AN ELEMENTARY PROOF OF THE WEITZENBÖCK THEOREM

BY

ANDRZEJ TYC (TORUŃ)

Introduction. The main aim of this paper is to give an elementary and self-contained proof of the following classical result.

THEOREM (Weitzenböck [8]). Let \( \mathbb{C}^+ \) be the additive group of the complex field \( \mathbb{C} \) and let \( V \) be a finite-dimensional rational representation of \( \mathbb{C}^+ \). Then the algebra \( \mathbb{C}[V]^{\mathbb{C}^+} \) of invariant polynomial functions on \( V \) is finitely generated.

The first modern proof of the theorem is due to Seshadri [6] and it is geometric. Our proof is an algebraic version of Seshadri’s proof.

As a consequence of our considerations and the main result of [5] for \( G = \text{SL}(2, \mathbb{C}) \) we get the following.

THEOREM. Let \( V \) be a finite-dimensional, rational, non-trivial representation of \( \mathbb{C}^+ \) determined by a nilpotent endomorphism \( f \) of the vector space \( V \). Then

1. \( \mathbb{C}[V]^{\mathbb{C}^+} \) is a Gorenstein ring.
2. \( \mathbb{C}[V]^{\mathbb{C}^+} \) is a polynomial algebra if and only if \( V = V_0 \oplus V' \) for some subrepresentations \( V_0, V' \) of \( V \) such that \( V_0 \) is trivial (that is, \( f(V_0) = 0 \)) and the Jordan matrix of \( f|_{V'} : V' \rightarrow V' \) is one of the following:

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

This theorem is equivalent to the following.

THEOREM. Let \( A = \mathbb{C}[X_1, \ldots, X_n] \) and let \( 0 \neq d : A \rightarrow A \) be a locally nilpotent derivation such that \( d(W) \subset W \), where \( W = \mathbb{C}X_1 + \ldots + \mathbb{C}X_n \subset A \). Then

1991 Mathematics Subject Classification: Primary 14A50.
1. $A^d$ (= Ker $d$) is a Gorenstein ring.
2. $A^d$ is a polynomial algebra if and only if $W = W_0 \oplus W'$ for some subspaces $W_0, W'$ of $W$ such that $d(W_0) = 0, d(W') \subset W'$, and the Jordan matrix of the endomorphism $d_{|W'} : W' \to W'$ is one of the above matrices.

1. Preliminaries and auxiliary lemmas. Throughout the paper all vector spaces, algebras, Lie algebras, and tensor products are defined over $\mathbb{C}$. All (associative) algebras are assumed to be commutative. We denote by $L$ the simple Lie algebra $\text{sl}(2, \mathbb{C}) = \{ (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in M_2(\mathbb{C}) : a + d = 0 \}$. Let $x = (0 1 0), y = (0 0 1 0), h = (1 0 0 -1)$.

Then $\{x, y, h\}$ is a linear basis of $L$ and $[x, y] = h, [h, x] = 2x, [h, y] = -2y$. It is known (see for instance [2, Chap. II]) that every finite-dimensional $L$-module is semisimple, and for each $m = 0, 1, \ldots$ there exists only one (up to isomorphism) simple $L$-module $V_m = \langle v_0, \ldots, v_m \rangle$ (= linear span of $v_0, \ldots, v_m$) of dimension $m + 1$ with

$$x.v_i = (m - i + 1)v_{i-1},$$
$$y.v_i = (i + 1)v_{i+1},$$
$$h.v_i = (m - 2i)v_i$$

for $i = 0, \ldots, m \ (v_{-1} = 0 = v_{m+1})$. In particular, it follows that if $W$ is a finite-dimensional $L$-module, then the endomorphism $w \to x.w$ of $W$, as a vector space, is nilpotent.

By a trivial $L$-module we mean an $L$-module $W$ such that $t.w = 0$ for all $t \in L$ and $w \in W$.

Given an $L$-module $W$, the trivial submodule $\{w \in W : \forall t \in L \ t.w = 0 \}$ of $W$ is called the module of invariants of $W$ and it is denoted by $W^L$. Notice that $W^L = \{w \in W : x.w = 0 = y.w\}$. If $f : W \to W'$ is a homomorphism of $L$-modules, then $f(W^L) \subset W'^L$, and $f^L : W^L \to W'^L$ will denote the restriction of $f$ to $W^L$. If $W$ is an $L$-module, then $W^*$ denotes the dual vector space provided with the $L$-module structure given by $(t.w^*)(w) = w^*(-t.w), t \in L, w^* \in W^*, w \in W$.

An $L$-module $W$ is said to be locally finite if $W$ is a union of its finite-dimensional submodules. It is obvious that each locally finite $L$-module $W$ is semisimple, that is, $W \cong \bigoplus_{i \in I} V_{m_i}$ for some set $I$. In particular, $W = W^L \oplus LW$, where $LW = \{ \sum t_i.w_i : t_i \in L, w_i \in W \}$. Let $R_W : W \to W^L$ denote the natural projection. Then the $R_W$’s define the Reynolds operator on the category of locally finite $L$-modules, which means that the following conditions hold.
(i) For any locally finite $L$-module $W$, $R_W : W \to W^L$ is a surjective homomorphism of $L$-modules and $R_W(w) = w$ for $w \in W^L$.

(ii) If $f : W \to W'$ is a homomorphism of locally finite $L$-modules, then $f^L \circ R_W = R_{W'} \circ f$.

In fact, (i) follows immediately from the definition of $R_W$, and (ii) holds because $f(LW) \subseteq LW'$.

An algebra $A$ is an $L$-module algebra if $A$ is an $L$-module and for each $t \in L$ the map $d_t : A \to A$, $d_t(a) = t.a$, is a derivation of $A$. If $A$ is an $L$-module algebra, then $A^L$ is a subalgebra of $A$ called the algebra of invariants. An $L$-module algebra $A$ is called locally finite if $A$ is locally finite as an $L$-module. If this is the case, then we have the Reynold operator $R = R_A : A \to A^L$. It turns out that $R$ is an $A^L$-linear map, that is, $R(ay) = aR(y)$ for $a \in A^L$ and $y \in A$. To see this, it suffices to apply the condition (ii) of the Reynold operator to the homomorphism of $L$-modules $f : A \to A$ given by $f(y) = ay$.

Let $W$ be an $L$-module. Then the symmetric algebra $S(W)$ will be viewed as an $L$-module algebra via

$$t.(w_1 \ldots w_m) = \sum_{i=1}^{m} w_1 \ldots w_{i-1}(t.w_i)w_{i+1} \ldots w_m$$

for $t \in L$ and $w_1, \ldots, w_m \in W \subseteq S(W)$. It is obvious that $S(W)$ is locally finite if $W$ is finite-dimensional. In particular, for any finite-dimensional $L$-module $W$ we have the locally finite $L$-module algebra $S(W^*)$.

**Lemma 1.** If $W$ is a finite-dimensional $L$-module, then the algebra $S(W)^L$ of invariants is finitely generated.

**Proof.** Notice that $S(W)^L$ is a graded subalgebra of the graded algebra $S(W) = \bigoplus_{n=0}^{\infty} S^n(W)$. Therefore, in order to show that $S(W)^L$ is finitely generated it suffices to prove that the ring $S(W)^L$ is noetherian.

Let $I$ be an ideal in $S(W)^L$. Since the ring $S(W)$ is noetherian, there are $a_1, \ldots, a_n \in I$ such that $IS(W) = a_1S(W) + \ldots + a_nS(W)$. Our claim is that $I = (a_1, \ldots, a_n)$. Obviously $(a_1, \ldots, a_n) \subseteq I$. Let $a \in I$. Then $a = a_1 = a_1y_1 + \ldots + a_ny_n$ for some $y_i \in S(W)$. Hence $a = R(a) = a_1R(y_1) + \ldots + a_nR(y_n) \in (a_1, \ldots, a_n)$, because $R = R_A$ is $A^L$-linear. This implies that $I = (a_1, \ldots, a_n)$, which means that the ring $S(W)^L$ is noetherian.

From now on, given a finite-dimensional vector space $V$ (respectively, a finite-dimensional $L$-module $V$), $\mathbb{C}[V]$ will stand for the algebra $S(V^*)$ (respectively, for the $L$-module algebra $S(V^*)$) considered as the algebra of polynomial functions on $V$. 

Lemma 2. Let $V$ be a finite-dimensional vector space.

(i) If $f : V \to V$ is a nilpotent endomorphism of $V$, then there exists a unique (up to isomorphism) $L$-module structure $\psi : L \times V \to V$ on $V$ such that $f(v) = x.v$, where $x.v = \psi(x, v)$. More precisely, $(V, \psi)$ is isomorphic to $V_{m_1} \oplus \ldots \oplus V_{m_s}$, where $m_1 + 1, \ldots, m_s + 1$ are the dimensions of the Jordan cells of $f$.

(ii) If $d : \mathbb{C}[V] \to \mathbb{C}[V]$ is a locally nilpotent derivation of $\mathbb{C}[V]$ with $d(V^*) \subset V^*$, then there exists a (unique) $L$-module structure $\psi : L \times V \to V$ on $V$ such that $d = d_x : \mathbb{C}[(V, \psi)] \to \mathbb{C}[(V, \psi)]$.

Proof. (i) The Jordan matrix of $f$ equals

$$
\begin{pmatrix}
A_1 & 0 \\
& \ddots \\
& & A_s \\
0 & & & \\
& & & 0
\end{pmatrix},
$$

where all $A_i$'s (the Jordan cells of $f$) are of the form

$$
\begin{pmatrix}
0 & 1 & 0 \\
& \ddots & \ddots \\
& & 0 & 1 \\
0 & & &
\end{pmatrix}.
$$

Let $m_i = \dim A_i - 1$ for $i = 1, \ldots, s$. We can assume that $m_1 \leq \ldots \leq m_s$. Then $V = W_1 \oplus \ldots \oplus W_s$ and $f = f_1 \oplus \ldots \oplus f_s$ for some subspaces $W_i$ of dimension $m_i + 1$ and nilpotent endomorphisms $f_i : W_i \to W_i$ with Jordan matrices $A_i$, $i = 1, \ldots, s$.

First assume that $s = 1$. Then there exists a basis $v'_0, \ldots, v'_m$, $m = \dim V - 1$, of $V$ with $f(v'_i) = v'_{i-1}$ for $i = 0, \ldots, m$ ($v'_{-1} = 0$). Set $v_i = v'_i/(m - i)!$, $i = 0, \ldots, m$. Then $f(v_i) = (m - i + 1)v_{i-1}$, so that putting $\psi(x, v_i) = (m - i + 1)v_{i-1}$, $\psi(y, v_i) = (i + 1)v_{i+1}$, and $\psi(h, v_i) = (m - 2i)v_i$, $i = 0, \ldots, m$ ($v_{m+1} = 0$), we get an $L$-module structure $\psi : L \times V \to V$ such that $(V, \psi) = V_m$.

If $s$ is arbitrary, then we apply the above procedure to each $f_i$, $i = 1, \ldots, s$. As a result one obtains an $L$-module structure $\psi : L \times V \to V$ such that $(V, \psi) = V_{m_1} \oplus \ldots \oplus V_{m_s}$.

It remains to prove the uniqueness of $\psi$. Suppose that $\psi' : L \times V \to V$ makes $V$ an $L$-module in such a way that $f(v) = \psi'(x, v)$ for all $v \in V$. Then $(V, \psi') \cong V_{n_1} \oplus \ldots \oplus V_{n_s}$ for some $0 \leq n_1 \leq \ldots \leq n_s$. But the relation $f(v) = \psi'(x, v)$, $v \in V$, implies that $r = s$ and $n_1 = m_1, \ldots, n_s = m_s$. This proves part (i).

(ii) Since the evaluation map $ev : V \to V^{**}$, $ev(v)(v^*) = v^*(v)$, $v^* \in V^*$, $v \in V$, is an isomorphism, there is an endomorphism $f$ of $V$ such that the following diagram commutes:
where $g(s)(v^*) = -s(d(v^*))$, $s \in V^*$, $v^* \in V^*$. It is obvious that $f$ is nilpotent because $d_{V^*} : V^* \to V^*$ is nilpotent. So, applying (i) to $f$ we get an $L$-module structure $\psi : L \times V \to V$ such that $f(v) = \psi(x, v)$ for all $v \in V$.

In particular, we have the induced derivation $d_x : \mathbb{C}[(V, \psi)] \to \mathbb{C}[(V, \psi)]$. For $v^* \in V^*$, $v \in V$,

$$d_x(v^*)(v) = v^*(-\psi(x, v)) = -v^*(f(v)) = -\mathrm{ev}(f(v))(v^*) = -g \circ \mathrm{ev}(v)(d(v^*)) = d(v^*)(v),$$

which means that $d_x = d$ on $V^* \subset \mathbb{C}[V]$. This, however, implies that $d_x = d$. □

Below, $U$ will denote the vector space $\mathbb{C}^2$ provided with the natural $L$-module structure given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} az_1 + bz_2 \\ cz_1 + dz_2 \end{pmatrix}.$$ 

Then $\mathbb{C}[U] = \mathbb{C}[X, Y]$, where $X, Y \in (\mathbb{C}^2)^*$, $X(z_1, z_2) = z_1, Y(z_1, z_2) = z_2$, and the $L$-module algebra structure on $\mathbb{C}[U]$ is determined by

(1) $d_x(X) = -Y, \quad d_x(Y) = 0 = d_y(X), \quad d_y(Y) = -X.$

If $A, B$ are $L$-module algebras, then the tensor product $A \otimes B$ is an $L$-module algebra via $t.(a \otimes b) = t.a \otimes b + a \otimes t.b$, where $t \in L$, $a \in A, b \in B$.

In particular, for any $L$-module algebra $A$ we have the $L$-module algebra $A[X, Y] = A \otimes \mathbb{C}[U]$. Observe that $A[X, Y] = \mathbb{C}[V \oplus U]$ whenever $A = \mathbb{C}[V]$ for some finite-dimensional $L$-module $V$.

**Lemma 3.** Let $A$ be a locally finite $L$-module algebra. Then the homomorphism of algebras $\Phi : A[X, Y] \to A, \Phi(f(X, Y)) = f(1, 0)$, induces an isomorphism of algebras $\Phi : A[X, Y]^L \to A^*$, where $A^* = \{a \in A : x.a = 0\} = \{a \in A : d_x(a) = 0\}$.

**Proof.** Let $f = \sum_{k=0}^{s} f_k(X)Y^k \in A[X, Y]$ and let $f_k(X) = \sum_{j \geq 0} a_j^{(k)}X^j$, $k = 0, \ldots, s$. Using the formulas (1), we easily verify that $d_x(f) = 0 = d_y(f)$ if and only if the following conditions hold:

(2) $d_x(a_j^{(0)}) = 0 = f'_s(X), \quad d_x(a_j^{(k)}) = (j + 1)a_j^{(k+1)} + a_j^{(k-1)}), \quad k = 1, \ldots, s, \quad j \geq 0,$

$\quad d_y(a_0^{(k)}) = 0, \quad k = 0, \ldots, s,$

(3) $d_y(a_j^{(k)}) = (k + 1)a_j^{(k+1)}, \quad k = 0, \ldots, s - 1, \quad j \geq 1.$
From (3), by induction on \( k \), we get
\[
(4) \quad a_j^{(k+1)} = \frac{1}{(k+1)!} d_y^{k+1}(a_j^{(0)}), \quad k = 0, \ldots, s - 1, \quad j \geq 0.
\]

It turns out that also
\[
(5) \quad d_h(a_j^{(0)}) = ja_j^{(0)} \quad \text{for } j \geq 0.
\]

In fact, by (2) and (3), \( d_h(a_j^{(0)}) = d_x d_y(a_j^{(0)}) - d_y d_x(a_j^{(0)}) = d_x(a_j^{(1)}) = ja_j^{(0)} \) if \( j \geq 1 \), and \( d_h(a_j^{(0)}) = 0 \) because \( d_y(a_j^{(0)}) = 0 \).

From (5) it follows that the set \( \{a_j^{(0)} : j \geq 0\} \setminus \{0\} \) is linearly independent (over \( \mathbb{C} \)). From (2) we know that if \( f \in A[X,Y]^L \), then \( \Phi(f) = f(1,0) = f_0(1) = \sum_{j \geq 0} a_j^{(0)} \in A^x \). Therefore, the homomorphism of algebras \( \Phi \) induces a homomorphism of algebras
\[
\Phi : A[X,Y]^L \to A^x.
\]

If \( \Phi(f) = 0 \) for some \( f \in A[X,Y]^L \), that is, \( \sum_{j \geq 0} a_j^{(0)} = 0 \), then \( a_j^{(0)} = 0 \) for all \( j \geq 0 \), because the set \( \{a_j^{(0)} : j \geq 0\} \setminus \{0\} \) is linearly independent. In view of (4), this yields \( f = 0 \).

It remains to prove that \( \Phi \) is surjective. Since \( A \) is locally finite as an \( L \)-module, \( A = \bigoplus_{i \in I} V_m \), for some set \( I \). It follows that \( A = \bigoplus_{j \in \mathbb{Z}} A_j \), where \( A_j = \{a \in A : d_h(a) = ja\} \). Observe also that \( \{v \in V_m : x.v = 0\} = \langle v_0 \rangle \) for each \( m \geq 0 \). Hence
\[
(6) \quad A^x = \bigoplus_{j \geq 0} A_j \cap A^x.
\]

Now we show the following:
\[
(7) \quad \text{If } a \in A_m \cap A^x \text{ for some } m \geq 0, \text{ then } d_y^{m+1}(a) = 0 \text{ and } d_x d_y^j(a) = (m - j + 1) j d_y^{j-1}(a) \text{ for } j = 1, \ldots, m + 1.
\]

Let \( d_h(a) = ma \) and \( d_x(a) = 0 \) for some \( a \in A \) and \( m \geq 0 \). We can assume that \( a \in V_m \), for some \( i \in I \). Then obviously \( m_i = m \) and \( a = \alpha v_0 \) for an \( \alpha \in \mathbb{C} \), whence \( d_y^j(a) = \alpha j! v_j \) for all \( j \geq 1 \) \((v_j = 0 \text{ if } j > m)\). In particular, \( d_y^{m+1}(a) = 0 \). Furthermore, \( d_x d_y^j(a) = d_x(\alpha j! v_j) = \alpha j! x.v_j = \alpha (m - j + 1) j (j - 1)! v_{j-1} = (m - j + 1) j d_y^{j-1}(a), j = 1, \ldots, m + 1 \). So, the statement (7) is proved.

In order to prove that \( \Phi : A[X,Y]^L \to A^x \) is surjective take an \( a \in A^x \).

By (6), we can assume that \( a \in A_s \cap A^x \) for some \( s \geq 0 \). Set
\[
f_k(X) = \frac{1}{k!} d_y^k(a) X^{s-k}, \quad k = 0, \ldots, s,
\]
and let
\[
f(X,Y) = f_0(X) + f_1(X)Y + \ldots + f_s(X)Y^s.
\]
Making use of (2), (3), and (7), one easily checks that $f \in A[X,Y]^L$. Moreover, $\psi(f) = f(1,0) = f_0(1) = a$. This completes the proof of Lemma 3. □

Given a derivation $d$ of an algebra $B$, $B^d$ will denote the algebra of constants of $d$, i.e., $B^d = \ker d$.

2. Results. Let $\mathbb{C}^+$ denote the additive group of the complex field $\mathbb{C}$. We consider $\mathbb{C}^+$ as an algebraic group with the algebra of regular functions $\mathbb{C}[X]$. Then a rational representation of $\mathbb{C}^+$ is a linear space $V$ together with an action of $\mathbb{C}^+$ on $V$ such that, given $z \in \mathbb{C}^+$, $v \in V$,

$$z.v = \sum_{i \geq 0} \frac{f_i(v)}{i!} z^i,$$

for some locally nilpotent endomorphism $f : V \to V$. The endomorphism $f$ is uniquely determined by the action, and $f$ is nilpotent whenever $V$ is finite-dimensional.

Let $V$ be a finite-dimensional rational representation of $\mathbb{C}^+$ determined by the endomorphism $f : V \to V$. Then we have the induced action of $\mathbb{C}^+$ on the algebra $\mathbb{C}[V]$ defined by $(z.a)(v) = a(-z.v)$ for $a \in \mathbb{C}[V]$, $z \in \mathbb{C}^+$, $v \in V$. It is easy to check that this action is given by

$$z.a = \sum_{i \geq 0} \frac{d^i(a)}{i!} z^i,$$

where $d$ is the derivation of $\mathbb{C}[V]$ determined by $d(v^*) = -v^* \circ f$ for $v^* \in V^* \subset \mathbb{C}[V]$. This implies that $\mathbb{C}[V][\mathbb{C}^+] = \{a \in \mathbb{C}[V] : \forall z \in \mathbb{C}^+ \ z.a = a\} = \mathbb{C}[V]^d$. Notice also that $d$ is locally nilpotent and $d(V^*) \subset V^*$.

**Theorem 1.** If $V$ is a finite-dimensional rational representation of $\mathbb{C}^+$, then the algebra $\mathbb{C}[V][\mathbb{C}^+]$ is finitely generated.

**Proof.** As stated above, the action of $\mathbb{C}^+$ on $\mathbb{C}[V]$ is given by $(\ast)$, where $d : \mathbb{C}[V] \to \mathbb{C}[V]$ is a locally nilpotent derivation such that $d(V^*) \subset V^*$ and $\mathbb{C}[V][\mathbb{C}^+] = \mathbb{C}[V]^d$.

Using Lemma 2(ii) we see that there exists an $L$-module structure on $V$ such that $d = d_x$. Applying Lemma 3 to $A = \mathbb{C}[V]$ and taking into account that $A[X,Y] = \mathbb{C}[V \oplus U]$ we obtain

$$\mathbb{C}[V][\mathbb{C}^+] = \mathbb{C}[V]^d = \mathbb{C}[V]^{d_x} \cong \mathbb{C}[V \oplus U]^L.$$

Now from Lemma 1 it follows that $\mathbb{C}[V][\mathbb{C}^+]$ is a finitely generated algebra. □

For the proof of the next theorem we have to recall some well-known links between locally finite $L$-modules and rational $G$-modules (= rational representations of $G$), where $G = \text{SL}(2,\mathbb{C}) = \{M \in M_2(\mathbb{C}) : \det M = 1\}$. Since $L$ is the Lie algebra of the algebraic group $G$, for any rational $G$-module
structure \( \varphi : G \times V \to V \) on a vector space \( V \) we have the associated locally finite \( L \)-module structure \( \tilde{\varphi} : L \times V \to V \) on \( V \) (analytically, \( \tilde{\varphi}(t,v) = (\partial/\partial s) \varphi(\exp(st),v)|_{s=0} \) for \( t \in L, v \in V \)). The map \( \tilde{\varphi} \) uniquely determines \( \varphi \) and \( (V,\varphi)^G = \{ v \in V' : \forall g \in G \varphi(g,v) = v \} = \{ v \in V : \forall t \in L \tilde{\varphi}(t,v) = 0 \} = (V,\tilde{\varphi})^L \). Moreover, if \((V,\varphi)\) is a rational \( G \)-module and \( \Phi : G \times S(V) \to S(V) \) is the induced action of \( G \) on the symmetric algebra \( S(V) \), then \( \tilde{\Phi} : L \times S(V) \to S(V) \) is the previously defined \( L \)-module algebra structure on \( S(V) \). In particular, \( S(V,\varphi)^G = S(V,\tilde{\varphi})^L \).

It is known ([7, Chap. 3]) that every rational \( G \)-module is semisimple and that for any \( m \geq 0 \) there exists a unique (up to isomorphism) simple rational \( G \)-module \( \varrho_m \) of dimension \( m+1 \). It is not difficult to show that the \( L \)-module associated with \( \varrho_m \) is isomorphic to \( V_m \) for all \( m \geq 0 \). As a consequence of the above facts we get the following.

**Corollary 4.** Let \( V \) be a finite-dimensional vector space. Then for any \( L \)-module structure \( \psi : L \times V \to V \) on \( V \) there exists a unique rational \( G \)-module structure \( \varphi : G \times V \to V \) on \( V \) such that the following conditions hold.

1. \( \tilde{\varphi} = \psi \).
2. \( (V,\varphi) \cong \varrho_{m_1} \oplus \ldots \oplus \varrho_{m_s} \) for some \( m_1,\ldots,m_s \) if and only if \( (V,\psi) \cong V_{m_1} \oplus \ldots \oplus V_{m_s} \).
3. \( S(V,\psi)^L \cong S(V,\varphi)^G \).

The \( G \)-module \((V,\varphi)\) is called the *lifting* of the \( L \)-module \((V,\psi)\).

**Theorem 2.** Let \( V \) be a finite-dimensional rational representation determined by a non-zero nilpotent endomorphism \( f \) of the vector space \( V \). Then

1. \( \mathbb{C}[V]^{G^+} \) is a Gorenstein ring.
2. \( \mathbb{C}[V]^{G^+} \) is a polynomial algebra if and only if \( V = V_{(0)} \oplus V' \) for some subrepresentations \( V_{(0)}, V' \) of \( V \) such that \( \mathbb{C}^+ \) acts trivially on \( V_{(0)} \) (i.e., \( f(V_{(0)}) = 0 \)) and the Jordan matrix of \( f' : V' \to V' \), \( f'(v) = f(v) \), is one of the following:

\[
\begin{pmatrix}
0 & 1 \\
0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

**Proof.** As in the proof of Theorem 1, \( \mathbb{C}[V]^{G^+} \cong \mathbb{C}[V \oplus U]^L \) for some \( L \)-module structure \( \psi : L \times V \to V \) such that \( \psi(x,v) = f(v) \) for \( v \in V \). But \( U \), being a simple \( L \)-module of dimension 2, is isomorphic to \( V_1 \), so that \( \mathbb{C}[V]^{G^+} \cong \mathbb{C}[V \oplus V_1]^L \). According to Corollary 4, there exists a unique rational \( G = \text{SL}(2,\mathbb{C}) \)-module structure \( \varphi \) on \( V \) such that \( \tilde{\varphi} = \psi \). This implies
that \( \mathbb{C}[V]^{C^+} \cong \mathbb{C}[(V, \varphi) \oplus g_1]^G \), because \( g_1 \) is the lifting of \( V_1 \). Now part 1 of the theorem follows, because, as is well known, \( \mathbb{C}[W]^G \) is a Gorenstein ring for any finite-dimensional rational \( G \)-module \( W \) (see [1, Remark 6.5.5]).

For part 2, in view of [5, Example following Thm. 3], \( \mathbb{C}[V]^{C^+} \cong \mathbb{C}[(V, \varphi) \oplus g_1]^G \) is a polynomial algebra if and only if there exists a trivial submodule \( V_t \) of the \( G \)-module \( (V, \varphi) \oplus g_1 \) isomorphic to one of the \( G \)-modules: \( V_t \oplus g_1 \), \( V_t \oplus g_2 \oplus g_1 \). It follows that \( \mathbb{C}[V]^{C^+} \) is a polynomial algebra if and only if \( (V, \varphi) \) is isomorphic to one of the \( G \)-modules: \( V_t \oplus g_1 \), \( V_t \oplus g_2 \), \( V_t \oplus g_1 \oplus g_1 \). By Corollary 4(b), this in turn implies that \( \mathbb{C}[V]^{C^+} \) is a polynomial algebra if and only if the \( L \)-module \( (V, \psi) \) is isomorphic to one of the \( L \)-modules: \( (V_0) \oplus V_1 \), \( V_0 \oplus V_2 \), \( V_0 \oplus V_1 \oplus V_1 \), where \( V_0 \) is the trivial \( L \)-module structure on \( V_t \) as a vector space. The conclusion now follows from Lemma 2(i) applied to \( f : V \to V \). The theorem is proved. ■

**Remark.** Part 1 of the theorem was announced in [4].

**Corollary 5.** Let \( A = \mathbb{C}[X_1, \ldots, X_n] \) and let \( d \neq 0 \) be a locally nilpotent derivation of \( A \) with \( d(W) \subset (W) \), where \( W = \mathbb{C}X_1 + \ldots + \mathbb{C}X_n \subset A \).

1. \( A^d \) is a Gorenstein ring.

2. \( A^d \) is a polynomial algebra if and only if \( W = W_0 \oplus W' \) for some subspaces \( W_0, W' \) of \( W \) such that \( d(W_0) = 0 \), \( d(W') \subset W' \), and the Jordan matrix of \( d|_{W'} : W' \to W' \) is one of the three matrices appearing in Theorem 2.

**Proof.** We can consider \( A \) as the algebra \( \mathbb{C}[V] \), where \( V = \mathbb{C}^n \). Then \( W = V^* \), and hence there exists an endomorphism \( f : V \to V \) such that \(-f^* = d|_W : W \to W \). Since \( d|_W \) is nilpotent, the endomorphism \( f \) is also nilpotent. Therefore, the formula

\[
z.v = \sum_{i \geq 0} \frac{f^i(v)}{i!}z^i, \quad z \in \mathbb{C}^+, \ v \in V,
\]

makes \( V \) a rational representation of \( \mathbb{C}^+ \) such that \( \mathbb{C}[V]^{C^+} = \mathbb{C}[V]^d = A^d \). Now, the corollary is a consequence of Theorem 2, because the Jordan matrices of \( f \) and \( \pm f^* \) coincide. ■

**Remark 6.** It is easy to see that the corollary is equivalent to Theorem 2.

**Remark 7.** Let \( M = (a_{ij}) \in M_n(\mathbb{C}) \) be a nilpotent matrix and let \( d : A \to A, A = \mathbb{C}[X_1, \ldots, X_n] \), be the locally nilpotent derivation defined by

\[
d(X_i) = \sum_{i=1}^n a_{ij}X_j, \quad i = 1, \ldots, n.
\]
Essentially, the implication $\iff$ in part 2 of Corollary 5 says that $A^d$ is a polynomial algebra if $M$ is one of the three matrices appearing in Theorem 2. But this follows also from [3], where it was shown that

- $A^d = \mathbb{C}[X_1]$ if $n = 2$ and $M = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (this is obvious),
- $A^d = \mathbb{C}[X_1, X_2^2 - 2X_1X_3]$ if $n = 3$ and $M = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ [3, Ex. 6.8.1], and
- $A^d = \mathbb{C}[X_1, X_3, X_2X_3 - X_1X_4]$ if $n = 4$ and $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ [3, Prop. 6.9.5].

In fact, since $\text{tr.deg}_\mathbb{C} A^D = n - 1$ for any locally nilpotent derivation $D : A \to A$, $D \neq 0$, the generators of $A^d$ in the above three cases are algebraically independent.

REFERENCES


Faculty of Mathematics and Informatics
Nicholas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: atyc@mat.uni.torun.pl

Received 8 January 1998;
revised 10 February 1998