COLLOQUIUM MATHEMATICUM

VOL. 78 1998 NO. 1

ON LIE GROUPS IN VARIETIES OF TOPOLOGICAL GROUPS

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1. Introduction. The primary aim of this note is to prove the following result, providing the solution (in the positive) to a problem that first appeared in [8] as P897.

THEOREM 1.1. *If Ω is any class of topological groups and $V(\Omega)$ the variety of topological groups generated by Ω, then every Banach–Lie group* (*in particular*, *every finite-dimensional Lie group and every additive topological group of a Banach space*) *in $V(\Omega)$ is contained in* $\mathrm{Q}\overline{\mathrm{S}}\mathrm{P}(\Omega)$.

Here *variety* [9] means a class of topological groups closed with respect to forming direct products of arbitrary subfamilies equipped with Tychonoff topology (which operation is denoted in the sequel by C), proceeding to topological subgroups (S), and quotient groups (Q). The symbol P denotes forming finite direct products of topological groups, while $\overline{\mathrm{S}}$ refers to taking closed topological subgroups.

The version of the above theorem stated for finite-dimensional Lie groups was announced in [5], however it appears that the proposed proof is incorrect. In our analysis of what went wrong in the original proof, we isolate a new concept playing a central role in the argument, that of a locally minimal topological group. While being similar to widely known minimal topological groups, locally minimal topological groups are found more often. In particular, every Banach–Lie group, including every finite-dimensional Lie group, every additive group of a Banach space, as well as every discrete group, is locally minimal. The major technical result which we obtain is of independent interest, and it states, in particular, that whenever a locally minimal group G having no small normal subgroups (in an obvious sense) isomorphically embeds into the product of a family of topological groups, it embeds isomorphically into the product of a finite subfamily. While it turns out that topological groups with no small subgroups (NSS groups) are not necessarily in this class—and this was essentially the flaw of the proof in [5]—the

1991 *Mathematics Subject Classification*: Primary 22A05.

so-called groups uniformly free from small subgroups, introduced by Enflo in [3] and very close in their properties to NSS groups, are. They contain, in particular, all Banach–Lie groups, whence Theorem 1.1 follows.

2. Locally minimal topological groups. A (Hausdorff) topological group $G = (G, \tau)$ is called *minimal* [2] if it admits no Hausdorff group topology strictly coarser than τ. We need a somewhat weaker version of this concept and for this reason we introduce the following new notion.

DEFINITION 2.1. We say that a topological group $G = (G, \tau)$ is *locally minimal* if there exists a neighbourhood of the identity, V, with the property that whenever σ is a Hausdorff group topology on G with $\sigma \subseteq \tau$ such that the σ-interior of V is nonempty, one has $\sigma = \tau$.

It is useful to observe that local minimality is indeed a *local* property in the sense traditionally used in the theory of topological groups: a neighbourhood of the identity, V, possessing the property from Definition 2.1, can be chosen so as to be arbitrarily small. In other words, for every neighbourhood of the identity, U, there is a neighbourhood of the identity W such that $W \subseteq U$ and whenever σ is a Hausdorff group topology on G coarser than the original topology τ and having a nonempty σ-interior of W, then $\sigma = \tau$. (The proof is in fact trivial: just put $W = U \cap V$, where V is as in the definition.)

Every minimal topological group is obviously locally minimal. To see that the converse is not true, notice that every discrete topological group G is locally minimal if one puts $V = \{e_G\}$. In particular, the additive group of integers, $\mathbb{Z}$, equipped with the discrete topology, is locally minimal, while this group is well known to support a wealth of nondiscrete group topologies. (See e.g. Chs. I and II in [2].)

We aim to show that the class of locally minimal topological groups includes all (underlying topological groups of) Banach–Lie groups. To prove this, we recall a concept introduced by Enflo [3]. A topological group G is said to be *uniformly free from small subgroups* if it contains a neighbourhood of the identity, U, such that for every neighbourhood of the identity, V, there exists a positive integer n_V with the property that $x \notin V \Rightarrow x^n \notin U$ for some $n \leq n_V$.

For any subset S of a group G and for any positive integer n we set

$$\frac{1}{n}S = \{x \in G : \forall k = 1, \ldots, n,\ x^k \in S\}.$$

The following is obvious.

PROPOSITION 2.2. *If V is a neighbourhood of the identity in a topological group G and n is a positive integer, then the set $(1/n)V$ is a neighbourhood of the identity in G.*

We wish to reformulate the concept of a group uniformly free from small subgroups in a more convenient form for our purposes. The following is immediate.

PROPOSITION 2.3. *A topological group G is uniformly free from small subgroups if and only if for some neighbourhood of the identity*, U, *the sets* $(1/n)U$ *form a neighbourhood basis at the identity.*

Recall that a topological group G *has no small subgroups*, or else is an *NSS group*, if some neighbourhood of the identity, V, contains no subgroups of G other than $\{e_G\}$. It is easy to see that every topological group uniformly free from small subgroups is an NSS group, but the converse is not true because, for example, each group uniformly free from small subgroups is metrizable, while an NSS group need not be so. (The simplest such example would be the abelian topological group from Example 2.1.1 in [3]. There exists, however, a vast class of NSS groups of importance that are not metrizable unless they are discrete—the free topological groups on submetrizable spaces, cf. [10], [12].)

REMARK 2.4. There exist metrizable NSS groups that are not uniformly free from small subgroups. Such is the additive group of any nonnormable locally convex Fréchet space admitting a continuous norm, e.g. the space $C^\infty(X)$ of all infinitely smooth real-valued functions on a compact manifold equipped with the usual topology of uniform convergence with all derivatives.

PROPOSITION 2.5. *If a topological group G is uniformly free from small subgroups*, *then it is locally minimal.*

Proof. Select as U the neighbourhood appearing in the definition of a group uniformly free from small subgroups, and denote by V any neighbourhood of the identity such that $VV^{-1} \subseteq U$. Let σ be a Hausdorff group topology on G such that $\sigma \subseteq \tau$ and the σ-interior of V is nonempty. Then the σ-interior of U is easily checked to contain e. Now for every $n \in \mathbb{N}$ the set $(1/n)U$ must be σ-open. But such sets form a basis for τ at the identity, which shows that $\sigma = \tau$.

REMARK 2.6. Not every locally minimal group—and in fact, not every minimal group—is uniformly free from small subgroups. The most widely known example is the group $S(X)$ of all permutations of an infinite set X equipped with the topology of pointwise convergence with respect to the discrete topology on X. It is minimal [2] but not even an NSS group, since open subgroups form a neighbourhood basis at the identity.

THEOREM 2.7. *Every Banach–Lie group is uniformly free from small subgroups.*

P r o o f. Let G be a Banach–Lie group, and denote by $\mathfrak{g}$ the corresponding Banach–Lie algebra. Equip $\mathfrak{g}$ with a submultiplicative norm. Let $\varepsilon > 0$ be so small that (i) the restriction of the exponential map $\exp : \mathfrak{g} \to G$ to the open ball $O_\varepsilon \subset \mathfrak{g}$ of radius ε centred at zero is a diffeomorphism onto its image, and (ii) the Hausdorff series $H(x, y)$ converges for any two elements $x, y \in O_\varepsilon$, making O_ε into a local Banach–Lie group with respect to the Hausdorff multiplication $*$. (For the basics of Banach–Lie theory we refer the reader to [1].) Now choose a $\delta > 0$ so that for every $x, y \in O_\delta$ one has $x * y \in O_\varepsilon$. Define $U = \exp O_\delta$. We claim that for every positive integer n,

$$\frac{1}{n}U = \exp(O_{\delta/n}).$$

The inclusion $\supseteq$ is immediate. Let now $x \in G$ be such that $x^k \in U$ for all $k = 1, \ldots, n$. Since in particular $x \in U$, there is a unique $\widetilde{x} \in O_\delta$ with $\exp \widetilde{x} = x$. Now we proceed by induction on k. Suppose that $\widetilde{x} \in O_{\delta/k}$ for some $k = 1, \ldots, n-1$. Then clearly $(k+1)\widetilde{x} \in O_{(k+1)\delta/k} \subseteq O_{2\delta} \subseteq O_\varepsilon$, because $(k+1)/k \leq 2$. Since $x^{k+1} = \exp((k+1)\widetilde{x})$ and $x^{k+1} \in U$, one must have $(k+1)\widetilde{x} \in O_\delta$, as $\exp|_{O_\varepsilon}$ is injective and therefore $(\exp|_{O_\varepsilon})^{-1}(\exp(O_\delta)) = O_\delta$. But this means exactly that $\widetilde{x} \in (1/(k+1))O_\delta = O_{\delta/(k+1)}$. We have thus established that $\widetilde{x} \in O_{\delta/n}$ and therefore $x \in \exp(O_{\delta/n})$.

Finally, observe that the open balls $O_{\varepsilon/n}$ form a neighbourhood basis in the Banach–Lie algebra $\mathfrak{g}$ and their images under exp form a neighbourhood basis in G.

COROLLARY 2.8. *Every Banach–Lie group* (*in particular*, *every finite-dimensional Lie group and the additive topological group of every Banach space*) *is a locally minimal topological group.*

REMARK 2.9. The above result does not seem to extend to more general classes of useful infinite-dimensional Lie groups. The additive topological group of the Fréchet space $C^\infty(X)$ (Remark 2.4) is an obvious example of a regular abelian Fréchet–Lie group in the sense of [7] whose underlying topological group is not locally minimal: if V is a neighbourhood of zero, then for some $n \in \mathbb{N}$ large enough the interior of V with respect to the C^n-topology is nonempty, and the latter topology is strictly coarser than the C^∞-topology.

Let us say that a topological group G *has no small normal subgroups* if there is a neighbourhood of the identity, V, containing no nontrivial normal subgroups of G. (Equivalently: no nontrivial closed normal subgroups of G.) This notion is perfectly in line with the well known and important concept of a group with no small subgroups. Clearly, every NSS group has no small normal subgroups, but the converse is not true. (As an example, consider again the full symmetric group, $S(X)$, of an infinite set X, equipped with the topology of pointwise convergence. It is known to be topologically simple, that is, to contain no proper nontrivial closed normal subgroups [4]. At the same time, it is not an NSS group, as noticed in Remark 2.4.)

It turns out that in the absence of small normal subgroups, the property of local minimality can be strengthened as follows.

PROPOSITION 2.10. *Let G be a locally minimal topological group having no small normal subgroups. Then each neighbourhood U of the identity in G contains a neighbourhood V of the identity such that whenever σ is a* (*not necessarily Hausdorff*) *group topology on G with $\sigma \subseteq \tau$ such that the σ-interior of V is nonempty*, *one has $\sigma = \tau$.*

Proof. Let W be a neighbourhood of the identity with the property taken from Definition 2.1 of local minimality; one can also assume without loss of generality that W contains no small normal subgroups and (since local minimality is a local property) $W \subseteq U$. Choose a symmetric neighbourhood V of the identity such that $V^2 \subseteq W$. Now let σ be a group topology on G with $\sigma \subseteq \tau$ and such that the σ-interior of V is nonempty. Denote by N the σ-closure of $\{e_G\}$. Then N is contained in the σ-closure of V, which is in turn a subset of $V^2 \subseteq W$. (Recall that the closure of a set X in a topological group is exactly the intersection of all sets of the form XO as O runs over a neighbourhood basis at the identity.) By the assumption, one must have $N = \{e_G\}$, that is, σ is a Hausdorff topology and therefore $\sigma = \tau$.

3. The main results. The following two are the central technical results of this note.

LEMMA 3.1. *Let H, F, G be Hausdorff topological groups and let $\pi : H \to G$ be an open continuous surjective homomorphism*, *$f : H \to F$ be a continuous homomorphism and $g : F \to G$ be a homomorphism such that $\pi = g \circ f$. Let G be locally minimal and have no small normal subgroups*, *and let V be a neighbourhood of the identity in G chosen as in Proposition* 2.10. *Suppose the interior of $g^{-1}(V)$ in F is nonempty. Then the surjective homomorphism $g : F \to G$ is continuous and open.*

Proof. Denote by σ the factor topology of the group topology on F formed with respect to the homomorphism $g : F \to G$. In other words, the σ-open subsets of G are exactly the images of open subsets of F under the homomorphism g. Since the latter is surjective, σ is a group topology on G (possibly non-Hausdorff). If W is an open subset of F, then $g(W) = \pi(f^{-1}(W))$ is an open subset of G with respect to its original topology, therefore σ is coarser than the original topology on G. Since $g^{-1}(V)$ is assumed to have a nonempty interior in F, the σ-interior of V is nonempty. Now we are under the assumptions of Proposition 2.10 and can conclude that σ coincides with the original topology of G. But this means exactly that $g : F \to G$ is continuous and open.

Now we are able to establish a rectified version of the flawed Lemma 5.2 of [5].

LEMMA 3.2. *Assume that a topological group G is a quotient group of a subgroup of the product of a family $\mathcal{G}$ of topological groups. Assume that G is locally minimal and has no small normal subgroups. Then G is a quotient group of a subgroup of the product of a finite subfamily of $\mathcal{G}$.*

Proof. Let $\mathcal{G} = \{G_\alpha : \alpha \in A\}$, let H be a topological subgroup of $\prod_{\alpha\in A} G_\alpha$, and denote by $\pi : H \to G$ the factor homomorphism with kernel N. Denote by V a neighbourhood of the identity in G small enough to contain no nontrivial normal subgroups of G and to satisfy the property stated in Proposition 2.10. There are a finite set $B=\{\alpha_1, \ldots, \alpha_n\}$ of indices and neighbourhoods $V_{\alpha_i} \subseteq G_{\alpha_i}$ of the identity, $i = 1, \ldots, n$, such that

$$p_B^{-1}(V_{\alpha_1} \times \ldots \times V_{\alpha_n}) \cap H \subseteq \pi^{-1}(V),$$

where

$$p_B : \prod_{\alpha\in A} G_\alpha \to \prod_{\alpha\in B} G_\alpha \equiv G_{\alpha_1} \times \ldots \times G_{\alpha_n}$$

is the canonical projection homomorphism.

Define $F = p_B(H)$; it is a topological subgroup of $G_{\alpha_1} \times \ldots \times G_{\alpha_n}$. Let $f = p_B|_H : H \to F$; it is a continuous homomorphism with kernel $p_B^{-1}(e) \cap H$. Since $p_B^{-1}(e) \cap H \subseteq \pi^{-1}(V)$ and therefore $\pi(p_B^{-1}(e))$ is a normal subgroup of G contained in V, it is trivial and one has $p_B^{-1}(e) \cap H \subseteq N$. Because of that, π factors through $p_B^{-1}(e) \cap H$ to give rise to a surjective homomorphism $g : F \to G$.

Notice that $\pi = g \circ f$, $\pi : H \to G$ is open, continuous and onto, $f : H \to F$ is continuous, and $g : F \to G$ is a group homomorphism. The interior of $g^{-1}(V)$ in F is nonempty, because it contains the set $W = (V_{\alpha_1} \times \ldots \ldots \times V_{\alpha_n}) \cap F$: indeed,

$$g(W) = \pi(f^{-1}(W)) = \pi(p_B^{-1}(V_{\alpha_1} \times \ldots \times V_{\alpha_n}) \cap H) \subseteq V.$$

We are now under the assumptions of Lemma 3.1, which result tells us that the surjective homomorphism g is continuous and open and therefore G is a topological factor group of $F < G_{\alpha_1} \times \ldots \times G_{\alpha_n}$. ∎

REMARK 3.3. Even though a topological subgroup F of a finite subproduct, having G as its factor group, is a continuous homomorphic image of the topological subgroup H of the infinite product, it need not be a topological factor group of H. Here is a counterexample. Let E be an infinite-dimensional normed space, and denote by E_σ the space E having its weak topology. Fix a nontrivial continuous linear functional $f : E \to \mathbb{C}$. Let H be a subgroup of the Tychonoff product $E_\sigma \times E^{\aleph_0}$ formed by all constant maps, that is, the image of E under the diagonal embedding $x \mapsto (x, x, x, \ldots)$. Clearly, H is topologically isomorphic to E. Let $G = \mathbb{C}$ and define a homomorphism $\pi : H \to G$ via $\pi(x, x, x, \ldots) = f(x)$. Certainly, G is a topological factor group of H. Choose as a finite subproduct the first factor, E_σ. The projection of the infinite product onto the first factor, restricted to $H \cong E$, is the canonical continuous map $E \to E_\sigma$, which is not open.

Remark 3.4. In the above example we have used an infinite product of topological groups to make the setting look more "generic". However, as was justly pointed out by the referee of this paper, the product $E \times E_\sigma$ would do just as well. In this case $H = \{(x, x) : x \in E\}$ is simply the graph of the continuous identity function $E \to E_\sigma$ which fails to be open since E is infinite-dimensional. The graph of any continuous function is always isomorpic to its domain (under the projection onto its domain). The projection onto the range is a continuous bijective morphism which is not an isomorphism of topological groups. This example also illustrates that not all finite partial projections preserve the property of the absence of small subgroups even if they project the subgroup H bijectively onto its image.

The following is an immediate consequence of Lemma 3.2, Proposition 2.5 and the fact that every group uniformly free from small subgroups is NSS and therefore has no small normal subgroups.

Corollary 3.5. *Let G be a topological group uniformly free from small subgroups. Then, whenever G is isomorphic to a topological subgroup of the direct product of a family $\mathcal{G}$ of topological groups, G is isomorphic to a subgroup of the product of a finite subfamily of $\mathcal{G}$.*

Remark 3.6. In view of the above Corollary 3.5, it is useful to remember that not every locally minimal topological group having no small normal subgroups is uniformly free from small subgroups. A counterexample is conveniently provided by the same infinite symmetric group $S(X)$.

Corollary 3.7. *Let G be a Banach–Lie group. Then whenever G is isomorphic to a topological subgroup of the direct product of a family $\mathcal{G}$ of topological groups, G is isomorphic to a subgroup of the product of a finite subfamily of $\mathcal{G}$.*

By simply reformulating Lemma 3.2, we obtain the following result, which is the corrected version of Proposition 5.3 in [5].

Proposition 3.8. *For a class Ω of topological groups, the members of* QSC(Ω) *which are locally minimal and have no small normal subgroups, are contained in* QSP(Ω).

REMARK 3.9. Proposition 5.3 of [5] claimed that for a class Ω of topological groups, the members of $\mathrm{QSC}(\Omega)$ having no small subgroups are contained in $\mathrm{QSP}(\Omega)$. This statement is not true, and the simplest way to see this is to observe that if applied to the class Ω of all metrizable topological groups, it yields immediately the wrong statement: every abelian NSS group is metrizable. (As the operations P, S, and Q all preserve the first axiom of countability, and every abelian topological group is isomorphic to a topological subgroup of the product of metrizable groups, see e.g. [6].) Now cf. the earlier comment on the issue preceding Remark 2.4.

Repeating word for word the argument contained in [5] on pp. 161–162 between the statement of Proposition 5.3 and the statement of Theorem 5.4, we obtain the following corrected version of Theorem 5.4.

THEOREM 3.10. *The class of members of $V(\Omega)$ that are locally minimal and have no small normal subgroups is contained in* $\mathrm{SPQ\overline{S}P}(\Omega) \subseteq \mathrm{QSP}(\Omega)$.

Now the proof of our Theorem 1.1 proceeds exactly as that of Theorem 5.5 in [5], but we replace the NSS property with that of being locally minimal and having no small normal subgroups, apply Corollary 2.8, and also observe that since a Banach–Lie group is complete in its two-sided uniformity [1], it is therefore closed in any topological group containing it as a topological subgroup [11].

Acknowledgments. The authors are grateful to the referee whose thoughtful and thorough reports on two preliminary versions of this paper have enabled the authors to substantially improve the presentation of results, especially as far as Theorem 2.7, Lemmas 3.1 and 3.2, and Remarks 3.3 and 3.4 are concerned.

The second author (V.P.) is thankful to the Mathematical Analysis Research Group (MARG) for support and hospitality extended during his visit to the University of Wollongong in November–December 1996.

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Received 19 February 1997;
revised 17 January 1998