilde{P}_1(x) = \left[T_{2k}(x) - \frac{\beta-\alpha}{(\alpha+\beta+2)}\right] + \frac{2\alpha}{\alpha+\beta+1}T_k(x)U_{k-2}(x). \tag{5.8}$$

Thus

$$\begin{aligned}
&p_{2nk}(x)\\
&\quad = 2^{(1-2k)n}\left[P_n^{(\alpha,\beta)}(T_{2k}(x)) + \frac{2\alpha}{\alpha+\beta+1}T_k(x)U_{k-2}(x)(P_{n-1}^{(\alpha,\beta)})^{(1)}(T_{2k}(x))\right]
\end{aligned} \tag{5.9}$$

for $n \geq 0$. Similar calculations give for the system $\{P_n^{(1)}(x)\}$ of the first associated polynomials of $\{P_n(x)\}$ the equality

$$P_n^{(1)}(x) = 2^{(1-2k)n}(P_n^{(\alpha,\beta)})^{(1)}(T_{2k}(x)), \qquad n \geq 0. \tag{5.10}$$

Hence, from (2.48),

$$p_{2nk-1}^{(1)}(x) = 2^{(1-2k)n}U_{2k-1}(x)(P_{n-1}^{(\alpha,\beta)})^{(1)}(T_{2k}(x)), \qquad n \geq 0. \tag{5.11}$$

Thus:

THEOREM 5.1. *The limit $X(z)$ of the continued fraction of $\{p_n(x)\}$ is*

$$X(z) := \lim_{n\to\infty} \frac{p^{(1)}_{2nk-1}(x)}{p_{2nk}(x)} = \frac{U_{2k-1}(z)X_{\alpha,\beta}(T_{2k}(z))}{1+\frac{2\alpha}{\alpha+\beta+1}T_k(z)U_{k-2}(z)X_{\alpha,\beta}(T_{2k}(z))}, \tag{5.12}$$

where

$$X_{\alpha,\beta}(T_{2k}(z)) = \frac{1}{T_{2k}(z)-1} F\left[\begin{matrix}1,\alpha+1\\ \alpha+\beta+2\end{matrix}\,\middle|\, \frac{2}{1-T_{2k}(z)}\right], \tag{5.13}$$

provided that $T_{2k}(z) \notin [-1,1]$, i.e., $z \notin [-1,1]$.

It follows from (5.9) that for $\alpha \neq 0$, $\{p_n(x)\}$ cannot be obtained through a polynomial mapping. If $\alpha=0$, then $\{p_n(x)\}$ is a special case of the sieved Jacobi polynomials of the first kind in Section 3 and originates from $\{P_n^{(0,\beta)}(x)\}$ via the polynomial mapping $x \to T_{2k}(x)$.

In order to determine explicitly the orthogonality measure of $\{p_n(x)\}$, precise information about $X(z)$ is needed. This is rather sticky to obtain in the full general case. Therefore we restrict ourselves to the case $\alpha=-\beta$. This is neat, leads to new and interesting systems of orthogonal polynomials, and sheds light on the sort of difficulties that arise in the general case. Clearly, this imposes on α the restriction $-1<\alpha<1$. We will write $X_\alpha(z)$ instead of $X_{\alpha,-\alpha}(z)$. If $\alpha \neq 0$ and $|z-1|>2$ then

$$\begin{aligned} X_\alpha(z) &= -\sum_{n=0}^{\infty} \frac{(1+\alpha)_n}{(n+1)!}\left(\frac{2}{1-z}\right)^{n+1} \\ &= \frac{1}{2\alpha}\left[1-\sum_{n=0}^{\infty}\frac{(\alpha)_n}{n!}\left(\frac{2}{1-z}\right)^n\right], \end{aligned} \tag{5.14}$$

and it follows from (2.33) that

$$X_\alpha(z) = \frac{1}{2\alpha}\left[1-\left(\frac{z-1}{z+1}\right)^\alpha\right], \qquad z \notin [-1,1]. \tag{5.15}$$

If $\alpha = 0$, then

$$X_0(z) = \frac{1}{2}\log\left(\frac{z-1}{z+1}\right), \qquad z \notin [-1,1]. \tag{5.16}$$

Throughout, we use the branch of log with imaginary part in $(-\pi,\pi]$. Hence, for $\alpha=-\beta$, (5.12) yields

$$X(z) = \frac{U_{2k-1}(z)X_\alpha(T_{2k}(z))}{1+2\alpha T_k(z)U_{k-2}(z)X_\alpha(T_{2k}(z))}, \qquad \alpha \neq 0,\ z \notin [-1,1], \tag{5.17}$$

and

$$X(z) = \frac{1}{2}U_{2k-1}(z)\log\left(\frac{T_{2k}(z)-1}{T_{2k}(z)+1}\right), \qquad \alpha = 0,\ z \notin [-1,1]. \tag{5.18}$$

If $\widehat{X}(z)$ is defined by (2.15) for $X(z)$ as above, then $\widehat{X}(z)$ is continuous on $\operatorname{Im} z \leq 0$, except possibly at the points $\zeta_0 = -1$, $\zeta_{2k} = 1$ and at the roots $\zeta_1, \ldots, \zeta_{2k-1}$ of $U_{2k-1}(x)$. Notice that, in fact, $\zeta_1, \ldots, \zeta_{2k-1}$ are the double roots of $T_{2k}^2(x)-1$ (the simple roots being $\zeta_0 = -1$ and $\zeta_{2k} = 1$). This follows from relations (2.23) and (2.24). Then observe that $X_\alpha(T_{2k}(z))$ may become infinite, according to the relative values of α, at the roots of $T_{2k}(x) + 1$ or of $T_{2k}(x) - 1$. However,

$$\lim_{z \to \zeta_j} (z - \zeta_j)\widehat{X}(z) = 0, \qquad j = 0, 1, \ldots, 2k. \tag{5.19}$$

On the other hand, from (5.17) it follows that $w(x) = \frac{1}{\pi} \operatorname{Im} \widehat{X}(x)$ is given by

$$\begin{aligned} w(x) \\ = \frac{\sin \pi\alpha}{2\pi\alpha} \frac{|U_{2k-1}(x)|(1 - T_{2k}(x))^\alpha (1 + T_{2k}(x))^{-\alpha} \chi(x)}{|1 + T_k(x)U_{k-2}(x)(1 - e^{i\pi\alpha}(1 - T_{2k}(x))^\alpha (1 + T_{2k}(x))^{-\alpha})|^2} \end{aligned} \tag{5.20}$$

for $x \neq \zeta_j$, $j = 0, \ldots, 2k$, where $\chi(x)$ is the characteristic function of $(-1, 1)$ and $\frac{\sin \pi\alpha}{\pi\alpha} = 1$ if $\alpha = 0$. Since the roots of $U_{2k-1}(x)$ are the double roots of $1 - T_{2k}(x)$ or of $1 + T_{2k}(x)$ and $-1 < \alpha < 1$, so that $1 \pm 2\alpha > -1$, it follows that $w(x)$ is integrable on $\mathbb{R}$. Thus:

Theorem 5.2. *The orthogonality measure μ of $\{p_n(x)\}$ is absolutely continuous and given by*

$$d\mu(x) = w(x)dx \tag{5.21}$$

with $w(x)$ as in (5.20). *In particular,*

$$d\mu(x) = \tfrac{1}{2}|U_{2k-1}(x)|\chi(x)dx \tag{5.22}$$

for $\alpha = 0$.

Remark 5.1. The analysis of μ would have been rather difficult without appealing to Theorem 2.1 and Remark 2.1. The usual arguments for this type of problems use Stieltjes' inversion formula (2.11), assume that the limit and the integral can be interchanged, and then, if $w(x)$ turns out to be integrable, check for masses at the points of discontinuity of $\widehat{X}(z)$. The test for a mass point at $x = \zeta$, i.e.,

$$\sum_{n=0}^{\infty} \frac{P_n^2(\zeta)}{\lambda_n} < \infty, \qquad \lambda_n = \mathcal{L}(P_n^2(x)), \tag{5.23}$$

is the usual resort for the latter purpose (see [7] for a discussion about (5.23) and for many examples). However, convergence of (5.23) is in general difficult to assert, and some sort of asymptotic analysis must be done, which makes it difficult to determine the exact value $(\sum_{n=0}^{\infty} P_n^2(\zeta)/\lambda_n)^{-1}$ of the mass at ζ. All this is avoided in our case.

Remark 5.2. When $k=2$, $\{p_n(x)\}$ can still be obtained through a polynomial mapping. In fact,

$$p_{2n}(x)=2^{-n}q_n(T_2(x)),\qquad n\ge 0, \tag{5.24}$$

where

$$q_n(x)=p_n^{(\alpha,\beta)}(x)+\frac{\alpha}{\alpha+\beta+1}(p_{n-1}^{(\alpha,\beta)})^{(1)}(x),\qquad n\ge 0, \tag{5.25}$$

$\{p_n^{(\alpha,\beta)}(x)\}$ being the system of little Jacobi polynomials.

Remark 5.3. The polynomials $\{p_n(x)\}$ in this section are related to the exceptional Jacobi polynomials of [26]. These, which are denoted by $P_n^{(\alpha)}(x)$, $n\ge 0$, are given by the recurrence relation (2.36) (with $\alpha=-\beta$) for $n\ge 1$ and by $P_0^{(\alpha)}(x)=1$, $P_1^{(\alpha)}(x)=x$ (instead of $P_1^{(\alpha,-\alpha)}(x)=x+\alpha$ for the usual Jacobi polynomials). Thus

$$P_n^{(\alpha)}(x)=P_n^{(\alpha,-\alpha)}(x)-\alpha(P_{n-1}^{(\alpha,-\alpha)})^{(1)}(x),\qquad n\ge 1, \tag{5.26}$$

and if $X^{(\alpha)}(z)$ denotes the limit of the continued fraction of $\{P_n^{(\alpha)}(x)\}$ then

$$X^{(\alpha)}(z)=\frac{X_\alpha(z)}{1-\alpha X_\alpha(z)},\qquad z\notin[-1,1], \tag{5.27}$$

with $X_\alpha(z)$ as in (5.15) for $\alpha\ne 0$ and as in (5.16) for $\alpha=0$ (clearly, $P_n^{(0)}(x)=P_n^{(0,0)}(x)$, $n\ge 0$). The function

$$\widehat{X}^{(\alpha)}(z)=\begin{cases} X^{(\alpha)}(z), & \operatorname{Im} z<0,\\ \lim\limits_{\zeta\to z,\,\operatorname{Im}\zeta<0} X^{(\alpha)}(\zeta), & \operatorname{Im} z=0,\end{cases}$$

is obviously continuous, except possibly at $z=\pm 1$, but

$$\lim_{z\to\pm 1}(z\pm 1)\widehat{X}^{(\alpha)}(z)=0.$$

Furthermore, if $w(x)=\frac{1}{\pi}\operatorname{Im}\widehat{X}^{(\alpha)}(x)$, $x\in\mathbb{R}$, $x\ne\pm 1$, then

$$w(x)=\frac{2\sin\pi\alpha(1-x^2)^\alpha\chi(x)}{\pi\alpha((1+x)^{2\alpha}+2\cos\pi\alpha(1-x^2)^\alpha+(1-x)^{2\alpha})}, \tag{5.29}$$

with $\frac{\sin\pi\alpha}{\pi\alpha}=1$ if $\alpha=0$, follows at once from (5.27), and it is as obtained in [26] by a different procedure. Evidently, $w(x)$ is continuous on $\mathbb{R}$ for $\alpha\ne 0$ and reduces to $\frac{1}{2}\chi(x)$ if $\alpha=0$. Thus, the orthogonality measure of $\{P_n^{(\alpha)}(x)\}$ is $d\mu(x)=w(x)dx$ and is absolutely continuous.

Remark 5.4. As a matter of fact, the system $\{\widetilde{p}_n(x)\}$ given by (2.40) with $\widetilde{p}_{-1}(x)=0$, $\widetilde{p}_0(x)=1$, $b_n^{(j)}=0$, $j=0,1,\dots,k-1$,

$$a_n^{(0)}=\frac{n+\alpha}{2n+4},\quad a_n^{(1)}=\frac{n-\alpha+1}{2n+4},\qquad n\ge 0, \tag{5.30}$$

and

$$a_n^{(j)} = 1/4, \quad j = 2, \dots, k-1,\ n \geq 0, \tag{5.31}$$

if $k > 2$, is a system of sieved polynomials of the first kind of the system $\{P_n^{(\alpha)}(x)\}$ in Remark 5.3. It is readily seen that

$$\begin{aligned} \widetilde{p}_{nk}(x) &= 2^{(1-k)n}[P_n^{(\alpha)}(T_k(x)) + \alpha U_{k-2}(x)(P_{n-1}^{(\alpha)})^{(1)}(T_k(x))], \\ \widetilde{p}_{nk-1}^{(1)}(x) &= 2^{(1-k)n} U_{k-1}(x)(P_{n-1}^{(\alpha)})^{(1)}(T_k(x)) \end{aligned} \tag{5.32}$$

for $n \geq 0$. The limit $\widetilde{X}(z)$ of their continued fraction in terms of $X^{(\alpha)}(T_k(z))$ in (5.27) is

$$\widetilde{X}(z) = \frac{U_{k-1}(z)X^{\alpha}(T_k(z))}{1 - \alpha U_{k-2}(z)X^{\alpha}(T_{2k}(z))}, \quad z \notin [-1,1], \tag{5.33}$$

and in terms of $X_\alpha(T_k(z))$,

$$\widetilde{X}(z) = \frac{U_{k-1}(z)X_{\alpha}(T_k(z))}{1 - \alpha(U_{k-2}(z) - 1)X_{\alpha}(T_k(z))}, \quad z \notin [-1,1]. \tag{5.34}$$

The same techniques as above yield for their orthogonality measure

$$\begin{aligned} &d\widetilde{\mu}(x) \\ &= 2\frac{\sin \pi\alpha}{\pi\alpha} \frac{|U_{k-1}(x)|(1 - T_k(x))^{\alpha}(1 + T_k(x))^{-\alpha}\chi(x)dx}{|2 + (U_{k-2}(x) - 1)(1 - e^{i\pi\alpha}(1 - T_k(x))^{\alpha}(1 + T_k(x))^{-\alpha})|^2} \end{aligned} \tag{5.35}$$

where $\chi(x)$ is the characteristic function of $(-1,1)$. The polynomials $\{\widetilde{p}_n(x)\}$ generalize $\{p_n(x)\}$ to the case of k odd. The polynomials $\{P_n^{(\alpha)}(x)\}$ correspond to $k = 1$.

6. A further system of the second kind. Let $\{p_n(x)\}$ be given by (1.8) with

$$\begin{aligned} a_0^{(0)} &= \frac{\alpha+1}{2(\alpha+\beta+1)}, & a_0^{(k-1)} &= \frac{1}{2(\alpha+\beta+2)}, \\ a_0^{(k)} &= \frac{\alpha+\beta+1}{2(\alpha+\beta+2)}, & a_0^{(2k-1)} &= \frac{\beta+1}{2(\alpha+\beta+3)} \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} a_n^{(0)} &= \frac{n+\alpha+1}{2(2n+\alpha+\beta+1)}, & a_n^{(k-1)} &= \frac{n}{2(2n+\alpha+\beta+2)}, \\ a_n^{(k)} &= \frac{n+\alpha+\beta+2}{2(2n+\alpha+\beta+2)}, & a_n^{(2k-1)} &= \frac{n+\beta+1}{2(2n+\alpha+\beta+3)} \end{aligned} \tag{6.2}$$

for $n \geq 1$, and with

$$a_n^{(j)} = 1/4, \quad 0 < j < 2k-1,\ j \neq k-1, k, \tag{6.3}$$

if $k > 2$. We assume $\alpha + \beta > -1,\ \alpha > -1,\ \beta > -1$.

This system is naturally related to the sieved Jacobi polynomials of the second kind in Section 4. They are likewise sieved random walk polynomials of the second kind: $a_{n+1}^{(0)} + a_n^{(2k-1)} = a_n^{(k)} + a_n^{(k-1)} = 1/2$ for all $n \geq 0$. They are also sieved polynomials of the second kind of the system $\{p_n^{(\alpha+1,\beta)}(x)\}$. This follows from (2.60), (2.61), (6.1), (6.2) and (6.3).

To determine their orthogonality measure we proceed as in Section 5. Let then $\{q_n(x)\}$ be such that $p_n(x) = q_n^{(1)}(x),\ n \geq 0$. We may take $\{q_n(x)\}$ given by

$$xq_{2nk+j}(x) = q_{2nh+j+1}(x) + b_n^{(j)} q_{2nk+j-1}(x), \qquad n \geq 0,$$

and $q_{-1}(x) = 0,\ q_0(x) = 1$, where $b_n^{(0)} = a_{n-1}^{(2k-1)},\ b_n^{(1)} = a_n^{(0)},\ b_n^{(k)} = a_n^{(k-1)},\ b_n^{(k+1)} = a_n^{(k)}$, the $a_n^{(j)}$ being as in (6.1) and (6.2), and $b_n^{(j)} = 1/4$ for $j = 2, \ldots, 2k-1, j \neq k, k+1$. For the system $\{P_n(x)\}$ of the link polynomials of $\{q_n(x)\}$ we argue as in Section 4 to obtain in this case

(6.4) $$P_n^{(1)}(x) = \frac{1}{2^{(2k-1)n}}\left[P_n^{(\alpha+1,\beta+1)}(T_{2k}(x)) + \frac{2(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)} P_n^{(\alpha+1,\beta+1)}(T_{2k}(x))\right]$$

and

$$P_n^{(2)}(x) = \frac{1}{2^{(2k-1)n}} (P_n^{(\alpha+1,\beta+1)})^{(1)}(T_{2n}(x)). \tag{6.5}$$

Then, from (2.57),

$$p_{2(n+1)k-1}(x) = \frac{1}{2^{(2k-1)}} U_{2k-1}(x) P_n^{(1)}(x)$$

and

$$p_{2(n+1)k-2}^{(1)}(x) = \Delta_0(3, 2k-1) P_n^{(1)}(x) + \frac{4^3 a_0^{(k-1)} a_0^{(k)} a_0^{(2k-1)}}{4^{(2k-1)}} P_n^{(2)}(x),$$

so that the limit $Y(z)$ of the continued fraction of $\{p_n(x)\}$ is

(6.6) $$Y(z) = \frac{2^{2k-1}\Delta_0(3, 2k-1)}{U_{2k-1}(z)} + \frac{64 a_0^{(k-1)} a_0^{(k)} a_0^{(2k-1)} X_{\alpha+1,\beta+1}(T_{2k}(z))}{U_{2k-1}(z)\left[1 + \frac{2(\beta+1)}{(\alpha+\beta+2)(\alpha+\beta+3)} X_{\alpha+1,\beta+1}(T_{2k}(z))\right]}$$

for $z \notin [-1, 1]$. Now we assume as in Section 5 that $\alpha = -\beta$ and $-1 < \alpha < 1$. Taking into account (2.39), (6.1), (6.2) and (6.3), relation (6.6) yields

$$Y(z) = \frac{4U_{k-1}(z)T_{k-1}(z) - 2}{U_{2k-1}(z)} \tag{6.7}$$

$$+\frac{(1-T_{2k}^2(z))X_\alpha(T_{2k}(z))+T_{2k}(z)-\alpha}{(1+\alpha)U_{2k-1}(z)\left[1+\dfrac{1}{2(1+\alpha)}[(1-T_{2k}^2(z))X_\alpha(T_{2k}(z))+T_{2k}(z)-\alpha]\right]}$$

with X_α as in (5.15) for $\alpha \neq 0$ and as in (5.16) for $\alpha = 0$. Let $X(z)$ be the second term on the right hand side of (6.7) and let $\zeta_1, \dots, \zeta_{2k-1}$ be the roots of $U_{2k-1}(z)$. Let $w(x)$ be given for $x \in \mathbb{R}$ by

$$w(x) = \lim_{z \to x,\, \operatorname{Im} z < 0} \frac{1}{\pi} \operatorname{Im}(X(z)), \qquad x \neq \zeta_j,\ 1 < j \leq 2k,$$

so that

(6.8) $\quad w(x) =$

$$\frac{(1+\alpha)\sin\pi\alpha(1-x^2)|U_{2k-1}(x)|(1-T_{2k}(x))^{\alpha}(1+T_{2k}(x))^{-\alpha}\chi(x)}{\pi\alpha|2+\alpha+T_{2k}(x)+(2\alpha)^{-1}(1-x^2)U_{2k-1}^2(x)(1-e^{i\pi\alpha}(1-T_{2k}(x))^{\alpha}(1+T_{2k}(x))^{-\alpha})|^2} =$$

$$\frac{(1+\alpha)\sin\pi\alpha(1-x^2)^{\alpha+1}|U_{2k-1}(x)|^{2\alpha+1}\chi(x)}{\pi\alpha|(2+\alpha+T_{2k}(x))(1+T_{2k}(x))^{\alpha}+(2\alpha)^{-1}(1-x^2)U_{2k-1}^2(x)((1+T_{2k}(x))^{\alpha}-e^{i\pi\alpha}(1-T_{2k}(x))^{\alpha})|^2}$$

for $\alpha \neq 0$ and

(6.9) $\quad w(x) =$

$$\frac{4(1-x^2)|U_{2k-1}(x)|\chi(x)}{|4+2T_{2k}(x)+(1-x^2)U_{2k-1}(x)(\log(1+T_{2k}(x))-\log(1-T_{2k}(x))+i\pi)|^2}$$

for $\alpha = 0$, $\chi(x)$ being the characteristic function of $(-1,1)$. Although $w(x)$ may be discontinuous at $\zeta_1, \dots, \zeta_{2k-1}$, it is obviously an integrable function on $\mathbb{R}$ (it can be easily verified that $w(x)$ is continuous for $-1/2 \leq \alpha \leq 1/2$). To examine the behavior of $\widehat{Y}(z) = \lim_{\zeta \to z} Y(\zeta)$ on $\operatorname{Im} z \geq 0$, $z \neq \zeta_1, \dots, \zeta_{2k-1}$, assume that $\zeta_1 < \dots < \zeta_{k-1}$ are the roots of $U_{k-1}(x)$, which in view of (2.24) are the double roots of $1 - T_{2k}(x)$, and $\zeta_k < \dots < \zeta_{2k-1}$ are the roots of $T_k(x)$ and the double roots of $1 + T_{2k}(x)$. Partial fraction decomposition yields

$$2\frac{2U_{k-1}(z)T_{k-1}(z)-1}{U_{2k-1}(z)} = \sum_{j=1}^{2k-1} \frac{1-\zeta_j^2}{k(z-\zeta_j)}, \tag{6.10}$$

so that if $\mathcal{L}^1$ is the moment functional of $\{p_n(x)\}$ and C is a positively oriented contour around $[-1,1]$, then

$$\mathcal{L}^1(P(x)) = \sum_{j=1}^{2k-1} \frac{(1-\zeta_j^2)}{k} P(\zeta_j) + \frac{1}{2\pi i}\int_C P(z)X(z)\,dz \tag{6.11}$$

for any polynomial $P(z) \in \mathbb{C}[x]$. This suggests that each ζ_j, $j = 1, \dots, 2k-1$, is a mass point of $\mathcal{L}^1$ with mass $\frac{1}{k}(1-\zeta_j^2)$. However, a simple calculation on $\widehat{X}(z)$ (defined for $X(z)$ as above by (2.15)) gives for $j = k, \dots, 2k-1$, where $T_{2k}(\zeta_j)=-1$, the following:

$$\lim_{z\to\zeta_j}(z-\zeta_j)\widehat{X}(z)=-\frac{2}{U'_{2k-1}(\zeta_j)}=-\frac{1-\zeta_j^2}{k}, \tag{6.12}$$

so that

$$\lim_{z\to\zeta_j}\operatorname{Re}(z-\zeta_j)\widehat{X}(z)=-\frac{1-\zeta_j^2}{k}. \tag{6.13}$$

Also,

$$\lim_{z\to\zeta_j}\operatorname{Re}(z-\zeta_j,)\widehat{X}(z)=\frac{2(1-\alpha)}{U'_{2k-1}(\zeta_j)(3+\alpha)}=-\frac{1-\alpha}{k(3+\alpha)}(1-\zeta_j^2) \tag{6.14}$$

for $j=1,\dots,k-1$, as in this case $T_{2k}(\zeta_j)=1$. Thus:

THEOREM 6.1. *The moment functional* $\mathcal{L}^1$ *of the system* $\{p_n(x)\}$ *given by* (1.8), (6.1), (6.2) *and* (6.3) *is*, *for* $-1<\alpha<1$,

$$\mathcal{L}^1(P(x))=\frac{2(1+\alpha)}{(3+\alpha)k}\sum_{j=1}^{k-1}(1-\zeta_j^2)P(\zeta_j)+\int_{-\infty}^{\infty}P(x)w(x)\,dx, \tag{6.15}$$

where $w(x)$ *is as in* (6.8) (*with* $\frac{\sin\pi\alpha}{\pi\alpha}=1$ *for* $\alpha=0$). *The orthogonal measure bears mass points at the roots* $\zeta_1,\dots,\zeta_{k-1}$ *of* $U_{2k-1}(x)$ (*i.e.*, *at the double roots of* $T_{2k}(x)-1$), *the mass at* ζ_j *having the value* $2(1+\alpha)(1-\zeta_j^2)/((\alpha+3)k)$.

REMARK 6.1. It follows that the masses at the roots ζ_j, $j=k,\dots,2k-1$, of $T_k(x)$ were only apparent. Clearly, $w(\zeta_j)=\infty$, $j=1,\dots,2k-1$, for $1/2<|\alpha|<1$.

REMARK 6.2. The system $\{p_n(x)\}$ shows that the condition $\lim_{n\to\infty}a_n^{(j)}=1/4$, $j=0,1,\dots,2k-1$, does not guarantee the absence of mass points embedded in $(-1,1)$ ([7], p. 102). Of course, in this case ([7], p. 13),

$$\sum_{j=0}^{2k-1}\sum_{j=1}^{\infty}|(a_n^{(j)})^{1/2}-1/2|=\infty,$$

as, for example,

$$(a_n^{(k)})^{1/2}-1/2=\sqrt{\frac{n}{2(2n+2)}}-\frac{1}{2}=\frac{\sqrt{2n(2n+1)}}{2(2n+1)}-\frac{1}{2}\sim\frac{1}{2n},$$

i.e.,

$$|(a_n^{(k)})^{1/2}-1/2|=O(1/n).$$

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