COLLOQUIUM MATHEMATICUM

VOL. 78 1998 NO. 1

THE GROTHENDIECK GROUP OF G-EQUIVARIANT MODULES OVER COORDINATE RINGS OF G-ORBITS

BY

J. KLIMEK (TORUŃ), W. KRAŚKIEWICZ (TORUŃ)
AND J. WEYMAN (BOSTON)

1. Introduction. Let V be a finite-dimensional representation of a complex reductive connected algebraic group G and let X be the closure of an orbit of G in V. We will always assume that the group G contains a one-dimensional torus acting on X by multiplication. Then the coordinate ring $\mathbb{C}[X]$ has a natural grading coming from that action. We investigate the category $\mathcal{C}(X)$ of graded finitely generated modules over $\mathbb{C}[X]$ with rational G-action compatible with the module structure. Denote by $K_0'(X)$ the Grothendieck group of the category $\mathcal{C}(X)$. We are interested in the structure of $K_0'(X)$.

In the case when V is the space of $m \times n$ matrices with natural action of $GL(m) \times GL(n)$, the structure of $K_0'(X)$ was described in [W]. In this case the orbit closures X_r consist of matrices with rank less than or equal to r. There exist three families of natural desingularizations of X_r. For any such family a set of free generators of $K_0'(X_r)$ was constructed. It turns out that the proofs of the results of [W] depend only on a few formal properties of desingularizations. We extract these properties defining the notion of coherent desingularization and show how to derive the generating result in a quite general set up. Then using the classification by Kac [K], we show how to construct suitable families of desingularizations for orbit closures in irreducible multiplicity free actions. In effect we obtain a description of the Grothendieck groups in that case.

Let G, V and X be as above. We fix a Borel subgroup B of G and a maximal torus $T \subset B$. Declare the roots of B to be positive and denote the set of them by R^+.

Definition. We say that X admits a *coherent desingularization* if for every G-orbit $\mathcal{O}$ in X there exist a parabolic subgroup $P_{\mathcal{O}} \supset B$ and a $P_{\mathcal{O}}$-submodule $W_{\mathcal{O}} \subset V$ such that the following conditions are satisfied.

1991 *Mathematics Subject Classification*: 16E20, 20G05.

Research of the third author partially supported by NSF, grant DMS 9403703.

(i) The collapsing map $q_{\mathcal{O}}$ from $Z_{\mathcal{O}} = G \times_{P_{\mathcal{O}}} W_{\mathcal{O}}$ to V is a desingularization of the orbit closure $\overline{\mathcal{O}}$.

(ii) Whenever $\overline{\mathcal{O}}_1 \subset \overline{\mathcal{O}}_2$ then $W_{\mathcal{O}_1} \subset W_{\mathcal{O}_2}$.

Assume that X admits a coherent desingularization. In particular, $X = \overline{\mathcal{O}}$ and we have a parabolic subgroup $P = P_{\mathcal{O}}$ and a P-submodule $W = W_{\mathcal{O}}$ of V satisfying the condition (i) of the above definition. Then the fiber product $Z = G \times_P W$ is a vector bundle over the homogeneous space G/P. Let $p : Z \to G/P$ be the canonical projection and denote by q the collapsing map from Z to X.

For every character $\lambda \in X(T)$ we denote by $\mathcal{L}_{P/B}(\lambda)$ the linear bundle on P/B generated by λ. The space of global sections of $\mathcal{L}_{P/B}(\lambda)$ has a natural structure of P-module, hence it induces a vector bundle $G \times_P H^0(P/B, \mathcal{L}_{P/B}(\lambda))$ on G/P. We define a (graded) sheaf

$$\mathcal{E}_P(\lambda) = p^*(G \times_P H^0(P/B, \mathcal{L}_{P/B}(\lambda))) \otimes \mathcal{O}_Z \tag{1}$$

on Z. Then the cohomology groups of $\mathcal{E}_P(\lambda)$ have a natural structure of a $\mathbb{C}[X]$-module. Let $\chi_W(\lambda)$ be the Euler character of $\mathcal{E}_P(\lambda)$ in the Grothendieck group $K'_0(X)$:

$$\chi_W(\lambda) = \chi(Z, \mathcal{E}_P(\lambda)) = \sum_{i \geq 0} (-1)^i [H^i(Z, \mathcal{E}_P(\lambda))]. \tag{2}$$

Let L be the Levi factor of P. The intersection $L \cap B$ is a Borel subgroup of L. Let R_L^+ be the set of positive roots for it. Denote by w_0^L the longest element in the Weyl group of L. It follows from Bott's theorem that the line bundle $\mathcal{L}_{P/B}(\lambda)$ has nonzero global sections if and only if $w_0^L \lambda$ is a dominant weight for L.

THEOREM 1. *If X admits a coherent desingularization then the Grothendieck group $K'_0(X)$ is generated by the shifts in grading of the Euler characters $\chi_W(w_0^L \lambda)$ where λ runs over the dominant weights for L.*

When a coherent desingularization satisfies some additional technical conditions (see Section 3 for details) we prove that we can assume λ to run over the set of dominant weights for G. If the action of G on V is irreducible and multiplicity free we can construct a coherent desingulariztion of every orbit closure $X \subset V$ satisfying this additional condition. As a consequence we obtain the following result.

THEOREM 2. *If V is an irreducible representation of a reductive group G and the action of G on V is multiplicity free then for every orbit closure $X \subset V$ the Grothendieck group $K'_0(X)$ is isomorphic to the additive group of the polynomial ring $\mathbb{Z}[e^{\omega_1}, \ldots, e^{\omega_r}][q, q^{-1}]$ where $\omega_1, \ldots, \omega_r$ are the fundamental weights for G.*

2. Some cohomology calculations. Assume that G, V and X are as in the introduction and that X admits a coherent desingularization. In order to prove Theorem 1 we have to calculate the cohomology groups $H^{\cdot}(Z, \mathcal{E}_P(\lambda))$. Our method is, generally speaking, to replace a calculation on Z by a calculation on G/P and then use Bott's theorem ([D]). To simplify the notation we will identify a P-module M with the vector bundle $G \times_P M$ on G/P.

LEMMA 3. *For every character $\lambda \in X(T)$,*

$$H^i(Z, \mathcal{E}_P(\lambda)) = H^i(G/P, H^0(P/B, \mathcal{L}_{P/B}(\lambda)) \otimes \mathrm{Sym}(W^*)).$$

Proof. The canonical projection $p : Z \to G/P$ is affine, hence $R^i p_* \mathcal{O}_Z = 0$ for $i > 0$. Since $p_* \mathcal{O}_Z = \mathrm{Sym}(W^*)$ by definition of Z, the result follows from the Grothendieck spectral sequence of composition.

Let L be the Levi factor of P. Denote by R_L^+ the set of those positive roots of G which are the roots for P and let W_L be the Weyl group of L with $w_0^L \in W_L$ being the element of maximal length. Let ϱ_L be half the sum of the positive roots of L. According to Bott's theorem the cohomology group $H^0(P/B, \mathcal{L}_{P/B}(\lambda))$ is nonzero if and only if $w_0^L \lambda$ is a dominant weight for L and in that case it is irreducible as an L-module with highest weight $w_0^L \lambda$ (with respect to P). More generally, the group $H^i(P/B, \mathcal{L}_{P/B}(\lambda))$ is nonzero for at most one i: if there exists an element $w \in W_L$ with length $\ell(w)$ such that $\lambda' = w(w_0^L \lambda + \varrho_L) - \varrho_L$ is a dominant weight of L then $H^{\ell(w)}(P/B, \mathcal{L}_{P/B}(\lambda)) = H^0(P/B, \mathcal{L}_{P/B}(w_0^L \lambda'))$.

LEMMA 4. *Assume that S is a graded P-module and that λ is a character of T. If $w \in W_L$ and λ' are as above then*

$$H^i(G/B, \mathcal{L}_{G/B}(\lambda) \otimes S) = H^{i-\ell(w)}(G/P, H^0(P/B, \mathcal{L}_{P/B}(\lambda') \otimes S)).$$

Proof. Let $v : G/P \to G/B$ and $u : G/B \to \{*\}$ be the natural projections. We have a spectral sequence of the composition of functors $u_* v_* = (uv)_*$ with $E^2_{i,j}$-term

$$H^i(G/P, H^j(P/B, \mathcal{L}_{P/B}(\lambda) \otimes S)), \tag{3}$$

which by the projection formula is equal to

$$H^i(G/P, H^j(P/B, \mathcal{L}_{P/B}(\lambda)) \otimes S). \tag{4}$$

Since $H^j(P/B, \mathcal{L}_{P/B}(\lambda))$ is nonzero for at most one j, the sequence degenerates and this gives the formula stated in the lemma.

Denote by K the subgroup of $K_0'(X)$ generated by the Euler characters of the sheaves $\mathcal{E}_P(w_0^L \lambda)$ with λ a dominant weight for the Levi factor L of P.

Let X' be the closure of a G-orbit properly contained in X. We identify the elements of $K_0'(X')$ with their images in $K_0'(X)$. Let P', W', $Z' = G \times_{P'} W'$, $p' : Z' \to G/P'$ and $q : Z' \to X'$ be the data describing the desingularization of X'.

PROPOSITION 5. *If $w_0^{L'}\lambda$ is a dominant weight for L' then the Euler character $\chi(Z', \mathcal{E}_{P'}(\lambda))$ is contained in K.*

P r o o f. According to the definition we have

$$\chi(Z', \mathcal{E}_{P'}(\lambda)) = \chi(Z', (p')^* H^0(P'/B, \mathcal{L}_{P'/B}\lambda) \otimes \mathcal{O}_{Z'}). \tag{5}$$

By Lemma 3 the right hand side of (5) equals

$$\chi(G/P', H^0(P'/B, \lambda) \otimes \mathrm{Sym}(W')^*). \tag{6}$$

Since $\mathrm{Sym}(W')^*$ is a P'-module and the higher cohomology groups of the sheaf $\mathcal{L}_{P'/B}(\lambda)$ vanish, by Lemma 4 we have

$$\begin{aligned} H^i(G/P', H^0(P'/B, \mathcal{L}_{P'/B}(\lambda) \otimes \mathrm{Sym}(W')^*)) \\ = H^i(G/B, \mathcal{L}_{G/B}(\lambda) \otimes \mathrm{Sym}(W')^*). \end{aligned} \tag{7}$$

Hence we can replace the character calculation on Z' by the calculation on G/B:

$$\chi(Z', \mathcal{E}_{P'}(\lambda)) = \chi(G/B, \mathcal{L}_{G/B}(\lambda) \otimes \mathrm{Sym}(W')^*). \tag{8}$$

The inclusion of W' in W gives us the following short exact sequence of B-modules (or vector bundles on G/B):

$$0 \to (W/W')^* \to W^* \to (W')^* \to 0.$$

Therefore, the symmetric algebra $\mathrm{Sym}(W')^*$, as a module over $\mathrm{Sym}\, W^*$, has a Koszul resolution. Applying this to (8) we obtain

$$\chi(Z', \mathcal{E}_{P'}(\lambda)) = \sum_i (-1)^i \chi(G/B, \mathcal{L}_{G/B}(\lambda) \otimes \textstyle\bigwedge^i (W/W')^* \otimes \mathrm{Sym}\, W^*). \tag{9}$$

Every B-module $\bigwedge^i (W/W')^*$ admits a filtration with the associated graded object isomorphic to the direct sum of 1-dimensional B-modules and the right hand side of (9) becomes a sum of characters $\chi(G/B, \mathcal{L}_{G/B}(\lambda) \otimes \alpha \otimes \mathrm{Sym}(W^*))$ were α is a weight of $\bigwedge^{\cdot}(W/W')^*$. Once again from Lemma 4 the cohomology groups of $H^{\cdot}(G/B, \mathcal{L}_{G/B}(\lambda) \otimes \alpha \otimes \mathrm{Sym}\, W^*)$ are equal, up to some shift, to the cohomology groups

$$H^{\cdot}(G/P, H^0(P/B, \mathcal{L}_{P/B}(\lambda_\alpha)) \otimes \mathrm{Sym}\, W^*) \tag{10}$$

for some characters λ_α. This expresses the character $\chi(Z', \mathcal{E}_{P'/B}(\lambda))$ as a combination of the characters $\chi(Z, \mathcal{E}_{P/B}(\lambda_\alpha))$ with integer coefficients, and the proposition follows.

To formulate the following proposition let us note that the category $\mathcal{C}(X)$ of graded finitely generated $\mathbb{C}[X]$-modules is equivalent to the category of coherent graded sheaves on X and the equivalence is given by the global sections functor.

PROPOSITION 6. *For any graded G-equivariant sheaf $\mathcal{M}$ on Z the class of $q_*\mathcal{M}$ in $K_0'(X)$ is contained in K.*

Proof. We want to prove that the class of $H^0(X, q^*\mathcal{M}) = H^0(Z, \mathcal{M})$ is in K. But the higher cohomology groups $H^i(Z, \mathcal{M})$, $i > 0$, are supported on a G-invariant subset strictly contained in X, so they belong to K by Proposition 5. Hence it is enough to prove that the class of the Euler character $\chi(Z, \mathcal{M})$ is in K. Since Z is nonsingular, the sheaf $\mathcal{M}$ has a finite resolution

$$0 \to \mathcal{F}_s \to \ldots \to \mathcal{F}_1 \to \mathcal{F}_0 \to \mathcal{M} \to 0 \tag{11}$$

of graded vector bundles $\mathcal{F}_i$ on Z. Every vector bundle $\mathcal{F}_i$ is given by a P-module F_i with an additional structure of a $\operatorname{Sym} W^*$-module. Therefore, up to filtration, every $\mathcal{F}_i$ is a direct sum of vector bundles $\mathcal{E}_P(\lambda)$, and $\chi(Z, \mathcal{M})$ is in K.

Using the above propositions we can prove Theorem 1 by induction on the dimension of X. If $\dim X = 0$ then X consists of a single point and $K_0'(X)$ is the group of graded representations of G.

Assume that $\dim X > 0$. Denote by P, W, $Z = G \times_P W$ and $q : Z \to X$ the data describing a desingularization of X and let K be as above. Let M be a graded G-equivariant $\mathbb{C}[X]$-module and let $\mathcal{M}$ be the sheaf corresponding to M. Then the kernel and the cokernel of the natural map

$$\mathcal{M} \to q_* q^* \mathcal{M} \tag{12}$$

are supported on G-invariant closed subsets properly contained in X. Up to filtration we can assume that both are supported on some orbit closure X' properly contained in X, so their classes are in K by induction. The class of $q_*q^*\mathcal{M}$ is in K by Proposition 6 and the conclusion follows.

3. Independent generators. Assume once again that X is the closure of a G-orbit in V and that X admits a coherent desingularization. For any orbit $\mathcal{O}$ let $P_{\mathcal{O}}$ and $W_{\mathcal{O}}$ be the data describing the desingularization of $\overline{\mathcal{O}}$ and let $L_{\mathcal{O}}$ be the Levi factor of $P_{\mathcal{O}}$. We will denote the irreducible representation of $L_{\mathcal{O}}$ of highest weight λ by $V_{L_{\mathcal{O}}}(\lambda)$. For dominant weights λ, μ, ν of $L_{\mathcal{O}}$ we will write $\lambda \in \mu \otimes \nu$ if λ is a highest weight occurring in the decomposition of the tensor product $V_{L_{\mathcal{O}}}(\mu) \otimes V_{L_{\mathcal{O}}}(\nu)$. For any dominant weight μ of the Levi factor $L_{\mathcal{O}}$ we define

$$s_{\mathcal{O}}(\lambda) = \max_{\lambda} \max_{\alpha} \{(\lambda' - \lambda, \alpha) \mid \alpha \in R^+ \setminus R_L^+, \quad \lambda \text{ a dominant weight of } L_{\mathcal{O}},\ \lambda' \in \lambda \otimes \mu\}. \tag{13}$$

The coordinate ring of $\overline{\mathcal{O}}$ is equal to $H^0(G/B, \operatorname{Sym} W_{\mathcal{O}}^*)$. We decompose $\operatorname{Sym} W^*$ into irreducible $L_{\mathcal{O}}$-modules. In terms of $L_{\mathcal{O}}$-decomposition we formulate two postulates for a desingularization of X:

(iii) If μ is a highest weight of $W_{\mathcal{O}}^*$ then $s_{\mathcal{O}}(\mu) \le 0$.

(iv) There exists a highest weight μ of $\operatorname{Sym} W^*$ such that $w_0 w_0^L \mu$ is dominant and $s_{\mathcal{O}}(\mu) < 0$.

If $V_{L_{\mathcal{O}}}(\mu) \subset \operatorname{Sym}(W^*)$ is such that μ satisfies (iv) then $H^0(G/P_{\mathcal{O}}, V_{L_{\mathcal{O}}}(\mu))$ generates a G-invariant ideal in the coordinate ring $\mathbb{C}[x]$. Its zero set is a closed G-invariant proper subset of X. If it is the closure of an orbit $\mathcal{O}' \subset V$ we will call $V_{L_{\mathcal{O}}}(\mu)$ the *relative equations* of $\overline{\mathcal{O}}'$ in $\overline{\mathcal{O}}$.

EXAMPLE. Let V be any irreducible representation of G and let μ be the highest weight of V. Denote by W the one-dimensional space of highest weight vectors and by P the stabilizer of W in G. Then $X = GW$ is the closure of the orbit $\mathcal{O}_1$ of highest weight vectors and the collapsing map $q : G \times_P W \to X$ is a desingularization. We will call it the *natural desingularization* of the highest weight orbit. Since the variety X consists of the orbit $\mathcal{O}_1$ and one-element orbit $\mathcal{O}_0 = \{0\}$, and we can represent the last orbit as a zero vector bundle on G/G, it follows that X admits a coherent desingularization. The coordinate ring of X is $H^0(G/P, \operatorname{Sym} W^*)$ and as a G-module it decomposes into a direct sum of irreducible modules $V(n\mu^*)$, where $\mu^* = -w_0\mu$ is the highest weight of V^*. The ideal of $\mathcal{O}_0$ in $\mathbb{C}[X]$ is generated by $V(\mu^*) = H^0(G/P, W^*)$.

If a positive root α is not a root of P then $\mu - \alpha$ is a weight of V and $(\mu, \alpha) > 0$. In order to check that the natural desingularization of the highest weight orbit satisfies conditions (iii) and (iv), we have to calculate the scalar products of the form $(\lambda - \lambda', \alpha)$ for $\lambda' \in \lambda \otimes (-\mu)$. Since W^* is a one-dimensional irreducible representation of the Levi factor L of P with highest weight $-\mu$, we have $\lambda - \lambda' = -\mu$ for every dominant weight λ of L. For a root $\alpha \in R^+ \setminus R_L^+$ we have $(-\mu, \alpha) = -(\mu, \alpha) < 0$.

PROPOSITION 7. *If X admits a coherent desingularization satisfying conditions* (iii) *and* (iv) *above*, *then the Grothendieck group $K_0'(X)$ is generated by the shifts of the Euler characters $\chi_W(w_0\lambda)$ for all dominant weights λ of G.*

Proof. In the sequel we omit the subscript $\mathcal{O}$ from all data concerning $\overline{\mathcal{O}} = X$. We start with the observation that R^+ is the disjoint union of $w_0 w_0^L R_L^+$ and $-w_0 w_0^L (R^+ \setminus R_L^+)$. In particular, it follows that if λ is a dominant weight for G then $w_0^L w_0 \lambda$ is a dominant weight for L. We know from Theorem 1 that the group $K_0'(X)$ is generated by the shifts of the Euler characters $\chi_W(w_0\lambda)$ for dominant weights λ of L. Thus we need to express $\chi_W(w_0^L\lambda)$ as a combination of the characters $\chi_W(w_0\lambda')$ with λ' dominant. We assume inductively that our proposition is true for all orbit closures X' properly contained in X and let

$$\chi_W(w_0^L\lambda) = \chi(G/P, H^0(P/B, \mathcal{L}_{P/B}(w_0^L\lambda)) \otimes \operatorname{Sym} W^*) \tag{14}$$

be a typical generator of $K_0'(X)$. Since every P-module has a filtration with factors isomorphic to irreducible L-modules, only the L-module structure of $M = H^0(P/B, \mathcal{L}_{P/B}(w_0^L\lambda) \otimes \operatorname{Sym} W^*)$ is important for the calculation of the Euler character. We decompose M into irreducible L-modules. Let μ be the highest weight of an irreducible component of M. We want to measure how far $w_0 w_0^L \mu$ is from being dominant. We have $(w_0 w_0^L \mu, \alpha) \geq 0$ for $\alpha \in w_0 w_0^L R_L^+$. Set

$$s(\mu) = -\max\{(\mu, \alpha) \mid \alpha \in R^+ \setminus R_L^+\}. \tag{15}$$

Then $s(\mu)$ is the minimum of the scalar products $(w_0 w_0^L \mu, \alpha)$ with $\alpha \in -w_0 w_0^L(R^+ \setminus R_L^+)$, and $s(\mu) \geq 0$ if and only if $w_0 w_0^L \mu$ is dominant. Define M' to be the direct sum of all irreducible components $V_L(\mu)$ of M with $s(\mu) > s$. By (iii), M' is a $\operatorname{Sym} W^*$-submodule of M. Proceeding by reverse induction on s we can assume that the Euler character of M' is an integer combination of $\chi(w_0\lambda')$ with λ' dominant. On the other hand, it follows from the condition (iv) that the support of the quotient module M/M' is strictly smaller than X and the assertion follows from the induction hypothesis.

Applying Proposition 7 to the example preceding that proposition we obtain the following fact.

Corollary 8. *If X is the closure of the G-orbit of highest weight vectors in an irreducible representation of G and if P is the stabilizer of a highest weight vector then the shifts of the Euler characters $\chi_P(w_0\lambda)$, λ a dominant weight of G, generate the Grothendieck group $K_0'(X)$.*

Assume now that X admits a coherent desingularization satisfying (iii) and (iv). Let $\omega_1, \ldots, \omega_r$ be the fundamental weights of G. We define a map Ψ from the additive group of the polynomial ring $\mathbb{Z}[e^{\omega_1}, \ldots, e^{\omega_r}][q, q^{-1}]$ to K_0' by

$$\Psi(e^\lambda q^i) = \chi(w_0\lambda)[i] \tag{16}$$

where $[i]$ means the corresponding shift in grading. It follows from Proposition 7 that Ψ is onto. Taking graded characters we see that the elements $\chi(w_0\lambda)[i]$ are independent over $\mathbb{Z}$. Hence we get

Corollary 9. *If X admits a coherent desingularization satisfying* (iii) *and* (iv) *then the Grothendieck group $K_0'(X)$ is isomorphic to the additive group of the polynomial ring $\mathbb{Z}[e^{\omega_1}, \ldots, e^{\omega_r}][q, q^{-1}]$.*

4. Multiplicity free actions. Multiplicity free irreducible actions of reductive groups were classified by Kac [K]. The list given below differs slightly from that of [K] because we always assume that there is a one-dimensional torus $\mathbb{C}^*$ in G acting on V by multiplication. Our goal is to prove Theorem 2. Many actions in Kac's list have only two or three orbits. In the case of two orbits they are the zero orbit and the highest weight orbit. In the case of three orbits we have in addition an open orbit. These two cases include the natural actions of $\mathrm{GL}(V)$, $\mathrm{Sp}(V)\times\mathbb{C}^*$, $\mathrm{SO}(V)\times\mathbb{C}^*$ and $G_2\times\mathbb{C}^*$. For an open orbit Theorem 2 follows immediately from Theorem 1 since we can take $P=G$, while for the highest weight orbit it follows from Corollary 8.

The natural action of the group $\mathrm{GL}(U)\times\mathrm{GL}(T)$ was discussed in detail in [W]. For other multiplicity free actions of a reductive group G on an irreducible G-module V we construct examples of coherent desingularizations of every orbit closure in V satisfying conditions (iii) and (iv) of Section 3. Again Theorem 2 follows from the existence of such coherent desingularizations.

4.1. *Symmetric forms.* Let U be a complex vector space of dimension n and set $G=\mathrm{GL}(U)$, $V=S_2(U)$. We can identify V with the space of symmetric forms on U^*. The G-orbits on V are parameterized by rank:

$$\mathcal{O}_i=\{\phi\in S_2(U)\mid \mathrm{rk}(\phi)=i\}$$

and $\overline{\mathcal{O}}_i\subset\overline{\mathcal{O}}_{i+1}$. In U we fix a full flag of subspaces $0=U_0\subset U_1\subset\ldots\subset U_n=U$. The stabilizer of this flag in G is a Borel subgroup. For every i, $0\le i\le n$, we define P_i to be the stabilizer of U_i and we write $W_i=S_2(U_i)$. Then $Z_i=G\times_{P_i}W_i$ is a desingularization of $\overline{\mathcal{O}}_i$. In this way we obtain a coherent desingularization of any orbit closure in V. The Levi factor L_i of P_i is naturally isomorphic to $\mathrm{GL}(U_i)\times\mathrm{GL}(U/U_i)$. Dominant weights of L_i are of the form $\lambda=(\lambda_1,\ldots,\lambda_i\mid\lambda_{i+1},\ldots,\lambda_n)$ with $\lambda_1\ge\ldots\ge\lambda_i$ and $\lambda_{i+1}\ge\ldots\ge\lambda_n$. The set $R^+\setminus R^+_{L_i}$ consists of the roots $\epsilon_k-\epsilon_l$ where $1\le k\le i<l\le n$. The representation W_i^* is an irreducible representation of L_i of highest weight $(0,\ldots,0,-2\mid 0,\ldots,0)$ and it follows from the Pieri formula for representations of $\mathrm{GL}(U_i)$ that every irreducible component of the tensor product $V_{L_i}(\lambda)\otimes W^*$ has a highest weight of the form $(\lambda_1',\ldots,\lambda_i'\mid\lambda_{i+1},\ldots,\lambda_n)$ with $\lambda_k'\le\lambda_k$ for $1\le k\le i$. Therefore condition (iii) is satisfied. Further, as a relative equation of $\overline{\mathcal{O}}_{i-1}$ in $\overline{\mathcal{O}}_i$ we can take $(\bigwedge^i U_i^*)^{\otimes 2}$ contained in $S_i(W_i^*)$. In matrix notation it corresponds to the $i\times i$ minors. The relative equation is a one-dimensional representation of L_i with highest weight $(-2,\ldots,-2\mid 0,\ldots,0)$. Hence $V_{L_i}(\lambda)\otimes(\bigwedge^i U_i^*)^{\otimes 2}$ is an irreducible L_i-module with highest weight $\lambda'=(\lambda_1-2,\lambda_2-2,\ldots,\lambda_i-2\mid\lambda_{i+1},\ldots,\lambda_n)$. This proves (iv).

4.2. *Antisymmetric forms.* Let U and G be as above and let $V=\bigwedge^2 U$. In this case, we can identify V with the space of antisymmetric forms on U^* and the G-orbits are again parameterized by rank. Since the rank of an antisymmetric form is even we have the orbits

$$\mathcal{O}_i=\{\phi\in{\textstyle\bigwedge}^2(U)\mid \mathrm{rk}(\phi)=i\}$$

for $i=0,2,\ldots,2[n/2]$ with the linear inclusion order on the closures. In analogy with the previous case $W_i = \bigwedge^2 U_i$ provides a desingularization of $\overline{\mathcal{O}}_i$ and $(\bigwedge^i U_i^*)^{\otimes 2}$ contained in $S_i(W_i^*)$ gives the relative equations of $\overline{\mathcal{O}}_{i-2}$ in $\overline{\mathcal{O}}_i$.

4.3. *Action of* $\mathrm{Sp}(T) \times \mathrm{GL}(R)$. Let R and T be complex vector spaces of dimensions r and $2t$ respectively. Assume that T is equipped with a nondegenerate symplectic form. The group $G = \mathrm{Sp}(T) \times \mathrm{GL}(R)$ acts in a natural way on the space $V = T \otimes R = T^* \otimes R = \mathrm{Hom}(T, R)$. The action of G is multiplicity free only in two cases:

Case 1: $\dim T = 4$, $\dim R \geq 4$.
Case 2: $\dim T \geq 4$, $\dim R = 2, 3$.

In general the orbits of G in V are parameterized by the rank of a linear map and the rank of the restriction of the symplectic form to the kernel of a linear map. For small dimensions the second parameter is essential only in few cases. We fix maximal flags $0 = R_0 \subset R_1 \subset \ldots \subset R_r = R$ of subspaces in R and $0 = T_0 \subset T_1 \subset \ldots \subset T_t$ of isotropic subspaces in T. This determines a Borel subgroup in G. We denote by $\mathrm{Grass}(i, R)$ the Grassmannian of i-dimensional subspaces in R and by $\mathrm{IGrass}(i, T)$ the Grassmanian of isotropic i-dimensional subspaces in T.

Case 1. The structure of the closures of G-orbits is as follows:

$$\overline{\mathcal{O}}_0 \subset \overline{\mathcal{O}}_1 \subset \overline{\mathcal{O}}_{2'} \subset \overline{\mathcal{O}}_2 \subset \overline{\mathcal{O}}_3 \subset \overline{\mathcal{O}}_4,$$

where

$$\mathcal{O}_i = \{f \in \mathrm{Hom}(T, R) \mid \dim \mathrm{Im}\, f = i\},$$
$$\mathcal{O}_{2'} = \{f \in \mathrm{Hom}(T, R) \mid \dim \mathrm{Im}\, f = 2 \text{ and } \mathrm{Ker}\, f \text{ is isotropic}\}.$$

In particular, $\overline{\mathcal{O}}_4 = \mathrm{Hom}(T, R)$ and $\mathcal{O}_0 = \{0\}$. Consider the flag

$$0 = W_0 \subset W_1 \subset W_{2'} \subset W_2 \subset W_3 \subset W_4 = V$$

of linear subspaces in V defined in the following way: $W_1 = T_1 \otimes R_1$, $W_{2'} = T_2 \otimes R_2$, $W_2 = T \otimes R_2$ and $W_3 = T \otimes R_3$. For $i = 0, 1, 2', 2, 3, 4$ we denote by P_i the stabilizer of W_i in G and we write $Z_i = G \times_{P_i} W_i$. Then Z_1 is the standard desingularization of the highest weight orbit,

$$Z_{2'} = \{(f, \overline{T}_2, \overline{R}_2) \mid f \in V,\ \overline{T}_2 \in \mathrm{IGrass}(2, T),\ \overline{R}_2 \in \mathrm{Grass}(2, R),\ f(\overline{T}_2) = 0,\ \mathrm{Im}\, f \subset \overline{R}_2\}$$

and

$$Z_i = \{(f, \overline{R}_i) \in V \times \mathrm{Grass}(i, R) \mid \mathrm{Im}\, f \subset \overline{R}_i\}$$

for $i = 2, 3, 4$, so they form a coherent family of desingularizations of orbits.

For weights μ satisfying (iv) we can take the highest weights of $\bigwedge^2 T_2^* \otimes \bigwedge^2 R_2^* \subset S_2(W_{2'}^*)$, $T \otimes \bigwedge^3 R_3^* \subset S_3(W_3^*)$ and $\mathbb{C} \otimes \bigwedge^2 R_2^* \subset S_2(W_2^*)$.

CASE 2. We have two subcases: $\dim R = 2$, and $\dim R = 3$. For $\dim R = 2$ we have $\mathcal{O}_0 \subset \overline{\mathcal{O}}_1 \subset \overline{\mathcal{O}}_{2'} \subset \overline{\mathcal{O}}_2$, where

$$\mathcal{O}_i = \{f \in \mathrm{Hom}(T, R) \mid \dim \mathrm{Im} f = i\},$$

and $\mathcal{O}_{2'}$ consists of the maps $f \in \mathrm{Hom}(T, R)$ of rank 2 such that Ker f contains a maximal isotropic subspace of T. A coherent family of orbit desingularizations can be obtained from $W_1 = T_1 \otimes R_1$, $W_{2'} = T_2 \otimes R_2$, $W_{3'} = T_3 \otimes R_3$ and $W_4 = T \otimes R$.

For $\dim R = 3$ we have six orbits with the following structure of closures: $\mathcal{O}_0 \subset \overline{\mathcal{O}}_1 \subset \overline{\mathcal{O}}'_2$, $\overline{\mathcal{O}}'_3$ and $\overline{\mathcal{O}}_2$ both contained in $\overline{\mathcal{O}}_3$ and containing $\overline{\mathcal{O}}'_2$. The desingularization of $\overline{\mathcal{O}}_i$ is given by

$$X_i = \{(f, \overline{R}_i) \in V \times \mathrm{Grass}(i, R) \mid \mathrm{Im} f \subset \overline{R}_i\}.$$

The desingularization of $\mathcal{O}'_i$ is given by

$$Y'_i = \{(f, \overline{T}_2, \overline{R}_2) \mid f \in X,\ \overline{T}_i \in \mathrm{IGrass}(i, T),\ \overline{R}_i \in \mathrm{Grass}(i, R),\\ f(\overline{T}_i^{\vee}) = 0,\ \mathrm{Im} f \subset \overline{R}_i\}.$$

4.4. *Spinor groups*. Let T be a complex vector space of dimension n with a nondegenerate symmetric form. By a *spinor representation* of the group $\mathrm{Spin}(T)$ we mean an irreducible representation V of highest weight $(1/2, \ldots, 1/2)$. For a spinor representation the orbit of highest weight vectors is called the *pure spinor orbit*. In fact, we will always assume that the group $G = \mathrm{Spin}(T) \times \mathbb{C}^*$ acts on V. This action is multiplicity free for $n = 7, 8, 9$ or 10 only. For n=7 and 10 there are only three orbits: 0, the pure spinors and an open orbit. Moreover, the actions for two other values of n are isomorphic, so we will work out the case $n = 9$. In this case there are four orbits with the following structure of closures: $\mathcal{O}_0 \subset \overline{\mathcal{O}}_1 \subset \overline{\mathcal{O}}_2 \subset \overline{\mathcal{O}}_3$, where $\mathcal{O}_0$ is the zero orbit, $\mathcal{O}_1$ consists of the pure spinors and $\overline{\mathcal{O}}_2$ is given by the invariant of degree two.

In order to describe a desingularization of $\overline{\mathcal{O}}_2$ we need a description of the spinor representation V. Let e_0 be a vector in T with $(e_0, e_0) = -1$ and let U and $\overline{U}$ be two maximal isotropic subspaces in T such that $T = U \oplus \mathbb{C}e_0 \oplus \overline{U}$ is a Witt decomposition. Let e_1, e_2, e_3, e_4 be a basis of U and let $\overline{e}_1$, $\overline{e}_2$, $\overline{e}_3$, $\overline{e}_4$ be the dual basis of $\overline{U}$. Let $C^+(T)$ be the even part of the Clifford algebra of T. Let $T' = U \oplus \overline{U}$. A map $i : T' \to C^+(T)$ given by $i(w) = we_0$ induces an isomorphism of the Clifford algebras $C(T')$ and $C^+(T)$. The group $\mathrm{Spin}(T)$ and the representation V can both be realized as subsets of $C(T')$. The representation V can be constructed as a left ideal of $C(T')$ generated by the product $f = \overline{e}_1\overline{e}_2\overline{e}_3\overline{e}_4$. The space V is isomorphic to the exterior algebra $\bigwedge^{\cdot} U$ and the isomorphism is given by multiplication by f. Hence the products $e_{i_1} \ldots e_{i_k} f$, $0 \le k \le 4$, $i_1 < \ldots < i_k$, form a basis of V.

For a spinor

$$s = x_0 f + \sum_{i<j} x_{ij} e_i e_j f + y_0 e_1 e_2 e_3 e_4 f + \sum_i x_i e_i f + \sum_i y_i e_i^* f \tag{18}$$

we write

$$g(s) = x_0 y_0 + Pf([x_{ij}]) + \sum_i x_i y_i. \tag{19}$$

Then g is a semiinvariant of G and $\overline{\mathcal{O}}_2$ is given by the equation $g(s) = 0$. Let W be the subspace of V spanned by all products containing e_1. The stabilizer P of W is generated by a Borel subgroup B of G and by the spinor group of the space T' spanned by e_2, e_3, e_4, $\overline{e}_2$, $\overline{e}_3$, $\overline{e}_4$ and e_0. The homogeneous space G/P is the Grassmannian of one-dimensional isotropic subspaces in T. We claim that the vector bundle $Z = G \times_P W$ over G/P is a desingularization of $\overline{\mathcal{O}}_2$. To see this we need a more geometric description of Z. In fact, we have

$$Z = \{(s, L) \in C^+(T) \times \mathrm{IGrass}(1, T) \mid g(s) = 0,\ Ls = 0\}.$$

LEMMA 10. *For every $s \in \mathcal{O}_2$ there exists a unique isotropic line $L \subset T$ such that $Ls = 0$ in $C(T)$.*

Proof. This follows by direct calculation for $s_0 = e_1 f + e_1 e_2 e_3 e_4 f$. The only line satisfying this condition is $L_0 = ke_1$. For arbitrary s in the orbit we have $s = gs_0$ for some $g \in \mathrm{Spin}(T)$ and $L = gL_0 g^{-1}$ is the only line satisfying $Ls = 0$.

The lemma gives a birational morphism $\overline{\mathcal{O}}_2 \to F$.

4.5. *Group of type E_6.* The following construction of an algebraic group of type E_6 is due to Dickson.

Let V be a complex vector space of triples (x, z, y) where x and y are vectors in $\mathbb{C}^6$ (we prefer to write them as columns) and $z = [z_{ij}]$ is a 6×6 antisymmetric matrix. It is a 27-dimensional vector space. We fix a basis e_i, $1 \le i \le 6$, g_{ij}, $1 \le i < j \le 6$, f_i, $1 \le i \le 6$, such that $(x, z, y) = \sum_i x_i e_i + \sum_{i<j} z_{ij} g_{ij} + \sum_i y_i f_i$. It is convenient to adopt the convention that $g_{ji} = -g_{ij}$. Let H be the subgroup of $\mathrm{GL}(V)$ consisting of the linear transformations preserving the cubic form

$$c(x, z, y) = x^t z y + Pf(z). \tag{20}$$

Then H is a simple algebraic group of type E_6. We need a more detailed description of its structure.

Let $d = d(t_1, \ldots, t_6)$, $t_i \in \mathbb{C}^*$, be the linear transformation of V given by the formula

$$\begin{aligned} d(e_i) &= t_1^{-1} t_2^{-1} \ldots t_i^2 \ldots t_6^{-1} e_i, \\ d(g_{ij}) &= t_1 \ldots t_i^{-2} \ldots t_j^{-2} \ldots t_6 g_{ij}, \\ d(f_i) &= t_i^3 f_i. \end{aligned}$$

The transformations $d = d(t_1, \dots, t_6)$, $d_i \in \mathbb{C}^*$, form a maximal torus T of H. Furthermore, we have a collection of one-parameter subgroups of H: L, L', A_{ik}, $1 \le i, k \le 6$, $i \ne k$, B_{ijk} and C_{ijk}, $1 \le i < j < k \le 6$. Elements of these groups are parameterized by $t \in \mathbb{C}$ and each of them acts identically on most basis elements of V. The essential values are as follows:

$$L_t(f_i) = f_i + te_i, \quad i = 1, \dots, 6,$$
$$L'_t(e_i) = e_i + tf_i, \quad i = 1, \dots, 6.$$

For $A = A_{ik,t}$ we have

$$A(e_k) = e_k + te_i, \quad A(f_k) = f_k + tf_i,$$
$$A(g_{ij}) = g_{ij} - tg_{kj}, \quad j \ne i, k.$$

If $i < j < k$ we complete the sequence (i, j, k) to an even permutation (i, j, k, l, m, n) of $\{1, \dots, 6\}$. Then for $B = B_{ijk,t}$ and $C = C_{ijk,t}$ we put

$$\begin{aligned}
B(f_l) &= f_l - tg_{mn}, & C(e_l) &= e_l + tg_{mn},\\
B(f_m) &= f_m - tg_{nl}, & C(e_m) &= e_m + tg_{nl},\\
B(f_n) &= f_n - tg_{lm}, & C(e_n) &= e_n + tg_{lm},\\
B(g_{jk}) &= g_{jk} + te_i, & C(g_{jk}) &= g_{jk} + tf_i,\\
B(g_{ki}) &= g_{ki} + te_j, & C(g_{ki}) &= g_{ki} + tf_j,\\
B(g_{ij}) &= g_{ij} + te_k, & C(g_{ij}) &= g_{ij} + tf_k.
\end{aligned}$$

The listed subgroups correspond to 72 roots of the root system of H. We can choose a basis α_1, α_2, α_3, α_4, α_5, α_6 consisting of the roots corresponding to the subgroups A_{12}, B_{456}, A_{23}, A_{34}, A_{45} and A_{56}, respectively. Then the maximal unipotent subgroup U^+ corresponding to the positive roots is generated by the subgroups A_{ij}, $1 \le j < i \le 6$, B_{ijk}, $1 \le i < j < k \le 6$, and L. We have a Borel subgroup $B = TU^+$. The simple roots α_i are identified with the characters of T in the following way:

$$\begin{aligned}
\alpha_1 &= (3, -3, 0, 0, 0, 0), & \alpha_4 &= (0, 0, 3, -3, 0, 0),\\
\alpha_2 &= (-2, -2, -2, 1, 1, 1), & \alpha_5 &= (0, 0, 0, 3, -3, 0),\\
\alpha_3 &= (0, 3, -3, 0, 0, 0), & \alpha_6 &= (0, 0, 0, 0, 3, -3).
\end{aligned}$$

Similarly, the fundamental weights are

$$\begin{aligned}
\omega_1 &= (2, -1, -1, -1, -1, -1), & \omega_4 &= (0, 0, 0, -3, -3, -3),\\
\omega_2 &= (-1, -1, -1, -1, -1, -1), & \omega_5 &= (0, 0, 0, 0, -3, -3),\\
\omega_3 &= (1, 1, -2, -2, -2, -2), & \omega_6 &= (0, 0, 0, 0, 0, -3).
\end{aligned}$$

Half of the sum of the positive roots is $\varrho = \omega_1 + \ldots + \omega_6 = (2, -1, -4, -7, -10, -13)$. The scalar product invariant under the action of the Weyl group is given by the matrix

$$\frac{1}{27}\begin{bmatrix} 4 & 1 & 1 & 1 & 1 & 1 \\ 1 & 4 & 1 & 1 & 1 & 1 \\ 1 & 1 & 4 & 1 & 1 & 1 \\ 1 & 1 & 1 & 4 & 1 & 1 \\ 1 & 1 & 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 1 & 1 & 4 \end{bmatrix}.$$

The space V is an irreducible representation of H of highest weight ω_1. We want to study the orbits of the action of the group $G = H \times \mathbb{C}^*$ on V. According to the results of [H-U] there are four orbits with the linear order on the closures: $\{0\} = \overline{\mathcal{O}}_0 \subset \overline{\mathcal{O}}_1 \subset \overline{\mathcal{O}}_2 \subset \overline{\mathcal{O}}_3 = V$. $\mathcal{O}_1$ is the orbit of the highest weight vector e_1. Hence $\overline{\mathcal{O}}_1$ has the standard desingularization and the ring of regular functions on it is $\mathbb{C}[\overline{\mathcal{O}}_1] = \bigoplus_{n\ge 0} V_{n\omega_6}$.

The closure of $\mathcal{O}_2$ is the zero place of the form c. Hence $\dim \mathcal{O}_2 = 26$. As a representative of the orbit $\mathcal{O}_2$ we choose $e_5 + g_{56}$. Let W be the linear subspace of V spanned by e_1, e_2, e_3, e_4, e_5 and g_{56}, g_{46}, g_{36}, g_{26}, g_{16}. Then W is stabilized by the parabolic subgroup P containing the subgroups A_{21}, A_{32}, A_{43}, A_{54} and C_{123}. The product $Z = G \times_P W$ is a vector bundle over G/P of rank 10 and its collapsing is a resolution of singularities of $\overline{\mathcal{O}}_2$. This follows from the fact that the line spanned by $e_5 + g_{56}$ is nonsingular in the sense of [A] and every element in its orbit is contained in exactly one subspace G-conjugate to W due to (6.7) of [A].

The Levi factor L of P acts on W by linear transformations preserving the nondegenerate bilinear symmetric form

$$(e_i, g_{j,6}) = \delta_{i,j}, \quad (e_i, e_j) = (g_{i,6}, g_{j,6}) = 0$$

and by the one-dimensional torus $d(1,1,1,1,1,t)$, $t \in \mathbb{C}^*$. Thus W is an irreducible L-module with highest weight $(2,-1,-1,-1,-1,-1)$ and W^* can be identified with $V_L(\lambda)$, where $\lambda = (2,-1,-1,-1,-1,1)$. The element $\sum_{i=1}^{6} e_i^* g_{i,6}^* \in S_2(W^*)$ spans a one-dimensional L-submodule on which L acts by the character $\mu' = (0,0,0,0,0,3)$. Since the positive root $\alpha = \sum_{i=1}^{6} a_i\alpha_i$ of H is not a root of L if and only if $a_6 > 0$, condition (iv) follows from the inequality $(\mu', \alpha_6) > 0$.

4. Remarks. It seems that Theorem 3 should be true in a more general context.

Conjecture. *If X is an affine spherical variety for the group G then the Grothendieck group $K_0'(X)$ is isomorphic to the additive group of the ring $\mathbb{Z}[e^{\omega_1}, \ldots, e^{\omega_r}][q, q^{-1}]$ where $\omega_1, \ldots, \omega_r$ are the fundamental weights for G.*

Example. Let U be a vector space of dimension n. Define $G = \mathrm{GL}(U)$ and $V = U \oplus \bigwedge^2 U$. We will identify V with the set of pairs (u, f) where u

is an element of U and $f: U^* \to U$ is an antisymmetric linear map. Orbits of G in V are parameterized by the rank of f and the relation of u to the image of the map f. More precisely, we have three types of orbits:

$$\mathcal{O}_{0,i} = \{(0,f) \mid \mathrm{rk}(f) = 2i\},$$
$$\mathcal{O}_{1,i} = \{(u,f) \mid \mathrm{rk}(f) = 2i,\ u \in \mathrm{Im}(f) \setminus \{0\}\},$$
$$\mathcal{O}_{2,i} = \{(u,f) \mid \mathrm{rk}(f) = 2i,\ u \notin \mathrm{Im}(f)\}.$$

There are 6 different types of minimal degenerations of orbits: $\overline{\mathcal{O}}_{0,i-1} \subset \overline{\mathcal{O}}_{0,i}$, $\overline{\mathcal{O}}_{1,i-1} \subset \overline{\mathcal{O}}_{1,i}$, $\overline{\mathcal{O}}_{2,i-1} \subset \overline{\mathcal{O}}_{2,i}$, $\overline{\mathcal{O}}_{0,i} \subset \overline{\mathcal{O}}_{1,i}$, $\overline{\mathcal{O}}_{1,i} \subset \overline{\mathcal{O}}_{2,i}$ and $\overline{\mathcal{O}}_{2,i-1} \subset \overline{\mathcal{O}}_{1,i}$.

Taking $W_{0,i} = 0 \oplus \bigwedge^2 U_{2i}$, $W_{1,i} = U_{2i} \oplus \bigwedge^2 U_{2i}$ and $W_{2,i} = U_{2i+1} \oplus \bigwedge^2 U_{2i}$ we obtain a coherent desingularization of orbit closures satisfying (iii). It also satisfies (iv) since for μ we can take the highest weight of $S_{2i}(\bigwedge^2 U_{2i}^*)$ in $S_{2i}(W_{k,i}^*)$ for $k = 0,1$, and the highest weight of $S_{2i}(\bigwedge^2 U_{2i}^* \otimes (U_{2i+1}/U_{2i})^*)$ in $S_{2i+1}(W_{k,i}^*)$ for $k = 2$.

REFERENCES

[A] M. Aschbacher, *The* 27-*dimensional module for* E_6. *I*, Invent. Math. 89 (1987), 159–195.

[D] M. Demazure, *A very simple proof of Bott's theorem*, ibid. 33 (1976), 271–272.

[H-U] R. Howe and T. Umeda, *The Capelli identity*, *the double commutant theorem*, *and multiplicity free actions*, Math. Ann. 290 (1991), 565–619.

[I] J. Igusa, *A classification of spinors up to dimension twelve*, Amer. J. Math. 92 (1970), 997–1028.

[K] V. Kac, *Some remarks on nilpotent orbits*, J. Algebra 64 (1980), 190–213.

[W] J. Weyman, *The Grothendieck group of* $\mathrm{GL}(F) \times \mathrm{GL}(G)$-*equivariant modules over the coordinate ring of determinantal varietes*, Colloq. Math. 76 (1998), 243–263.

Faculty of Mathematics and Informatics
Nicolas Copernicus University
Chopina 12/18
87-100 Toruń, Poland
E-mail: klimek@mat.uni.torun.pl
wkras@mat.uni.torun.pl

Department of Mathematics
Northeastern University
Boston, Massachusetts 02115
U.S.A.
E-mail: weyman@neu.edu

Received 18 December 1997;
revised 4 February 1998