COLLOQUIUM MATHEMATICUM

VOL. 78 1998 NO. 1

LIMITS OF FAMILIES OF MEASURE ALGEBRAS

BY

JOJI TAKAHASHI (KOBE)

The limit of a directed family of measure algebras is characterized as the unique complete Boolean algebra having a dense subset that is isomorphic to a canonical poset constructed from the given family.

Suppose that we are given a family $\mathcal{A} = \langle A_\zeta, \varrho_{\zeta\eta}\rangle_{\zeta,\eta\in W\wedge\zeta\trianglelefteq\eta}$ satisfying the following conditions:

(GF1) $\langle W, \trianglelefteq\rangle$ is a nonempty directed poset.
(GF2) For each $\eta \in W$, A_ζ is a complete Boolean algebra.
(GF3) For each pair $\langle\zeta,\eta\rangle$ ($\zeta,\eta \in W$ and $\zeta \trianglelefteq \eta$), $\varrho_{\zeta\eta}$ is a complete embedding from A_ζ to A_η.
(GF4) For each triple $\langle\zeta,\eta,\xi\rangle$ ($\zeta,\eta,\xi \in W$ and $\zeta \trianglelefteq \eta \trianglelefteq \xi$), $\varrho_{\zeta\xi} = \varrho_{\eta\xi}\circ\varrho_{\zeta\eta}$.
(GF5) For each $\zeta \in W$ and each $a \in A_\zeta$, $\varrho_{\zeta\zeta}(a) = a$.

It is then interesting to investigate limits of $\mathcal{A}$, i.e. families $\langle A, \varrho_\zeta\rangle_{\zeta\in W}$ having the following properties:

(L1) A is a complete Boolean algebra.
(L2) For each $\zeta \in W$, ϱ_ζ is a complete embedding from A_ζ to A.
(L3) For each pair $\langle\zeta,\eta\rangle$ ($\zeta,\eta \in W$ and $\zeta \trianglelefteq \eta$), $\varrho_\zeta = \varrho_\eta \circ \varrho_{\zeta\eta}$.
(L4) A is completely generated by $\bigcup_{\zeta\in W} \mathrm{Ran}(\varrho_\zeta)$.

Many different types of limits are known, the two best-known being the direct limit and the inverse limit ([Je, §§23 and 36], [Ku, VIII, §5]).

Now suppose that, in addition to $\mathcal{A}$, we are given a family $\langle\mu_\zeta\rangle_{\zeta\in W}$ such that:

(GF6) For each $\zeta \in W$, μ_ζ is a countably additive strictly positive probability measure on A_ζ.
(GF7) For each pair $\langle\zeta,\eta\rangle$ ($\zeta,\eta \in W$ and $\zeta \trianglelefteq \eta$) and each $a \in A_\zeta$, $\mu_\zeta(a) = \mu_\eta(\varrho_{\zeta\eta}(a))$.

We are then interested in limits of the expanded family

$$\widetilde{\mathcal{A}} = \langle A_\zeta, \mu_\zeta, \varrho_{\zeta\eta}\rangle_{\zeta,\eta\in W\wedge\zeta\trianglelefteq\eta},$$

1991 *Mathematics Subject Classification*: Primary 03E40.

which should consist of a limit $\langle A, \varrho_\zeta\rangle_{\zeta\in W}$ of $\mathcal{A}$ and an associated countably additive strictly positive probability measure μ on A such that

(L5) For each $\zeta \in W$ and each $a \in A_\zeta$, $\mu_\zeta(a) = \mu(\varrho_\zeta(a))$.

The question of existence of limits in this sense is easily settled affirmatively. All one has to do is to express each A_ζ as the measure algebra of a probability measure space ([Fr, 2.6]) and apply Kolmogorov's extension theorem ([Bo, 5.1]) to the family of these measure spaces. The objective of this note is to gain a more direct understanding of the limits of $\widetilde{\mathcal{A}}$. It will be shown that, like most of the known limits of the family $\mathcal{A}$ of plain complete Boolean algebras, the complete Boolean algebras A in the limits $\langle A, \mu, \varrho_\zeta\rangle_{\zeta\in W}$ of $\widetilde{\mathcal{A}}$ can be characterized by the property of having a dense subset isomorphic to a certain poset constructed from $\widetilde{\mathcal{A}}$ in a natural fashion.

For each pair $\langle\zeta,\eta\rangle$ ($\zeta,\eta \in W$ and $\zeta \trianglelefteq \eta$), let $\pi_{\zeta\eta}$ denote the projection associated with $\varrho_{\zeta\eta}$, i.e. that map from A_η to A_ζ such that

$$\forall b \in A_\eta\colon \pi_{\zeta\eta}(b) = \bigwedge\nolimits^{A_\zeta}\{a \in A_\zeta \mid b \sqsubseteq \varrho_{\zeta\eta}(a)\}.$$

Let Π denote the set of all functions p defined on the index set W such that

$$\forall \zeta \in W\colon p(\zeta) \in A_\zeta - \{0_{A_\zeta}\} \quad \text{and}$$
$$\forall \zeta,\eta \in W\ (\zeta \trianglelefteq \eta)\colon p(\zeta) = \pi_{\zeta\eta}(p(\eta)).$$

Define the partial order $\sqsubseteq$ on Π by

$$\forall p,q \in \Pi\colon [p \sqsubseteq q \Leftrightarrow \forall \zeta \in W\colon p(\zeta) \sqsubseteq q(\zeta)].$$

Many of the known limits $\langle A, \varrho_\zeta\rangle_{\zeta\in W}$ of $\mathcal{A}$ satisfy condition (L4) in a rather strong sense. They have a dense subset arising from some set $P \subset \Pi$. More precisely, one can find a set $P \subset \Pi$ so that the map $p \mapsto \bigwedge^{A}_{\zeta\in W} \varrho_\zeta(p(\zeta))$ ($p \in P$) is an isomorphism from the poset $\langle P, \sqsubseteq\rangle$ onto a dense subset of $A - \{0_A\}$. For example, we have the set

$$\{p \in \Pi \mid \exists \alpha \in W\colon \forall \zeta \in W\ (\alpha \trianglelefteq \zeta)\colon p(\zeta) = \varrho_{\alpha\zeta}(p(\alpha))\}$$

for the direct limit of $\mathcal{A}$, and the set Π itself for the inverse limit. We will see that the relationship between the family $\widetilde{\mathcal{A}}$ of measure algeras and their measure algebra limits can also be captured in this way by means of a suitably defined subset of Π.

Let us define the set $P \subset \Pi$ that gives rise to a dense subset of the limit of $\widetilde{\mathcal{A}}$. We need to make sure that P consists only of those $p \in \Pi$ such that the Boolean values $p(\zeta)$ shrink nicely as ζ increases with respect to $\trianglelefteq$. Those p such that $p(\zeta)$ contract too rapidly or in too unruly a manner must be weeded out. However, the standard method of selecting those p for which the set

$$\{\alpha \in W \mid \exists \zeta \in W\ (\alpha \trianglelefteq \zeta \wedge \alpha \neq \zeta)\colon p(\zeta) = \varrho_{\alpha\zeta}(p(\alpha))\},$$

called the *support* of p, is in a suitable ideal of subsets of W is known not to work for measure algebras ([Ku, VIII, Exercise K6, p. 302]). It is necessary to take advantage of the measures μ_ζ as a means of assessing the manner of contraction of $p(\zeta)$ so that we can distinguish correctly between those p to be allowed into P and those to be kept out. Thus, for each triple $\langle p, \alpha, a\rangle$ such that $p \in \varPi$, $\alpha \in W$ and $a \in A_\alpha$, we put

$$\inf(p, \alpha, a) = \inf\{\mu_\zeta(p(\zeta) \wedge \varrho_{\alpha\zeta}(a)) \mid \alpha \trianglelefteq \zeta \in W\},$$

and define P to be the set of all $p \in \varPi$ such that

$$\forall\alpha \in W\colon \forall a \in A_\alpha\,(p(\alpha) \wedge a \neq 0_{A_\alpha})\colon\ \inf(p, \alpha, a) > 0.$$

Throughout the remainder of this note, we assume that

$$\widetilde{\mathcal{A}} = \langle A_\zeta, \mu_\zeta, \varrho_{\zeta\eta}\rangle_{\zeta,\eta\in W\wedge\zeta\trianglelefteq\eta}$$

is a family satisfying (GF1)–(GF7), $\mathcal{A} = \langle A_\zeta, \varrho_{\zeta\eta}\rangle_{\zeta,\eta\in W\wedge\zeta\trianglelefteq\eta}$ is the plain complete Boolean algebra portion of $\widetilde{\mathcal{A}}$, and $\langle A, \varrho_\zeta\rangle_{\zeta\in W}$ is a limit of $\mathcal{A}$ as defined by (L1)–(L4). Also, let P denote the set defined as in the preceding paragraph, and θ the map $p \mapsto \bigwedge^{A}_{\zeta\in W} \varrho_\zeta(p(\zeta))$ from P to A. We will prove:

THEOREM 1. *Suppose that θ"P is a dense subset of $A - \{0_A\}$. Then we have*:

(a) *For any p_1, $p_2 \in P$, $p_1 \sqsubseteq p_2$ if and only if $\theta(p_1) \sqsubseteq \theta(p_2)$.*

(b) *There is a countably additive strictly positive probability measure μ on A satisfying* (L5).

THEOREM 2. *If there is a countably additive strictly positive probability measure μ on A satisfying* (L5), *then θ"P is a dense subset of $A - \{0_A\}$.*

It follows from these two theorems that A carries a countably additive strictly positive probability measure μ satisfying (L5) if and only if θ is an order isomorphism from $\langle P, \sqsubseteq\rangle$ onto a dense subset of A–$\{0_A\}$. In particular, the limits of $\widetilde{\mathcal{A}}$ are all isomorphic to each other. Also note that Theorem 1 gives a direct proof of the existence of the limit of $\widetilde{\mathcal{A}}$ that does not depend on Kolmogorov's extension theorem.

Part (a) of Theorem 1 is easy to prove. All we need is the following fact.

LEMMA 1. *For any $p_1 \in P$, any $\zeta \in W$ and any $a \in A_\zeta$ $(p_1(\zeta)\wedge a \neq 0_{A_\zeta})$, there is a $p_2 \in P$ such that $p_2 \sqsubseteq p_1$ and $p_2(\zeta) \sqsubseteq a$.*

Proof. Suppose that p_1, ζ and a are as above. Then there is a unique function p_2 on W such that

$$\forall\eta \in W\colon \forall\xi \in W\,(\zeta, \eta \trianglelefteq \xi)\colon\ p_2(\eta) = \pi_{\eta\xi}(p_1(\xi) \wedge \varrho_{\zeta\xi}(a)).$$

We easily check that $p_2 \in \varPi$, $p_2 \sqsubseteq p_1$ and $p_2(\zeta) \sqsubseteq a$. Also, given any $\alpha \in W$ and any $a' \in A_\alpha$ $(p_2(\alpha) \wedge a' \neq 0_{A_\alpha})$, we can choose a $\beta \in W$ with ζ, $\alpha \trianglelefteq \beta$, and see that

$$\inf(p_2,\alpha,a') = \inf(p_2,\beta,\varrho_{\alpha\beta}(a')) = \inf(p_1,\beta,\varrho_{\zeta\beta}(a)\wedge\varrho_{\alpha\beta}(a')), \quad \text{and}$$
$$p_1(\beta)\wedge\varrho_{\zeta\beta}(a)\wedge\varrho_{\alpha\beta}(a') \neq 0_{A_\alpha}.$$

Since $p_1 \in P$, it follows that $\inf(p_2,\alpha,a') > 0$. Therefore $p_2 \in P$. ■

Proof of Theorem 1(a). Let $p_1, p_2 \in P$. The "only if" part is obvious. For the converse, if $p_1 \not\sqsubseteq p_2$, then $p_1(\zeta) \not\sqsubseteq p_2(\zeta)$ for some $\zeta \in W$. Using Lemma 1, we can choose a $p_3 \in P$ so that $p_3 \sqsubseteq p_1$ and $p_3(\zeta)\wedge p_2(\zeta) = 0_{A_\zeta}$. It follows that $\theta(p_3) \sqsubseteq \theta(p_1)$ and $\theta(p_3)\wedge\theta(p_2) = 0_A$. Furthermore, $\theta(p_3) \neq 0_A$. Thus $\theta(p_1) \not\sqsubseteq \theta(p_2)$. ■

Proving part (b) of Theorem 1 and Theorem 2 requires more preliminary work. We need to know more about the structure of the poset $\langle P, \sqsubseteq\rangle$.

The elements of Π are characterized by the property that $p(\zeta) \in A_\zeta - \{0_{A_\zeta}\}$, $p(\eta) \in A_\eta - \{0_{A_\eta}\}$ and $p(\zeta) = \pi_{\zeta\eta}(p(\eta))$ whenever $\zeta, \eta \in W$ and $\zeta \trianglelefteq \eta$. Sometimes it will turn out necessary to deal with functions p having the somewhat weaker peoperty that $p(\zeta) \in A_\zeta$, $p(\eta) \in A_\eta$ and $p(\eta) \sqsubseteq \varrho_{\zeta\eta}(p(\zeta))$ for all $\zeta, \eta \in W$ with $\zeta \trianglelefteq \eta$. We denote the set of all functions having this latter property by $\Pi^{\#}$, and extend the partial order $\sqsubseteq$ on Π to one on $\Pi^{\#}$. Note that the operation $\inf(p,\alpha,a)$ makes sense not only for $p \in \Pi$ but for $p \in \Pi^{\#}$.

In what follows, $\mathbb{R}^{>0}$ and $\mathbb{R}^{\geq 0}$ will denote the set of all positive real numbers and that of all nonnegative real numbers respectively.

LEMMA 2. *For any $q \in \Pi^{\#}$, any $\alpha \in W$ and any pairwise disjoint $X \subset A_\alpha$, we have*

$$\inf\Big(q,\alpha,\bigvee^{A_\alpha} X\Big) = \sum_{a\in X} \inf(q,\alpha,a).$$

Proof. Let $q \in \Pi^{\#}$ and $\alpha \in W$. It is easily checked that the equality above holds for any finite pairwise disjoint $X \subset A_\alpha$. Let X be an arbitrary pairwise disjoint subset of A_α, and put $a_1 = \bigvee^{A_\alpha} X$.

Let us first show that the left-hand side is less than or equal to the right-hand side. For this, it suffices to prove that

$$\forall\delta\in\mathbb{R}^{>0}\colon\ \inf(q,\alpha,a_1) < \sum_{a\in X}\inf(q,\alpha,a) + \delta.$$

Given a $\delta \in \mathbb{R}^{>0}$, choose a finite $Y \subset X$ so that

$$\mu_\alpha(a_1\wedge(-a_2)) < \delta,$$

where $a_2 = \bigvee^{A_\alpha} Y$. Then

$$\inf(q,\alpha,a_1) = \inf(q,\alpha,a_2) + \inf(q,\alpha,a_1\wedge(-a_2)).$$

But

$$\inf(q, \alpha, a_2) = \sum_{a \in Y} \inf(q, \alpha, a) \le \sum_{a \in X} \inf(q, \alpha, a),$$

and

$$\inf(q, \alpha, a_1 \wedge (-a_2)) \le \mu_\alpha(a_1 \wedge (-a_2)) < \delta.$$

Thus

$$\inf(q, \alpha, a_1) < \sum_{a \in X} \inf(q, \alpha, a) + \delta.$$

On the other hand, we have

$$\begin{aligned}\sum_{a \in X} \inf(q, \alpha, a) &= \sup\Big\{ \sum_{a \in Y} \inf(q, \alpha, a) \,\Big|\, Y \subset X \wedge Y \text{ is finite}\Big\} \\ &= \sup\Big\{ \inf\Big(q, \alpha, \bigvee\nolimits^{A_\alpha} Y\Big) \,\Big|\, Y \subset X \wedge Y \text{ is finite}\Big\} \\ &\le \inf(q, \alpha, a_1). \ \blacksquare\end{aligned}$$

For each $p \in \Pi^{\#}$, put $\inf(p) = \inf\{\mu_\zeta(p(\zeta)) \mid \zeta \in W\}$.

LEMMA 3. *For any* $q \in \Pi^{\#}$ *with* $\inf(q) > 0$, *there is a* $p \in P$ *such that* $p \sqsubseteq q$ *and* $\inf(p) = \inf(q)$.

Proof. Suppose that $q \in \Pi^{\#}$ and $\inf(q) > 0$. Define the functions q' and p on W as follows:

$$\forall \zeta \in W\colon q'(\zeta) = \bigvee\nolimits^{A_\zeta} \{a \in A_\zeta \mid \inf(q, \zeta, a) = 0\}, \qquad \forall \zeta \in W\colon p(\zeta) = -q'(\zeta).$$

Clearly, $p \in \Pi^{\#}$ and $p \sqsubseteq q$.

CLAIM 1. *For any* $\zeta \in W$ *and* $a \in A_\zeta$, $a \sqsubseteq q'(\zeta)$ *if and only if* $\inf(q, \zeta, a) = 0$.

Proof. The "if" part is immediate from the definition of q'. To prove the "only if" part, suppose that $a_1 \sqsubseteq q'(\zeta)$ ($\zeta \in W$ and $a_1 \in A_\zeta$). Then there is a pairwise disjoint $X \subset A_\zeta$ such that

$$a_1 = \bigvee\nolimits^{A_\zeta} X \quad \text{and} \quad \forall a \in X\colon \inf(q, \zeta, a) = 0,$$

and it follows from Lemma 2 that $\inf(q, \zeta, a_1) = 0$. ■

CLAIM 2. *For any* $\alpha \in W$ *and* $a \in A_\alpha$, $\inf(p, \alpha, a) = \inf(q, \alpha, a)$.

Proof. Let $\alpha \in W$ and $a \in A_\alpha$. Since $p \sqsubseteq q$, we have

$$\inf(p, \alpha, a) \le \inf(q, \alpha, a).$$

On the other hand, for any $\zeta \in W$ with $\alpha \trianglelefteq \zeta$,

$$\begin{aligned}\inf(q, \alpha, a) &= \inf(q, \zeta, \varrho_{\alpha\zeta}(a)) \\ &= \inf(q, \zeta, p(\zeta) \wedge \varrho_{\alpha\zeta}(a)) + \inf(q, \zeta, q'(\zeta) \wedge \varrho_{\alpha\zeta}(a)).\end{aligned}$$

But

$$\inf(q, \zeta, p(\zeta) \wedge \varrho_{\alpha\zeta}(a)) \le \mu_\zeta(q(\zeta) \wedge p(\zeta) \wedge \varrho_{\alpha\zeta}(a)) = \mu_\zeta(p(\zeta) \wedge \varrho_{\alpha\zeta}(a)),$$

and, by Claim 1,

$$\inf(q, \zeta, q'(\zeta) \wedge \varrho_{\alpha\zeta}(a)) = 0.$$

Therefore

$$\inf(q, \alpha, a) \le \mu_\zeta(p(\zeta) \wedge \varrho_{\alpha\zeta}(a)).$$

Thus $\inf(q, \alpha, a) \le \inf(p, \alpha, a)$. ∎

By Claim 2, $\inf(p) = \inf(q)$.

CLAIM 3. $p \in \Pi$.

Proof. Since $\inf(p) = \inf(q) > 0$, we see that

$$\forall \zeta \in W \colon p(\zeta) \in A_\zeta - \{0_{A_\zeta}\}.$$

Also, for any $\zeta, \eta \in W$ $(\zeta \trianglelefteq \eta)$ and any $a \in A_\zeta$,

$$\begin{aligned} p(\eta) \sqsubseteq \varrho_{\zeta\eta}(a) &\Leftrightarrow \varrho_{\zeta\eta}(-a) \sqsubseteq q'(\eta) \\ &\Leftrightarrow \inf(q, \eta, \varrho_{\zeta\eta}(-a)) = 0 \quad \text{(by Claim 1)} \\ &\Leftrightarrow \inf(q, \zeta, -a) = 0 \\ &\Leftrightarrow -a \sqsubseteq q'(\zeta) \quad \text{(by Claim 1)} \\ &\Leftrightarrow p(\zeta) \sqsubseteq a, \end{aligned}$$

whence $\forall \zeta, \eta \in W$ $(\zeta \trianglelefteq \eta)$: $p(\zeta) = \pi_{\zeta\eta}(p(\eta))$. ∎

CLAIM 4. *For any $\alpha \in W$ and $a \in A_\alpha$ $(p(\alpha) \wedge a \ne 0_{A_\alpha})$, $\inf(p, \alpha, a) > 0$.*

Proof. If $\alpha \in W$, $a \in A_\alpha$ and $\inf(p, \alpha, a) = 0$, then $a \sqsubseteq q'(\alpha)$ by Claims 1 and 2, so that $p(\alpha) \wedge a = 0_{A_\alpha}$. ∎

By Claims 3 and 4, $p \in P$. Lemma 3 is proved. ∎

LEMMA 4. *For any $p_1, p_2 \in P$, p_1 and p_2 are compatible in the poset $\langle P, \sqsubseteq \rangle$ if and only if $\inf(p_1 \wedge p_2) > 0$, where $p_1 \wedge p_2$ is that element of $\Pi^\#$ such that*

$$\forall \zeta \in W \colon (p_1 \wedge p_2)(\zeta) = p_1(\zeta) \wedge p_2(\zeta).$$

Proof. The "if" direction follows from Lemma 3, while the "only if" direction is obvious. ∎

LEMMA 5. *For any $X \subset P$ and $q \in P$, we have*:

(a) *If $p \sqsubseteq q$ for all $p \in X$ and X is pairwise incompatible in $\langle P, \sqsubseteq \rangle$, then $\sum_{p \in X} \inf(p) \le \inf(q)$.*

(b) *If X is predense below q in $\langle P, \sqsubseteq \rangle$, then $\sum_{p \in X} \inf(p) \ge \inf(q)$.*

Proof. (a) Suppose that $p \sqsubseteq q$ for all $p \in X$ and X is pairwise incompatible in $\langle P, \sqsubseteq \rangle$. Without loss of generality, we may assume that X is finite. We will show that

$$\forall \delta \in \mathbb{R}^{>0}\colon \sum_{p \in X} \inf(p) \le \inf(q) + 2\delta.$$

Let $\delta \in \mathbb{R}^{>0}$. By Lemma 4,

$$\forall p_1, p_2 \in X\ (p_1 \neq p_2)\colon\ \inf(p_1 \wedge p_2) = 0.$$

So, since X is finite, there is a $\xi \in W$ such that

$$t = \sum_{p_1, p_2 \in X \wedge p_1 \neq p_2} \mu_\xi(p_1(\xi) \wedge p_2(\xi)) \le \delta \quad \text{and} \quad \mu_\xi(q(\xi)) \le \inf(q) + \delta.$$

We then have

$$\begin{aligned}\sum_{p \in X} \inf(p) &\le \sum_{p \in X} \mu_\xi(p(\xi)) \le \mu_\xi\Big(\bigvee\nolimits^{A_\xi}_{p \in X} p(\xi)\Big) + t \\ &\le \mu_\xi(q(\xi)) + t \le \inf(q) + 2\delta.\end{aligned}$$

(b) Suppose that $\sum_{p \in X} \inf(p) < \inf(q)$, and let $\delta \in \mathbb{R}^{>0}$ be such that

$$\inf(q) - \sum_{p \in X} \inf(p) \ge 2\delta.$$

Since $\sum_{p \in X} \inf(p)$ is finite, X must be at most countable. So there are numbers $\delta_p \in \mathbb{R}^{>0}$ $(p \in X)$ such that

$$\sum_{p \in X} \delta_p \le \delta.$$

Then we can choose elements $\xi_p \in W$ $(p \in X)$ so that

$$\forall p \in X\colon \mu_{\xi_p}(p(\xi_p)) \le \inf(p) + \delta_p.$$

Now define the function q' on W by

$$\forall \zeta \in W\colon q'(\zeta) = q(\zeta) \wedge \Big(-\bigvee\nolimits^{A_\zeta} \{\varrho_{\xi_p \zeta}(p(\xi_p)) \mid p \in X \wedge \xi_p \trianglelefteq \zeta\}\Big).$$

Clearly, $q' \in \Pi^{\#}$ and $q' \sqsubseteq q$. Also, for any $\zeta \in W$,

$$\begin{aligned}\mu_\zeta(q'(\zeta)) &\ge \mu_\zeta(q(\zeta)) - \sum_{p \in X \wedge \xi_p \trianglelefteq \zeta} \mu_{\xi_p}(p(\xi_p)) \\ &\ge \inf(q) - \sum_{p \in X}(\inf(p) + \delta_p) = \Big(\inf(q) - \sum_{p \in X} \inf(p)\Big) - \sum_{p \in X} \delta_p \ge \delta.\end{aligned}$$

Hence $\inf(q') > 0$. Therefore, by Lemma 3, we get a $q'' \in P$ such that $q'' \sqsubseteq q'$. Then $q'' \sqsubseteq q$, and since

$$\forall p \in X\colon \exists \zeta \in W\colon p(\zeta) \wedge q'(\zeta) = 0_{A_\zeta},$$

we also have

$$\forall p \in X\colon \exists \zeta \in W\colon p(\zeta) \wedge q''(\zeta) = 0_{A_\zeta},$$

whence q'' is incompatible with all $p \in X$. Thus X is not predense below q. ■

Proof of Theorem 1(b). The natural way to define a measure μ as required is as follows:

Given an $a \in A$, choose a pairwise incompatible $X \subset P$ such that $a = \bigvee^A \theta\text{''} X$, and put $\mu(a) = \sum_{p\in X} \inf(p)$.

This is, in fact, the approach that we will take. First we have to show that the value of $\mu(a)$ does not depend on the choice of the set X.

CLAIM. *If X and Y $(X, Y \subset P)$ are pairwise incompatible and $\bigvee^A \theta\text{''} X = \bigvee^A \theta\text{''} Y$, then $\sum_{p\in X} \inf(p) = \sum_{q\in Y} \inf(q)$.*

Proof. Without loss of generality, assume that X is a refinement of Y, i.e.

$$\forall p \in X\colon \exists q \in Y\colon p \sqsubseteq q,$$

so that we have

$$\forall q \in Y\colon \theta(q) = \bigvee^A \theta\text{''} X_q,$$

where for each $q \in Y$,

$$X_q = \{p \in X \mid p \sqsubseteq q\}.$$

It follows from Lemma 5 that

$$\forall q \in Y\colon \sum_{p\in X_q} \inf(p) = \inf(q).$$

Hence

$$\sum_{p\in X} \inf(p) = \sum_{q\in Y} \sum_{p\in X_q} \inf(p) = \sum_{q\in Y} \inf(q). \ ■$$

By the claim, we can define the map $\mu : A \to \mathbb{R}^{\geq 0}$ so that for any $a \in A$ and any $X \subset P$,

$$X \text{ is pairwise incompatible in } \langle P, \sqsubseteq\rangle \wedge a = \bigvee^A \theta\text{''} X \Rightarrow \mu(a) = \sum_{p\in X} \inf(p).$$

It is then routine to check that μ is a countably additive strictly positive probability measure on A such that

$$\forall \zeta \in W\colon \forall a \in A_\zeta\colon \mu_\zeta(a) = \mu(\varrho_\zeta(a)). \ ■$$

Proof of Theorem 2. We will show that

$$\forall a \in A\colon \exists X \subset P\colon a = \bigvee^A \theta\text{''} X.$$

By (L4), it suffices to prove that the set A' of all $a \in A$ such that

$$\exists X \subset P\colon a = \bigvee^A \theta\text{''} X$$

is a complete subalgebra of A including $\bigcup_{\zeta \in W} \operatorname{Ran}(\varrho_\zeta)$ as a subset.

Since, for any $\zeta \in W$ and any $a \in A_\zeta - \{0_{A_\zeta}\}$, that function p in Π such that $p(\eta) = \varrho_{\zeta\eta}(a)$ for all $\eta \in W$ $(\zeta \trianglelefteq \eta)$ is an element of P, we have $\bigcup_{\zeta \in W} \operatorname{Ran}(\varrho_\zeta) \subset A'$. It is obvious that A' is closed under the join operation $\bigvee$.

Showing that A' is closed under Boolean complementation requires two claims.

CLAIM 1. *For any incompatible* $p_1, p_2 \in P$, $\theta(p_1) \wedge \theta(p_2) = 0_A$.

Proof. If p_1 and p_2 are incompatible, then

$$\mu(\theta(p_1) \wedge \theta(p_2)) = \inf(p_1 \wedge p_2) = 0 \quad \text{(by Lemma 4)}.$$

Hence $\theta(p_1) \wedge \theta(p_2) = 0_A$. ∎

CLAIM 2. *For any predense* $X \subset P$, $\bigvee^A \theta\text{''} X = 1_A$.

Proof. Let X be an arbitrary predense subset of P. Since there is a pairwise incompatible predense set $X' \subset P$ such that

$$\forall p' \in X'\colon \exists p \in X\colon p' \sqsubseteq p,$$

there is no loss of generality in assuming that X is pairwise incompatible to begin with. Then, by Claim 1, the elements $\theta(p)$ $(p \in X)$ are pairwise disjoint in A. So

$$\mu\Big(\bigvee^A \theta\text{''} X\Big) = \sum_{p \in X} \mu(\theta(p)).$$

But

$$\sum_{p \in X} \mu(\theta(p)) = \sum_{p \in X} \inf(p) \geq 1 \quad \text{(by Lemma 5(b))}.$$

Thus $\mu(\bigvee^A \theta\text{''} X) \geq 1$, and we conclude that $\bigvee^A \theta\text{''} X = 1_A$. ∎

Proving that A' is also closed under the complement operation on the basis of Claims 1 and 2 is quite standard. Consider an arbitrary element $a = \bigvee^A \theta\text{''} X$ $(X \subset P)$ of A'. If we put

$$Y = \{q \in P \mid q \text{ is incompatible with all } p \in X\} \quad \text{and} \quad b = \bigvee^A \theta\text{''} Y,$$

then $a \wedge b = 0_A$ by Claim 1, and $a \vee b = \bigvee^A \theta\text{''}(X \cup Y) = 1_A$ by Claim 2, so that $-a = b \in A'$.

REFERENCES

[Bo] S. Bochner, *Harmonic Analysis and the Theory of Probability*, Univ. of California Press, 1955.
[Fr] D. H. Fremlin, *Measure Algebras*, in: Handbook of Boolean Algebras, Vol. 3, J. D. Monk (ed.), Elsevier, Amsterdam, 1989, 877–980.
[Je] T. Jech, *Set Theory*, Pure Appl. Math. 79, Academic Press, New York, 1978.
[Ku] K. Kunen, *Set Theory*, Stud. Logic Found. Math. 102, North-Holland, Amsterdam, 1980.

Tsurukabuto 3-11
Nada Ku
Kobe
657-8501 Japan
E-mail: takaj@kobe-u.ac.jp

Received 14 October 1997