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AN ELEMENTARY PROOF OF THE WEITZENBÖCK THEOREM

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Introduction. The main aim of this paper is to give an elementary and self-contained proof of the following classical result.

THEOREM (Weitzenböck [8]). *Let $\mathbb{C}^+$ be the additive group of the complex field $\mathbb{C}$ and let V be a finite-dimensional rational representation of $\mathbb{C}^+$. Then the algebra $\mathbb{C}[V]^{\mathbb{C}^+}$ of invariant polynomial functions on V is finitely generated.*

The first modern proof of the theorem is due to Seshadri [6] and it is geometric. Our proof is an algebraic version of Seshadri's proof.

As a consequence of our considerations and the main result of [5] for $G = \mathrm{SL}(2,\mathbb{C})$ we get the following.

THEOREM. *Let V be a finite-dimensional, rational, non-trivial representation of $\mathbb{C}^+$ determined by a nilpotent endomorphism f of the vector space V. Then*

1. *$\mathbb{C}[V]^{\mathbb{C}^+}$ is a Gorenstein ring.*
2. *$\mathbb{C}[V]^{\mathbb{C}^+}$ is a polynomial algebra if and only if $V = V_0 \oplus V'$ for some subrepresentations V_0, V' of V such that V_0 is trivial (that is, $f(V_0) = 0$) and the Jordan matrix of $f_{|V'} : V' \to V'$ is one of the following:*

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

This theorem is equivalent to the following.

THEOREM. *Let $A = \mathbb{C}[X_1, \ldots, X_n]$ and let $0 \neq d : A \to A$ be a locally nilpotent derivation such that $d(W) \subset W$, where $W = \mathbb{C}X_1 + \ldots + \mathbb{C}X_n \subset A$. Then*

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1. A^d ($=\operatorname{Ker} d$) *is a Gorenstein ring.*

2. A^d *is a polynomial algebra if and only if* $W = W_0 \oplus W'$ *for some subspaces* W_0, W' *of* W *such that* $d(W_0) = 0$, $d(W') \subset W'$, *and the Jordan matrix of the endomorphism* $d_{|W'} : W' \to W'$ *is one of the above matrices.*

1. Preliminaries and auxiliary lemmas. Throughout the paper all vector spaces, algebras, Lie algebras, and tensor products are defined over $\mathbb{C}$. All (associative) algebras are assumed to be commutative. We denote by L the simple Lie algebra $\mathrm{sl}(2,\mathbb{C}) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{C}) : a + d = 0 \right\}$. Let

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\{x, y, h\}$ is a linear basis of L and $[x, y] = h$, $[h, x] = 2x$, $[h, y] = -2y$. It is known (see for instance [2, Chap. II]) that every finite-dimensional L-module is semisimple, and for each $m = 0, 1, \ldots$ there exists only one (up to isomorphism) simple L-module $V_m = \langle v_0, \ldots, v_m \rangle$ (= linear span of $v_0, \ldots, v_m$) of dimension $m + 1$ with

$$\begin{aligned} x.v_i &= (m - i + 1)v_{i-1}, \\ y.v_i &= (i + 1)v_{i+1}, \\ h.v_i &= (m - 2i)v_i \end{aligned}$$

for $i = 0, \ldots, m$ ($v_{-1} = 0 = v_{m+1}$). In particular, it follows that if W is a finite-dimensional L-module, then the endomorphism $w \to x.w$ of W, as a vector space, is nilpotent.

By a *trivial* L-module we mean an L-module W such that $t.w = 0$ for all $t \in L$ and $w \in W$.

Given an L-module W, the trivial submodule $\{w \in W : \forall t \in L\ t.w = 0\}$ of W is called the *module of invariants* of W and it is denoted by W^L. Notice that $W^L = \{w \in W : x.w = 0 = y.w\}$. If $f : W \to W'$ is a homomorphism of L-modules, then $f(W^L) \subset W'^L$, and $f^L : W^L \to W'^L$ will denote the restriction of f to W^L. If W is an L-module, then W^* denotes the dual vector space provided with the L-module structure given by $(t.w^*)(w) = w^*(-t.w)$, $t \in L$, $w^* \in W^*$, $w \in W$.

An L-module W is said to be *locally finite* if W is a union of its finite-dimensional submodules. It is obvious that each locally finite L-module W is semisimple, that is, $W \cong \bigoplus_{i \in I} V_{m_i}$ for some set I. In particular, $W = W^L \oplus LW$, where $LW = \{\sum t_i.w_i : t_i \in L,\ w_i \in W\}$. Let $R_W : W \to W^L$ denote the natural projection. Then the R_W's define the *Reynold operator* on the category of locally finite L-modules, which means that the following conditions hold.

(i) For any locally finite L-module W, $R_W : W \to W^L$ is a surjective homomorphism of L-modules and $R_W(w) = w$ for $w \in W^L$.

(ii) If $f : W \to W'$ is a homomorphism of locally finite L-modules, then $f^L \circ R_W = R_{W'} \circ f$.

In fact, (i) follows immediately from the definition of R_W, and (ii) holds because $f(LW) \subset LW'$.

An algebra A is an *L-module algebra* if A is an L-module and for each $t \in L$ the map $d_t : A \to A$, $d_t(a) = t.a$, is a derivation of A. If A is an L-module algebra, then A^L is a subalgebra of A called the *algebra of invariants*. An L-module algebra A is called *locally finite* if A is locally finite as an L-module. If this is the case, then we have the Reynold operator $R = R_A : A \to A^L$. It turns out that R is an A^L-linear map, that is, $R(ay) = aR(y)$ for $a \in A^L$ and $y \in A$. To see this, it suffices to apply the condition (ii) of the Reynold operator to the homomorphism of L-modules $f : A \to A$ given by $f(y) = ay$.

Let W be an L-module. Then the symmetric algebra $S(W)$ will be viewed as an L-module algebra via

$$t.(w_1 \ldots w_m) = \sum_{i=1}^{m} w_1 \ldots w_{i-1}(t.w_i)w_{i+1} \ldots w_m$$

for $t \in L$ and $w_1, \ldots, w_m \in W \subset S(W)$. It is obvious that $S(W)$ is locally finite if W is finite-dimensional. In particular, for any finite-dimensional L-module W we have the locally finite L-module algebra $S(W^*)$.

LEMMA 1. *If W is a finite-dimensional L-module, then the algebra $S(W)^L$ of invariants is finitely generated.*

Proof. Notice that $S(W)^L$ is a graded subalgebra of the graded algebra $S(W) = \bigoplus_{n=0}^{\infty} S^n(W)$. Therefore, in order to show that $S(W)^L$ is finitely generated it suffices to prove that the ring $S(W)^L$ is noetherian.

Let I be an ideal in $S(W)^L$. Since the ring $S(W)$ is noetherian, there are $a_1, \ldots, a_n \in I$ such that $IS(W) = a_1S(W) + \ldots + a_nS(W)$. Our claim is that $I = (a_1, \ldots, a_n)$. Obviously $(a_1, \ldots, a_n) \subset I$. Let $a \in I$. Then $a = a_1 = a_1y_1 + \ldots + a_ny_n$ for some $y_i \in S(W)$. Hence $a = R(a) = a_1R(y_1) + \ldots + a_nR(y_n) \in (a_1, \ldots, a_n)$, because $R = R_A$ is A^L-linear. This implies that $I = (a_1, \ldots, a_n)$, which means that the ring $S(W)^L$ is noetherian. ∎

From now on, given a finite-dimensional vector space V (respectively, a finite-dimensional L-module V), $\mathbb{C}[V]$ will stand for the algebra $S(V^*)$ (respectively, for the L-module algebra $S(V^*)$) considered as the algebra of polynomial functions on V.

LEMMA 2. *Let V be a finite-dimensional vector space.*

(i) *If $f : V \to V$ is a nilpotent endomorphism of V, then there exists a unique* (*up to isomorphism*) *L-module structure $\psi : L \times V \to V$ on V such that $f(v) = x.v$, where $x.v = \psi(x, v)$. More precisely, (V, ψ) is isomorphic to $V_{m_1} \oplus \ldots \oplus V_{m_s}$, where $m_1 + 1, \ldots, m_s + 1$ are the dimensions of the Jordan cells of f.*

(ii) *If $d : \mathbb{C}[V] \to \mathbb{C}[V]$ is a locally nilpotent derivation of $\mathbb{C}[V]$ with $d(V^*) \subset V^*$, then there exists a* (*unique*) *L-module structure $\psi : L \times V \to V$ on V such that $d = d_x : \mathbb{C}[(V, \psi)] \to \mathbb{C}[(V, \psi)]$.*

Proof. (i) The Jordan matrix of f equals

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_s \end{pmatrix},$$

where all A_i's (the Jordan cells of f) are of the form

$$\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

Let $m_i = \dim A_i - 1$ for $i = 1, \ldots, s$. We can assume that $m_1 \leq \ldots \leq m_s$. Then $V = W_1 \oplus \ldots \oplus W_s$ and $f = f_1 \oplus \ldots \oplus f_s$ for some subspaces W_i of dimension $m_i + 1$ and nilpotent endomorphisms $f_i : W_i \to W_i$ with Jordan matrices A_i, $i = 1, \ldots, s$.

First assume that $s = 1$. Then there exists a basis $v'_0, \ldots, v'_m$, $m = \dim V - 1$, of V with $f(v'_i) = v'_{i-1}$ for $i = 0, \ldots, m$ ($v'_{-1} = 0$). Set $v_i = v'_i/(m - i)!$, $i = 0, \ldots, m$. Then $f(v_i) = (m - i + 1)v_{i-1}$, so that putting $\psi(x, v_i) = (m - i + 1)v_{i-1}$, $\psi(y, v_i) = (i + 1)v_{i+1}$, and $\psi(h, v_i) = (m - 2i)v_i$, $i = 0, \ldots, m$ ($v_{m+1} = 0$), we get an L-module structure $\psi : L \times V \to V$ such that $(V, \psi) = V_m$.

If s is arbitrary, then we apply the above procedure to each f_i, $i = 1, \ldots, s$. As a result one obtains an L-module structure $\psi : L \times V \to V$ such that $(V, \psi) = V_{m_1} \oplus \ldots \oplus V_{m_s}$.

It remains to prove the uniqueness of ψ. Suppose that $\psi' : L \times V \to V$ makes V an L-module in such a way that $f(v) = \psi'(x, v)$ for all $v \in V$. Then $(V, \psi') \cong V_{n_1} \oplus \ldots \oplus V_{n_r}$ for some $0 \leq n_1 \leq \ldots \leq n_r$. But the relation $f(v) = \psi'(x, v)$, $v \in V$, implies that $r = s$ and $n_1 = m_1, \ldots, n_s = m_s$. This proves part (i).

(ii) Since the evaluation map $\mathrm{ev} : V \to V^{**}$, $\mathrm{ev}(v)(v^*) = v^*(v)$, $v^* \in V^*$, $v \in V$, is an isomorphism, there is an endomorphism f of V such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}} \\ V^{**} & \xrightarrow[g]{} & V^{**} \end{array}$$

where $g(s)(v^*) = -s(d(v^*))$, $s \in V^{**}$, $v^* \in V^*$. It is obvious that f is nilpotent because $d_{|V^*} : V^* \to V^*$ is nilpotent. So, applying (i) to f we get an L-module structure $\psi : L \times V \to V$ such that $f(v) = \psi(x, v)$ for all $v \in V$. In particular, we have the induced derivation $d_x : \mathbb{C}[(V, \psi)] \to \mathbb{C}[(V, \psi)]$. For $v^* \in V^*$, $v \in V$,

$$\begin{aligned} d_x(v^*)(v) &= v^*(-\psi(x, v)) = -v^*(f(v)) = -\operatorname{ev}(f(v))(v^*) \\ &= -g \circ \operatorname{ev}(v)(d(v^*)) = d(v^*)(v), \end{aligned}$$

which means that $d_x = d$ on $V^* \subset \mathbb{C}[V]$. This, however, implies that $d_x = d$. ■

Below, U will denote the vector space $\mathbb{C}^2$ provided with the natural L-module structure given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}.$$

Then $\mathbb{C}[U] = \mathbb{C}[X, Y]$, where $X, Y \in (\mathbb{C}^2)^*$, $X(z_1, z_2) = z_1$, $Y(z_1, z_2) = z_2$, and the L-module algebra structure on $\mathbb{C}[U]$ is determined by

$$(1) \qquad d_x(X) = -Y, \quad d_x(Y) = 0 = d_y(X), \quad d_y(Y) = -X.$$

If A, B are L-module algebras, then the tensor product $A \otimes B$ is an L-module algebra via $t.(a \otimes b) = t.a \otimes b + a \otimes t.b$, where $t \in L$, $a \in A$, $b \in B$. In particular, for any L-module algebra A we have the L-module algebra $A[X, Y] = A \otimes \mathbb{C}[U]$. Observe that $A[X, Y] = \mathbb{C}[V \oplus U]$ whenever $A = \mathbb{C}[V]$ for some finite-dimensional L-module V.

LEMMA 3. *Let A be a locally finite L-module algebra. Then the homomorphism of algebras $\Phi : A[X, Y] \to A$, $\Phi(f(X, Y)) = f(1, 0)$, induces an isomorphism of algebras $\Phi : A[X, Y]^L \to A^x$, where $A^x = \{a \in A : x.a = 0\}$ $= \{a \in A : d_x(a) = 0\}$.*

Proof. Let $f = \sum_{k=0}^s f_k(X) Y^k \in A[X, Y]$ and let $f_k(X) = \sum_{j \geq 0} a_j^{(k)} X^j$, $k = 0, \ldots, s$. Using the formulas (1), we easily verify that $d_x(f) = 0 = d_y(f)$ if and only if the following conditions hold:

$$(2) \quad d_x(a_j^{(0)}) = 0 = f_s'(X), \quad d_x(a_j^{(k)}) = (j+1) a_{j+1}^{(k-1)}, \quad k = 1, \ldots, s,\ j \geq 0,$$

$$(3) \quad \begin{aligned} & d_y(a_0^{(k)}) = 0, && k = 0, \ldots, s, \\ & d_y(a_j^{(k)}) = (k+1) a_{j-1}^{(k+1)}, && k = 0, \ldots, s-1,\ j \geq 1. \end{aligned}$$

From (3), by induction on k, we get

$$a_j^{(k+1)} = \frac{1}{(k+1)!} d_y^{k+1}(a_{j+k+1}^{(0)}), \qquad k = 0, \ldots, s-1,\ j \geq 0. \tag{4}$$

It turns out that also

$$d_h(a_j^{(0)}) = j a_j^{(0)} \qquad \text{for } j \geq 0. \tag{5}$$

In fact, by (2) and (3), $d_h(a_j^{(0)}) = d_x d_y(a_j^{(0)}) - d_y d_x(a_j^{(0)}) = d_x d_y(a_j^{(0)}) = d_x(a_{j-1}^{(1)}) = j a_j^{(0)}$ if $j \geq 1$, and $d_h(a_0^{(0)}) = 0$ because $d_y(a_0^{(0)}) = 0$.

From (5) it follows that the set $\{a_j^{(0)} : j \geq 0\} \setminus \{0\}$ is linearly independent (over $\mathbb{C}$). From (2) we know that if $f \in A[X,Y]^L$, then $\Phi(f) = f(1,0) = f_0(1) = \sum_{j\geq 0} a_j^{(0)} \in A^x$. Therefore, the homomorphism of algebras Φ induces a homomorphism of algebras

$$\Phi : A[X,Y]^L \to A^x.$$

If $\Phi(f) = 0$ for some $f \in A[X,Y]^L$, that is, $\sum_{j\geq 0} a_j^{(0)} = 0$, then $a_j^{(0)} = 0$ for all $j \geq 0$, because the set $\{a_j^{(0)} : j \geq 0\} \setminus \{0\}$ is linearly independent. In view of (4), this yields $f = 0$.

It remains to prove that Φ is surjective. Since A is locally finite as an L-module, $A = \bigoplus_{i\in I} V_{m_i}$ for some set I. It follows that $A = \bigoplus_{j\in\mathbb{Z}} A_j$, where $A_j = \{a \in A : d_h(a) = ja\}$. Observe also that $\{v \in V_m : x.v = 0\} = \langle v_0 \rangle$ for each $m \geq 0$. Hence

$$A^x = \bigoplus_{j\geq 0} A_j \cap A^x. \tag{6}$$

Now we show the following:

(7) If $a \in A_m \cap A^x$ for some $m \geq 0$, then $d_y^{m+1}(a) = 0$ and $d_x d_y^j(a) = (m-j+1) j d_y^{j-1}(a)$ for $j = 1, \ldots, m+1$.

Let $d_h(a) = ma$ and $d_x(a) = 0$ for some $a \in A$ and $m \geq 0$. We can assume that $a \in V_{m_i}$ for some $i \in I$. Then obviously $m_i = m$ and $a = \alpha v_0$ for an $\alpha \in \mathbb{C}$, whence $d_y^j(a) = \alpha j! v_j$ for all $j \geq 1$ ($v_j = 0$ if $j > m$). In particular, $d_y^{m+1}(a) = 0$. Furthermore, $d_x d_y^j(a) = d_x(\alpha j! v_j) = \alpha j! x.v_j = \alpha(m-j+1) j (j-1)! v_{j-1} = (m-j+1) j d_y^{j-1}(a)$, $j = 1, \ldots, m+1$. So, the statement (7) is proved.

In order to prove that $\Phi : A[X,Y]^L \to A^x$ is surjective take an $a \in A^x$. By (6), we can assume that $a \in A_s \cap A^x$ for some $s \geq 0$. Set

$$f_k(X) = \frac{1}{k!} d_y^k(a) X^{s-k}, \qquad k = 0, \ldots, s,$$

and let

$$f(X,Y) = f_0(X) + f_1(X)Y + \ldots + f_s(X)Y^s.$$

Making use of (2), (3), and (7), one easily checks that $f \in A[X,Y]^L$. Moreover, $\psi(f) = f(1,0) = f_0(1) = a$. This completes the proof of Lemma 3. ∎

Given a derivation d of an algebra B, B^d will denote the algebra of constants of d, i.e., $B^d = \operatorname{Ker} d$.

2. Results. Let $\mathbb{C}^+$ denote the additive group of the complex field $\mathbb{C}$. We consider $\mathbb{C}^+$ as an algebraic group with the algebra of regular functions $\mathbb{C}[X]$. Then a *rational representation* of $\mathbb{C}^+$ is a linear space V together with an action of $\mathbb{C}^+$ on V such that, given $z \in \mathbb{C}^+$, $v \in V$,

$$z.v = \sum_{i \geq 0} \frac{f^i(v)}{i!} z^i$$

for some locally nilpotent endomorphism $f : V \to V$. The endomorphism f is uniquely determined by the action, and f is nilpotent whenever V is finite-dimensional.

Let V be a finite-dimensional rational representation of $\mathbb{C}^+$ determined by the endomorphism $f : V \to V$. Then we have the induced action of $\mathbb{C}^+$ on the algebra $\mathbb{C}[V]$ defined by $(z.a)(v) = a(-z.v)$ for $a \in \mathbb{C}[V]$, $z \in \mathbb{C}^+$, $v \in V$. It is easy to check that this action is given by

$$(*) \qquad z.a = \sum_{i \geq 0} \frac{d^i(a)}{i!} z^i,$$

where d is the derivation of $\mathbb{C}[V]$ determined by $d(v^*) = -v^* \circ f$ for $v^* \in V^* \subset \mathbb{C}[V]$. This implies that $\mathbb{C}[V]^{\mathbb{C}^+} = \{a \in \mathbb{C}[V] : \forall z \in \mathbb{C}^+ \ z.a = a\} = \mathbb{C}[V]^d$. Notice also that d is locally nilpotent and $d(V^*) \subset V^*$.

THEOREM 1. *If V is a finite-dimensional rational representation of $\mathbb{C}^+$, then the algebra $\mathbb{C}[V]^{\mathbb{C}^+}$ is finitely generated.*

Proof. As stated above, the action of $\mathbb{C}^+$ on $\mathbb{C}[V]$ is given by $(*)$, where $d : \mathbb{C}[V] \to \mathbb{C}[V]$ is a locally nilpotent derivation such that $d(V^*) \subset V^*$ and $\mathbb{C}[V]^{\mathbb{C}^+} = \mathbb{C}[V]^d$.

Using Lemma 2(ii) we see that there exists an L-module structure on V such that $d = d_x$. Applying Lemma 3 to $A = \mathbb{C}[V]$ and taking into account that $A[X,Y] = \mathbb{C}[V \oplus U]$ we obtain

$$\mathbb{C}[V]^{\mathbb{C}^+} = \mathbb{C}[V]^d = \mathbb{C}[V]^{d_x} \cong \mathbb{C}[V \oplus U]^L.$$

Now from Lemma 1 it follows that $\mathbb{C}[V]^{\mathbb{C}^+}$ is a finitely generated algebra. ∎

For the proof of the next theorem we have to recall some well-known links between locally finite L-modules and rational G-modules (= rational representations of G), where $G = \mathrm{SL}(2,\mathbb{C}) = \{M \in M_2(\mathbb{C}) : \det M = 1\}$. Since L is the Lie algebra of the algebraic group G, for any rational G-module structure $\varphi : G \times V \to V$ on a vector space V we have the associated locally finite L-module structure $\widetilde{\varphi} : L \times V \to V$ on V (analytically, $\widetilde{\varphi}(t,v) = (\partial/\partial s)\,\varphi(\exp(st),v)_{|s=0}$ for $t \in L$, $v \in V$). The map $\widetilde{\varphi}$ uniquely determines φ and $(V,\varphi)^G = \{v \in V' : \forall g \in G\ \varphi(g,v) = v\} = \{v \in V : \forall t \in L\ \widetilde{\varphi}(t,v) = 0\} = (V,\widetilde{\varphi})^L$. Moreover, if (V,φ) is a rational G-module and $\Phi : G \times S(V) \to S(V)$ is the induced action of G on the symmetric algebra $S(V)$, then $\widetilde{\Phi} : L \times S(V) \to S(V)$ is the previously defined L-module algebra structure on $S(V)$. In particular, $S(V,\varphi)^G = S(V,\widetilde{\varphi})^L$.

It is known ([7, Chap. 3]) that every rational G-module is semisimple and that for any $m \ge 0$ there exists a unique (up to isomorphism) simple rational G-module ϱ_m of dimension $m+1$. It is not difficult to show that the L-module associated with ϱ_m is isomorphic to V_m for all $m \ge 0$. As a consequence of the above facts we get the following.

Corollary 4. *Let V be a finite-dimensional vector space. Then for any L-module structure $\psi : L \times V \to V$ on V there exists a unique rational G-module structure $\varphi : G \times V \to V$ on V such that the following conditions hold.*

(a) $\widetilde{\varphi} = \psi$,

(b) $(V,\varphi) \cong \varrho_{m_1} \oplus \ldots \oplus \varrho_{m_s}$ *for some* $m_1, \ldots, m_s$ *if and only if* $(V,\psi) \cong V_{m_1} \oplus \ldots \oplus V_{m_s}$.

(c) $S(V,\psi)^L \cong S(V,\varphi)^G$.

The G-module (V,φ) is called the *lifting* of the L-module (V,ψ).

Theorem 2. *Let V be a finite-dimensional rational representation determined by a non-zero nilpotent endomorphism f of the vector space V. Then*

1. $\mathbb{C}[V]^{\mathbb{C}^+}$ *is a Gorenstein ring.*

2. $\mathbb{C}[V]^{\mathbb{C}^+}$ *is a polynomial algebra if and only if $V = V_{(0)} \oplus V'$ for some subrepresentations $V_{(0)}$, V' of V such that $\mathbb{C}^+$ acts trivially on $V_{(0)}$ (i.e., $f(V_{(0)}) = 0$) and the Jordan matrix of $f' : V' \to V'$, $f'(v) = f(v)$, is one of the following*:

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Proof. As in the proof of Theorem 1, $\mathbb{C}[V]^{\mathbb{C}^+} \cong \mathbb{C}[V \oplus U]^L$ for some L-module structure $\psi : L \times V \to V$ such that $\psi(x,v) = f(v)$ for $v \in V$. But U, being a simple L-module of dimension 2, is isomorphic to V_1, so that $\mathbb{C}[V]^{\mathbb{C}^+} \cong \mathbb{C}[V \oplus V_1]^L$. According to Corollary 4, there exists a unique rational $G = \mathrm{SL}(2,\mathbb{C})$-module structure φ on V such that $\widetilde{\varphi} = \psi$. This implies that $\mathbb{C}[V]^{\mathbb{C}^+} \cong \mathbb{C}[(V,\varphi)\oplus\varrho_1]^G$, because ϱ_1 is the lifting of V_1. Now part 1 of the theorem follows, because, as is well known, $\mathbb{C}[W]^G$ is a Gorenstein ring for any finite-dimensional rational G-module W (see [1, Remark 6.5.5]).

For part 2, in view of [5, Example following Thm. 3], $\mathbb{C}[V]^{\mathbb{C}^+} \cong \mathbb{C}[(V,\varphi)\oplus \varrho_1]^G$ is a polynomial algebra if and only if there exists a trivial submodule V_t of the G-module (V,φ) such that $(V,\varphi)\oplus\varrho_1$ is isomorphic to one of the G-modules: $V_t\oplus\varrho_1\oplus\varrho_1$, $V_t\oplus\varrho_2\oplus\varrho_1$, $V_t\oplus\varrho_1\oplus\varrho_1\oplus\varrho_1$. It follows that $\mathbb{C}[V]^{\mathbb{C}^+}$ is a polynomial algebra if and only if (V,φ) is isomorphic to one of the G-modules: $V_t\oplus\varrho_1$, $V_t\oplus\varrho_2$, $V_t\oplus\varrho_1\oplus\varrho_1$. By Corollary 4(b), this in turn implies that $\mathbb{C}[V]^{\mathbb{C}^+}$ is a polynomial algebra if and only if the L-module (V,ψ) is isomorphic to one of the L-modules: $V_{(0)}\oplus V_1$, $V_{(0)}\oplus V_2$, $V_{(0)}\oplus V_1\oplus V_1$, where $V_{(0)}$ is the trivial L-module structure on V_t as a vector space. The conclusion now follows from Lemma 2(i) applied to $f: V\to V$. The theorem is proved. ■

REMARK. Part 1 of the theorem was announced in [4].

COROLLARY 5. *Let $A=\mathbb{C}[X_1,\ldots,X_n]$ and let $d\neq 0$ be a locally nilpotent derivation of A with $d(W)\subset(W)$, where $W=\mathbb{C}X_1+\ldots+\mathbb{C}X_n\subset A$.*

1. *A^d is a Gorenstein ring.*
2. *A^d is a polynomial algebra if and only if $W=W_0\oplus W'$ for some subspaces W_0, W' of W such that $d(W_0)=0$, $d(W')\subset W'$, and the Jordan matrix of $d_{|W'}: W'\to W'$ is one of the three matrices appearing in Theorem 2.*

Proof. We can consider A as the algebra $\mathbb{C}[V]$, where $V=\mathbb{C}^n$. Then $W=V^*$, and hence there exists an endomorphism $f: V\to V$ such that $-f^*=d_{|W}: W\to W$. Since $d_{|W}$ is nilpotent, the endomorphism f is also nilpotent. Therefore, the formula

$$z.v=\sum_{i\geq 0}\frac{f^i(v)}{i!}z^i,\qquad z\in\mathbb{C}^+,\ v\in V,$$

makes V a rational representation of $\mathbb{C}^+$ such that $\mathbb{C}[V]^{\mathbb{C}^+}=\mathbb{C}[V]^d=A^d$. Now, the corollary is a consequence of Theorem 2, because the Jordan matrices of f and $\pm f^*$ coincide. ■

REMARK 6. It is easy to see that the corollary is equivalent to Theorem 2.

REMARK 7. Let $M=(a_{ij})\in M_n(\mathbb{C})$ be a nilpotent matrix and let $d: A\to A$, $A=\mathbb{C}[X_1,\ldots,X_n]$, be the locally nilpotent derivation defined by

$$d(X_i)=\sum_{i=1}^{n}a_{ij}X_j,\qquad i=1,\ldots,n.$$

Essentially, the implication $\Leftarrow$ in part 2 of Corollary 5 says that A^d is a polynomial algebra if M is one of the three matrices appearing in Theorem 2. But this follows also from [3], where it was shown that

- $A^d = \mathbb{C}[X_1]$ if $n = 2$ and $M = \begin{pmatrix} 0&1 \\ 0&0 \end{pmatrix}$ (this is obvious),
- $A^d = \mathbb{C}[X_1, X_2^2 - 2X_1X_3]$ if $n = 3$ and $M = \begin{pmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{pmatrix}$ [3, Ex. 6.8.1], and
- $A^d = \mathbb{C}[X_1, X_3, X_2X_3 - X_1X_4]$ if $n = 4$ and $M = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{pmatrix}$ [3, Prop. 6.9.5].

In fact, since $\operatorname{tr} \deg_{\mathbb{C}} A^D = n - 1$ for any locally nilpotent derivation $D : A \to A$, $D \neq 0$, the generators of A^d in the above three cases are algebraically independent.

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