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EVALUATION MAPS, RESTRICTION MAPS, AND COMPACTNESS

BY

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1. Introduction. If K is a bounded subset of the Banach space X and $B(K)$ is the Banach space (sup norm) of all bounded real-valued functions defined on K, then the natural evaluation map $E : X^* \to B(K)$ defined by $E(x^*)(k) = x^*(k)$ has been used by many authors to study properties of K. Specifically we mention Propositions 1 and 5 of Bator [4], Theorem 1 of Saab [31], and Proposition 1 of Pełczyński [27]. Similarly, if K is a bounded subset of X^*, then one may define two natural evaluation maps: (1) $E_X : X \to B(K)$ and (2) $E_{X^{**}} : X^{**} \to B(K)$. Properties of K are reflected in both E_X and $E_{X^{**}}$, as well as in the restriction of these operators to certain subspaces. In this paper we study connections among certain compactness properties of K, evaluation maps, and restriction operators. In particular, we use the notion of bibasic sequences to study limited sets which fail certain compactness conditions.

2. Definitions and terminology. Throughout the paper, X and Y will denote real Banach spaces with continuous linear duals denoted by X^* and Y^*. The unit ball of X will normally be denoted by B_X; for simplicity, we denote the unit ball of $B(K)^*$ by B^*. If $T : X \to Y$ is a bounded linear transformation (= operator), then T^* will denote the adjoint of T. The space of all bounded linear operators from X to Y will be denoted by $B(X, Y)$. A subset K of X will be termed *weakly precompact* if every bounded sequence in K has a weakly Cauchy subsequence. Thus a bounded weakly precompact set is weakly conditionally compact. We denote the closed linear span of K by $[K]$. Further, a subset K of X (resp. K of X^*) is called a V^*-*set* (resp. V-*set*) if

$$\lim_n (\sup\{|x_n^*(x)| : x \in K\}) = 0,$$

respectively

$$\lim_n (\sup\{|x^*(x_n)| : x^* \in K\}) = 0,$$

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for each weakly unconditionally converging series $\sum x_n^*$ in X^* (resp. $\sum x_n$ in X).

A bounded subset K of the Banach space X is called a *Dunford–Pettis set* (or *DP-set*) if

$$\lim_n(\sup\{|x_n^*(x)| : x \in K\}) = 0$$

for every weakly null sequence (x_n^*) in X^*. The set K is called a *limited set* if

$$\lim_n(\sup\{|x_n^*(x)| : x \in K\}) = 0$$

for each w^*-null sequence (x_n^*) in X^*. Certainly, every relatively compact subset of X is limited. We also remark that Kevin Andrews [2] showed that a bounded subset K of X is a DP-set iff $T(K)$ is relatively norm compact for each weakly compact operator $T : X \to Y$. We shall use this equivalent formulation for Dunford–Pettis sets whenever it is convenient.

Closely related to the notions of DP-sets and limited sets is the idea of an L-set, e.g., see Bator [4] and Emmanuele [17], [18]. A bounded subset K of X^* is called an *L-set* if

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in K\}) = 0$$

for each weakly null sequence (x_n) in X.

Recall that the Banach space X is said to have the *Dunford–Pettis property* (or *DPP*) if every weakly compact operator $T : X \to Y$ is completely continuous, and X is said to have the *reciprocal Dunford–Pettis property* (or *RDPP*) if every completely continuous operator $T : X \to Y$ is weakly compact. We refer the reader to Diestel [12] or Diestel and Uhl [13] for any unexplained notation or terminology.

3. Evaluation maps. The following theorem explicitly motivates our consideration of evaluation maps. We use E to represent a generic evaluation map.

THEOREM 3.1. (i) *A bounded subset K of X^* is an L-set in X^* iff $E : X \to B(K)$ is completely continuous.*

(ii) *An operator $T : X \to Y$ is completely continuous iff $T^*(B_{Y^*})$ is an L-set in X^*.*

(iii) *If K is a bounded subset of X^*, then K is an L-set iff $T(K)$ is relatively compact in Y for each (w^*, w)-continuous operator $T : X^* \to Y$.*

(iv) *A bounded subset K of X is a Dunford–Pettis set iff $E : X^* \to B(K)$ is completely continuous.*

(v) *A subset K of X is a Dunford–Pettis set iff there is a Banach space Y and an operator $T : Y \to X$ so that T and T^* are completely continuous and $K \subseteq T(B_Y)$.*

(vi) *The Banach space X has RDPP iff each evaluation map $E_K : X \to B(K)$ which is completely continuous is also weakly compact.*

(vii) *A bounded subset K of X^* is a V-set iff $E : X \to B(K)$ is unconditionally converging.*

(viii) *A bounded subset K of X is a V^*-set iff $E : X^* \to B(K)$ is unconditionally converging.*

(ix) *A bounded subset K of X is a limited set iff $E : X^* \to B(K)$ is $(w^*, norm)$-sequentially continuous.*

Proof. (i) Suppose that K is a bounded subset of X^*. The evaluation map $E : X \to B(K)$ is completely continuous iff $\|E(x_n)\| \to 0$ for each weakly null sequence (x_n) in X iff

$$\lim_n (\sup\{|x^*(x_n)| : x^* \in K\}) = 0$$

for each weakly null sequence (x_n) in X iff K is an L-set.

(ii) Suppose that $T : X \to Y$ is an operator. Clearly, $T^*(B_{Y^*})$ is an L-set iff

$$\begin{aligned} &\lim_n (\sup\{|\langle T^*(y^*), x_n\rangle : y^* \in B_{Y^*}\}) \\ &\qquad = \lim_n (\sup\{|\langle y^*, T(x_n)\rangle : y^* \in B_{Y^*}\}) = \lim \|T(x_n)\| = 0 \end{aligned}$$

for each weakly null sequence (x_n) in X, i.e., iff T is completely continuous.

(iii) Suppose that K is an L-subset of X^* and $T : X^* \to Y$ is (w^*, w)-continuous. Therefore T is weakly compact and $T^*(Y^*) \subseteq X$. Moreover, T^* is (w^*, w)-continuous.

Now let (x_n^*) be a sequence in K, and (without loss of generality) suppose that $(T(x_n^*)) \to y$ weakly. In order to obtain a contradiction, we suppose that $(T(x_n^*))$ has no norm convergent subsequence. In fact, we suppose that $\varepsilon > 0$ and $\|T(x_n^*) - y\| > \varepsilon$ for all n. For each n, choose $y_n^* \in B_{Y^*}$ so that

$$(*) \qquad \langle y_n^*, T(x_n^*) - y\rangle > \varepsilon.$$

We may (and do) assume that $x \in X$ and $(T^*(y_n^*)) \to x$ weakly. Now let y^* be a w^*-cluster point of (y_n^*). The (w^*, w)-continuity of T^* ensures that $T^*(y^*) = x$. Thus $(T^*(y_n^* - y^*))$ is weakly null, and

$$\begin{aligned} 0 &= \lim\langle x_n^*, T^*(y_n^* - y^*)\rangle = \lim\langle T(x_n^*), y_n^* - y^*\rangle \\ &= \lim(\langle T(x_n^*) - y, y_n^* - y^*\rangle + \langle y, y_n^* - y^*\rangle) \\ &= \lim(\langle T(x_n^*) - y, -y^*\rangle + \langle T(x_n^*) - y, y_n^*\rangle + \langle y, y_n^* - y^*\rangle). \end{aligned}$$

Since $(T(x_n^*)) \to y$ weakly, we appeal to $(*)$ above and conclude that

$$|\langle y, y_n^* - y^*\rangle| > \varepsilon/2$$

for sufficiently large n. This is a clear contradiction of the fact that y^* is a w^*-cluster point of (y_n^*).

Conversely, suppose that if Y is a Banach space and $T : X^* \to Y$ is a (w^*, w)-continuous operator, then $T(K)$ is relatively compact. Let (x_n) be a weakly null sequence in X, and define $T : X^* \to c_0$ by $T(x^*) = (x^*(x_n))$. Then T is (w^*, w)-continuous, $T(K)$ is relatively compact in c_0, and

$$\lim_n(\sup\{|\langle x_n, x^*\rangle| : x^* \in K\}) = 0.$$

(iv) Suppose that K is a bounded subset of X and $E : X^* \to B(K)$ is completely continuous. Thus E^* maps B^*, the unit ball of $B(K)^*$, to an L-set in X^{**}. However, if $k \in K$ and δ_k denotes the point mass at k, then

$$E^*(\{\delta_k : k \in K\}) = K,$$

and K is an L-set in X^{**}. Therefore K is a Dunford–Pettis set in X.

Conversely, suppose that K is a DP-set in X, and let $E : X^* \to B(K)$ be the evaluation map. If $(x_n^*) \to 0$ weakly in X^*, then

$$\lim(\sup\{|x_n^*(x)| : x \in K\}) = \lim_n \|E(x_n^*)\| = 0,$$

and E is completely continuous.

(v) Suppose that K is a Dunford–Pettis set, and let $\mathrm{cach}(K)$ denote the closed absolutely convex hull of K. Note that $\mathrm{cach}(K)$ is also a DP-set. Let $Y = \ell^1(K)$, and define $T : Y \to X$ by $T(f) = \sum_{k\in K} f(k)k$ for $f \in \ell^1(K)$. Then T is a bounded linear operator, and $K \subseteq T(B_{\ell^1(K)}) \subseteq \mathrm{cach}(K)$. Since $\ell^1(K)$ is a Schur space, T is completely continuous. Further, T^* is the evaluation map $E : X^* \to B(K)$, and T^* is completely continuous by (iv).

As was noted in Section 2, the converse is immediate from the complete continuity of T^*.

(vi) If X has RDPP, then every completely continuous map on X is weakly compact. Conversely, suppose that each evaluation map $E_K : X \to B(K)$ which is completely continuous is also weakly compact. Let $T : X \to Y$ be completely continuous. Thus $K = T^*(B_{Y^*})$ is an L-set by (ii), $E_K : X \to B(K)$ is completely continuous by (i), and E_K is weakly compact by hypothesis. Hence $E_K^* : B(K)^* \to X^*$ is a weakly compact operator, and, as in (iii) above, K is relatively weakly compact. Consequently, T^* and T are weakly compact, and X has RDPP.

(vii) Suppose that K is a V-subset of X^*, i.e.,

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in K\}) = 0$$

for each wuc series $\sum x_n$ in X. However, since all rearrangements and all subseries of a wuc series are wuc, the preceding equality implies that

$$\lim_n \Big(\sup\Big\{\sum_{i=n}^{\infty} |x^*(x_i)| : x^* \in K\Big\}\Big) = 0$$

for each wuc series $\sum x_n$ in X. Thus $\sum E_K(x_i)$ is bounded multiplier convergent, and $\sum E_K(x_i)$ is unconditionally convergent whenever $\sum x_i$ is wuc.

Conversely, if E_K is an unconditionally converging operator and $\sum x_n$ is wuc in X, then certainly $\|E_K(x_n)\| \to 0$ as $n \to \infty$. That is,

$$\lim_n(\sup\{|x^*(x_n)| : x^* \in K\}) = 0.$$

(viii) The proof of (viii) is identical to that of (vii) and will be omitted.

(ix) A bounded subset K of X is a limited set iff

$$\lim_n(\sup\{|x_n^*(x)| : x \in K\}) = 0$$

for each w^*-null sequence (x_n^*) in X^* iff

$$\lim_n(\sup\{|\langle y_n^* - y^*, x\rangle| : x \in K\}) = 0$$

whenever $(y_n^*) \to y^*$ in the w^*-topology of X^* iff $E : X^* \to B(K)$ is (w^*, norm)-sequentially continuous. ∎

Corollary 3.2. (a) *A limited subset of a separable Banach space X is relatively compact.*

(b) *A limited subset of a reflexive Banach space X is relatively compact.*

(c) *If ℓ^1 does not embed in X^*, then a limited subset of X is relatively compact.*

(d) *If X is any subspace of a weakly compactly generated Banach space, then a limited subset of X is relatively compact.*

(e) *If X^* has the Radon–Nikodym property, then a limited subset of X is relatively compact.*

(f) *The subset K of X is relatively compact iff $\{x_n : n \in \mathbb{N}\}$ is limited in $[x_n : n \in \mathbb{N}]$ for each sequence (x_n) from K.*

(g) *If $T : X \to Y$ is a limited operator (i.e., $T(B_X)$ is limited in Y), then T is strictly cosingular.*

Proof. The proofs of (a)–(e) are the same. In each case, each sequence from B_{X^*} has a w^*-Cauchy subsequence. (Standard beginning techniques from functional analysis handle (a) and (b), Rosenthal's ℓ^1-theorem [29] furnishes the subsequence in (c), results of Amir and Lindenstrauss [1] take care of (d), and a theorem of Johnson and Hagler [22], [12, p. 230] provides the subsequence in (e).) Therefore E and E^* are compact. Since $K \subseteq E^*(B^*)$, K is relatively compact. Part (f) follows immediately from 3.1(ix). Finally, the Josefson–Nissenzweig theorem [12, Chap. XII] and 3.1(ix) immediately yield (g). Specifically, if Y is infinite-dimensional, $T : X \to Y$ is a surjection, and $K = T(B_X)$, then let (y_n^*) be a w^*-null sequence of norm one members of Y^* and note that $\|E(y_n^*)\| \not\to 0$. ∎

We remark that (a), (b), and (g) of 3.2 appeared in Bourgain and Diestel [11] with much different proofs; the arguments above seem to be significantly simpler.

We also note that (vii) of 3.1 was observed by Pełczyński [27]. The proof of (vii) presented here, however, uses an observation of Bombal [8].

If K is a bounded subset of X^*, $E : X^{**} \to B(K)$ is the evaluation map, and $T = E|_X$, then certainly T^{**} is not necessarily E. However, if K is an L-subset, then the restriction of T^{**} to the w^*-sequential extension of X in X^{**} is E. In the following theorem, let $B_1(X) = \{x^{**} \in X^{**} : x^{**}$ is a w^*-limit of a *sequence* in $X\}$.

THEOREM 3.3. *If K is an L-subset of X^*, $E : X^{**} \to B(K)$ is the evaluation map*, *and $T = E|_X$, then*

$$T^{**}|_{B_1(X)} = E|_{B_1(X)}.$$

Proof. Suppose that K is an L-subset of X^*. By 3.1(i), $T : X \to B(K)$ is completely continuous. Let $x^{**} \in B_1(X)$, and let (x_n) be a sequence in X so that $(x_n) \to x^{**}$ in the w^*-topology. Certainly, (x_n) is weakly Cauchy in X, as well as in X^{**}. Therefore $(T(x_n))$ is norm convergent. Let $f \in B(K)$ so that $(E(x_n)) = (T(x_n)) \to f$. If $k \in K$, then

$$f(k) = \lim\langle T(x_n), k\rangle = k(x_n) = x^{**}(k) = \langle E(x^{**}), k\rangle.$$

Consequently, $f = E(x^{**})$ and $\|E(x_n) - E(x^{**})\| \to 0$.

Moreover, since adjoints are (w^*, w^*)-continuous,

$$T^{**}(x^{**}) = w^*\text{-}\lim T(x_n) = w^*\text{-}\lim E(x_n) = E(x^{**}).$$

Thus $T^{**}(x^{**}) = E(x^{**})$ for each $x^{**} \in B_1(X)$. ∎

An immediate corollary of 3.3 is that if $B_1(X) = X^{**}$, then every L-subset of X^* is relatively weakly compact. In [26] Odell and Rosenthal show that if X is separable, then $B_1(X) = X^{**}$ iff ℓ^1 does not embed in X. However, the proof that ℓ^1 does not embed in X if $B_1(X) = X^{**}$ does not use the separability of X. Thus if $B_1(X) = X^{**}$, then ℓ^1 does not embed in X, and every L-subset (and consequently every DP-subset) of X^* is relatively compact.

The following theorem continues our study of the connection between L-subsets and DP-subsets of dual spaces.

THEOREM 3.4. *Every L-subset of X^* is a Dunford–Pettis set in X^* iff T^{**} is completely continuous whenever Y is an arbitrary Banach space and $T : X \to Y$ is a completely continuous operator.*

Proof. Suppose that every L-subset of X^* is Dunford–Pettis, and let $T : X \to Y$ be a completely continuous operator. Therefore $T^*(B_{Y^*})$ is an

L-set and hence a Dunford–Pettis set. Let (x_n^{**}) be a w-null sequence in X^{**}. Therefore

$$\sup\{|\langle T^*(y^*), x_n^{**}\rangle| : y^* \in Y^*, \|y^*\| \le 1\} = \|T^{**}(x_n^{**})\|, \qquad \|T^{**}(x_n^{**})\| \to 0,$$

and T^{**} is completely continuous.

Conversely, suppose that T^{**} is completely continuous whenever T is. Let K be an L-subset of X^*, and let $E : X \to B(K)$ be the evaluation map. Since E is completely continuous (Theorem 3.1(i)), E^{**} is completely continuous. Therefore $E^*(B^*)$ is a Dunford–Pettis set. Since $K \subseteq E^*(B^*)$, K is a Dunford–Pettis set. ∎

COROLLARY 3.5. *If K is a compact Hausdorff space, then every L-subset of $C(K)^*$ is a Dunford–Pettis set.*

Proof. Suppose that K is a compact Hausdorff space, Y is a Banach space, and $T : C(K) \to Y$ is completely continuous. Therefore T is weakly compact [6, Theorem 1] and T^{**} is weakly compact. Since $C(K)^{**}$ is also a continuous function space, T^{**} is completely continuous, and L-subsets of $C(K)^*$ are Dunford–Pettis by the preceding theorem. ∎

Professor G. Emmanuele recently contributed significantly to the understanding of Dunford–Pettis sets and L-sets. In Theorem 3 of [19] he showed that if X and Y contain no copies of ℓ^1 and all operators from X to Y^* are compact, then $X \otimes_\gamma Y$ does not contain a copy of ℓ^1 either. A key step in the proof of this theorem involves the following assertion about weak convergence in the greatest crossnorm tensor product completion. If X and Y are as above, (x_n) is weakly null in X, and (y_n) is bounded in Y, then $(x_n \otimes y_n)_{n=1}^\infty$ is weakly null in $X \otimes_\gamma Y$.

Bilyeu and Lewis [6] showed that if $1 \le p < \infty$, $\sum x_n$ is weakly p-summable in X, and (y_n) is bounded in Y, then $\sum x_n \otimes y_n$ is weakly p-summable in $X \otimes_\lambda Y$, the least crossnorm tensor product completion of X and Y. Consequently, it is natural to ask if Emmanuele's result always holds in the γ-norm. We first show that this assertion is false in general, and then we investigate necessary and sufficient conditions which will ensure its validity.

We recall that $B(X,Y)$ is certainly isometrically and isomorphically contained in $B(X,Y^{**})$, and $B(X,Y^{**})$ is isometrically isomorphic to $(X \otimes_\gamma Y^*)^*$ in a natural way. (We refer the reader to Chapter VIII of Diestel and Uhl [13] for a discussion of tensor products.) Now let $X = Y =$ an infinite-dimensional reflexive space. Apply the Josefson–Nissenzweig Theorem [12] to obtain a sequence (x_n^*) in X^* so that $\|x_n^*\| = 1$ for each n and $(x_n^*) \to 0$ in the w^*-topology on X^*. Let (x_n) be a sequence in B_X so that $x_n^*(x_n) = 1$ for each n. Let I denote the identity map on X, and note that $\langle x_n^* \otimes x_n, I\rangle = x_n^*(x_n) \not\to 0$. Thus $(x_n^* \otimes x_n)_{n=1}^\infty$ is not weakly null in $X \otimes_\gamma Y^*$

in this case. The following theorem establishes necessary and sufficient conditions (alluded to above) which ensure that $(x_n \otimes y_n) \to 0$ weakly whenever (x_n) is bounded and (y_n) is weakly null.

THEOREM 3.6. *The conditions that (x_n) is bounded in X and (y_n) is weakly null in Y guarantee that $(x_n \otimes y_n) \to 0$ weakly in $X \otimes_\gamma Y$ iff $T^*|_Y$ is completely continuous for every operator $T : X \to Y^*$.*

Proof. Suppose that $T : X \to Y^*$ is an operator and $T^*|_Y$ is not completely continuous. Let (y_n) be a weakly null sequence in Y so that $\|T^*(y_n)\| > 1$ for all n. Choose a sequence (x_n) in B_X so that $\langle T^*(y_n), x_n\rangle > 1$ for each n. Therefore $\langle T, x_n \otimes y_n\rangle \not\to 0$, and $(x_n \otimes y_n)_{n=1}^\infty$ is not weakly null.

Conversely, suppose that if $T : X \to Y^*$ is an operator, then $T^*|_Y$ is completely continuous. If (x_n) is bounded (with bound B) and (y_n) is weakly null in Y, then

$$|\langle T, x_n \otimes y_n\rangle| = |\langle T^*(y_n), x_n\rangle| \le B\|T^*(y_n)\|,$$

and $B\|T^*(y_n)\| \to 0$. Since $(X \otimes_\gamma Y)^* \cong B(X, Y^*)$, it follows that $(x_n \otimes y_n) \to 0$ weakly. ∎

COROLLARY 3.7. *If (y_n^*) weakly null in Y^* and (x_n) bounded in X guarantee that $(x_n \otimes y_n^*)_{n=1}^\infty$ is weakly null in $X \otimes_\gamma Y^*$, then $T^* : Y^* \to X^*$ is completely continuous for each operator $T : X \to Y$.*

Proof. Suppose the hypotheses are satisfied. By the previous theorem, $T^*|_{Y^*}$ is completely continuous for every operator $T : X \to Y^{**}$. Thus $T^*|_{Y^*}$ is completely continuous for every operator $T : X \to Y$. ∎

4. Classes of sets and bibasic sequences. The bounded subset K of X is defined to be a *reciprocal Dunford–Pettis set* (or *RDP-set*) if $T(K)$ is relatively weakly compact for each completely continuous operator T with domain X. We begin this section with a sequential characterization of RDP-sets similar in spirit to some of the results in Section 3.

THEOREM 4.1. *If K is a bounded subset of X, then the following are equivalent*:

(i) *K is an RDP-set.*

(ii) *If M is a w^*-compact and convex L-subset of X^* and (x_n) is a sequence from K, then there is a subsequence (x_{n_i}) of (x_n) and a point $x^{**} \in X^{**}$ so that $(x^*(x_{n_i})) \to x^{**}(x^*)$ for $x^* \in M$ and $x^{**}|_{(M,w^*)}$ is continuous.*

Proof. Suppose that K is an RDP-subset of X, and let M be a w^*-compact and convex L-subset of X^*. Let $C(M, w^*)$ denote the Banach space (with sup norm) of all real-valued functions on M which are continuous with respect to the w^*-topology, and let $E : X \to C(M, w^*)$ be the evaluation map. Then E is completely continuous, and $E(K)$ is relatively weakly compact. Let (x_n) be a sequence in K and let (x_{n_i}) be a subsequence so that

$$(E(x_{n_i})) \to \phi \in C(M, w^*)$$

in the weak topology.

Next we choose $x^{**} \in X^{**}$ so that $x^{**}|_M = \phi$. Therefore

$$(x^*(x_{n_i})) \to x^{**} = \phi(x^*)$$

for all $x^* \in M$, and $x^*|_M$ is w^*-continuous.

Conversely, suppose that (ii) holds, and let $T : X \to Y$ be a completely continuous operator. Thus $M = T^*(B_{Y^*})$ is a w^*-compact and convex L-subset of X^*. Let (x_n) be a sequence in K, and choose a subsequence (x_{n_i}) of (x_n) and a point x^{**} which satisfy (ii). Therefore

$$\langle T(x_{n_i}), y^* \rangle \to \langle x^{**}, T^*(y^*) \rangle$$

for each $y^* \in Y^*$.

Now suppose that $\|y^*_\alpha\| \le 1$ for each α and $(y^*_\alpha) \to y^*$ in the w^*-topology. We have

$$\langle T^{**}(x^{**}), y^*_\alpha \rangle = \langle x^{**}, T^*(y^*_\alpha) \rangle \to \langle x^{**}, T^*(y^*) \rangle = \langle T^{**}(x^{**}), y^* \rangle$$

by the w^*-continuity of $x^{**}|_M$. Further, by V.5.6 or V.5.7 of Dunford and Schwartz [15], the preceding convergence ensures that $T^{**}(x^{**})$ is w^*-continuous. Hence $T^{**}(x^{**}) \in Y$, $(T(x_{n_i})) \to T^{**}(x^{**})$ weakly, and $T(K)$ is relatively weakly compact. ∎

The next result establishes containment relationships that exist among some of the classes of sets that we have studied. These relationships will be useful in subsequent theorems. We use obvious acronyms to denote the classes.

THEOREM 4.2. *The limited sets*, *Dunford–Pettis sets*, *bounded weakly precompact sets*, *reciprocal Dunford–Pettis sets*, *and the* V^* *sets form five distinct classes of sets. More specifically*, $LS \subsetneq DP \subsetneq BWPC \subsetneq RDP \subsetneq V^*$, *and each containment is proper.*

Proof. Since every weakly null sequence in X is w^*-null, the definitions yield that every limited set is a DP-set. To see that the containment is proper, note that if (e_i) denotes the canonical basis of c_0 then

$$(\phi_n) = \Big(\sum_{i=1}^{n} e_i \Big), \qquad n \in \mathbb{N},$$

is a DP-set which is not a limited set in c_0, i.e., the canonical basis (e^*_n) of ℓ^1 is w^*-null, and $\langle \phi_n, e^*_n \rangle = 1$ for each n.

Odell's argument at the end of [30] shows that DP $\subseteq$ BWPC. The unit ball of any infinite-dimensional reflexive Banach space provides us with an example of a bounded weakly precompact set which is not Dunford–Pettis.

Clearly, the very definition of the two classes guarantees that BWPC $\subseteq$ RDP. Since every bounded subset of $C[0,1]$ is an RDP-set (completely continuous operators on $C[0,1]$ are weakly compact), this containment is also proper.

To easily see that RDP $\subseteq$ V*, we note that F. Bombal [8, Proposition 1.1] showed that a subset K of X is a V*-set iff $T(K)$ is relatively compact in ℓ^1 for each operator $T : X \to \ell^1$. Since relatively weakly compact subsets of a Schur space are relatively norm compact and all operators with domain or range a Schur space are completely continuous, Bombal's result immediately yields the desired containment.

It is more delicate to show that this last containment is proper. Let X be the first of the two major examples constructed by Bourgain and Delbaen in [10]. The space X is an infinite-dimensional Schur space with the property that X^* is weakly sequentially complete. Note that X does not contain a complemented copy of ℓ^1. (If it did, X^* would contain a copy of the non-weakly sequentially complete Banach space ℓ^∞.)

Now let I be the identity map. Certainly, I is completely continuous. Since B_X is not relatively compact, B_X is not an RDP-set. However, we assert that B_X is a V*-set. For if $L : X \to \ell^1$ is an arbitrary operator, then L must be compact by the "Pełczyński theory" of [28] and Chapter VII of Diestel [12]. Appealing to Bombal's characterization again [8], we see that B_X is a V*-set. ∎

We remark that if Ω is any compact Hausdorff space then one cannot use $C(\Omega)$ to differentiate between RDP-sets and V*-sets: Every completely continuous operator on $C(\Omega)$ is weakly compact and every bounded subset of $C(\Omega)$ is an RDP-set. Further, Bombal showed in [9] that the RDP-sets and the V*-sets in the dual of $C(\Omega)$ coincide.

Recall that a sequence (x_n, f_n^*) in $X \times X^*$ is called *bibasic* [32, p. 85] if (x_n) is basic in X, (f_n^*) is basic in X^*, and $f_m^*(x_n) = \delta_{mn}$. If (x_n, f_n^*) is a bibasic sequence, then f_n^* is an extension of the coefficient functional $x_n^* \in [x_i : i \in \mathbb{N}]^*$. If (x_n) is a basic sequence, then in the remainder of this section we shall denote the sequence of coefficient functionals by (x_n^*), and f_n^* will be a continuous linear extension of x_n^* to all of X for each n. Further, the bibasic sequence (x_n, f_n^*) is said to be *semi-normalized* if there are positive numbers p and q so that $p \le \|x_n\| \le q$ and $p \le \|f_n^*\| \le q$ for each n.

We note that Dineen [14] and Diestel [12] have discussed the compactness and weak compactness of limited sets in some detail. The remainder of this section deals with connections between bibasic sequences and the compactness of such sets.

THEOREM 4.3. *Suppose that K is a limited subset of X. The set K fails to be relatively compact if and only if there is a semi-normalized bibasic sequence (x_n, f_n^*) in $X \times X^*$ so that (x_n) is in $K - K$ and (f_n^*) is equivalent to the canonical unit vector basis (e_n^*) of ℓ^1.*

Proof. If (x_n, f_n^*) is a bibasic sequence which satisfies the first conclusion, then there is an $\varepsilon > 0$ so that $\|x_n - x_m\| > \varepsilon$ for $m \neq n$. Therefore $K - K$ is not relatively compact, and, consequently, K is not relatively compact.

Conversely, suppose that K is not relatively compact. Let $\varepsilon > 0$ and (y_n) be a sequence in K so that $\|y_n - y_m\| > \varepsilon$ for $m \neq n$. Let $x_n = y_n - y_{n+1}$. Moreover, since K is limited, by 4.2 we may (and do) assume that $(x_n) \to 0$ weakly. By a classical result of Bessaga and Pełczyński [5], [12, Chapter V], some subsequence of (x_n) is basic. Without loss of generality, suppose that (x_n) is basic, let (x_n^*) be the coefficient functionals, and, for each n, let f_n^* be a Hahn–Banach extension of x_n^* to all of X.

Suppose that $(f_{n_i}^*)$ is a weakly Cauchy subsequence of (f_n^*), and let $z_i^* = f_{n_i}^* - f_{n_{i+1}}^*$ for each i. Then $(z_i^*) \to 0$ weakly, and

$$\lim_i (\sup\{|z_i^*(y)| : y \in K - K\}) = 0.$$

However, $z_i^*(x_{n_i}) = 1$ for all i. Therefore no subsequence of (f_n^*) is weakly Cauchy. By Rosenthal's ℓ^1-theorem, some subsequence $(f_{n_i}^*)$ is equivalent to (e_i^*). The bibasic sequence $(x_{n_i}, f_{n_i}^*)$ satisfies the conclusion of the theorem. ■

If (x_n, f_n^*) is a bibasic sequence in $X \times X^*$, then much of the latter part of Section 1 of [32] is concerned with studying when (f_n^*) is equivalent to (x_n^*) (i.e., $(f_n^*) \sim (x_n^*)$). The following theorem completely resolves this question when (x_n, f_n^*) is produced by Theorem 4.3. To simplify notation, let $\mathrm{BBS} = \mathrm{BBS}(K)$ be the set of all semi-normalized bibasic sequences (x_n, f_n^*) so that (x_n) is from $K - K$ and $(f_n^*) \sim (e_n^*)$.

THEOREM 4.4. *If K is a non-relatively compact limited subset of X, then there is an element $(x_n, f_n^*) \in \mathrm{BBS}(K)$ so that (f_n^*) is equivalent to (x_n^*) if and only if there is an isomorphism $T : c_0 \to X$ so that $\{T(e_n) : n \in \mathbb{N}\} \subseteq K - K$.*

Proof. Suppose that $T : c_0 \to X$ is an isomorphism so that $T(e_n) = x_n \in K - K$ for each n. Let (x_n^*) be the corresponding sequence of coefficient functionals, and for each n let f_n^* be a Hahn–Banach extension of x_n^* to all of X. The proof of 4.3 shows that there is a subsequence of (x_n, f_n^*) which belongs to $\mathrm{BBS}(K)$. Since every subsequence of (e_n) is equivalent to (e_n) and the corresponding coefficient functionals are equivalent to (e_n^*), we assume that $(x_n, f_n^*) \in \mathrm{BBS}$. Therefore $(x_n^*) \sim (e_n^*)$ and $(x_n^*) \sim (f_n^*)$.

Conversely, suppose that $(x_n, f_n^*) \in \mathrm{BBS}$ and $(f_n^*) \sim (x_n^*)$. Let J be a positive integer so that

$$\frac{1}{J}\Big\|\sum_{n=1}^{m}\alpha_n x_n^*\Big\| \le \sum_{n=1}^{m}|\alpha_n| \le J\Big\|\sum_{n=1}^{m}\alpha_n x_n^*\Big\|$$

for each finite sequence $(\alpha_1, \ldots, \alpha_m)$ of scalars. Let M be the basis constant for (x_n). If $x \in [x_n : n \in \mathbb{N}]$ and $x^* \in [x_n : n \in \mathbb{N}]^*$, then

$$x^*(x) = \lim_{k\to\infty}\Big\langle\sum_{n=1}^{k}x^*(x_n)x_n^*, x\Big\rangle = \lim_{k\to\infty}\langle P_k^*(x^*), x\rangle,$$

where (P_k) is the sequence of projections associated with the basic sequence (x_n). Therefore

$$\|x\| \le \sup\Big\{\Big|\Big\langle\sum_{n=1}^{k}\alpha_n x_n^*, x\Big\rangle\Big| : k \in \mathbb{N}, \Big\|\sum_{n=1}^{k}\alpha_n x_n^*\Big\| \le M\Big\}.$$

Consequently, if $x = \sum_{n=1}^{p} x_n$, then

$$\begin{aligned}\|x\| &\le \sup\Big\{\Big|\Big\langle\sum_{n=1}^{k}\alpha_n x_n^*, x\Big\rangle\Big| : k \in \mathbb{N}, \Big\|\sum_{n=1}^{k}\alpha_n x_n^*\Big\| \le M\Big\}\\ &\le \sup\Big\{\sum_{n=1}^{k}|\alpha_n| : k \in \mathbb{N}, \Big\|\sum_{n=1}^{k}\alpha_n x_n^*\Big\| \le M\Big\} \le JM.\end{aligned}$$

Therefore by Johnson's lemma [16], [12, p. 245], some subsequence of (x_n) is equivalent to (e_n). Hence there is an isomorphism $T : c_0 \to X$ so that $T(e_n) \in K - K$ for each n. ■

Examples 1.3 and 1.4 on p. 89 of [32] produce bibasic sequences (x_n, f_n^*) for which (f_n^*) is not equivalent to (x_n^*). Our next result shows that an application of Pełczyński's version of the Eberlein–Shmul'yan theorem to non-relatively *weakly* compact limited sets automatically produces such bibasic sequences.

THEOREM 4.5. *If K is a non-relatively weakly compact limited subset of X, then there is a bibasic sequence (x_n, f_n^*) in $X \times X^*$ so that (x_n) is from K and (f_n^*) is not equivalent to (x_n^*).*

Proof. Suppose that K is limited and not relatively weakly compact, and let (y_n) be a sequence in K with no weakly convergent subsequence. Apply Pełczyński's version of the Eberlein–Shmul'yan theorem [12, p. 41], and let (z_n) be a basic subsequence of (y_n). Let (z_n^*) be the associated sequence of coefficient functionals, and for each n let h_n^* be a Hahn–Banach extension of z_n^* to all of X. Use the proof of 4.3, and let $(h_{n_i}^*)$ be a subsequence of (h_n^*) so that $(h_{n_i}^*) \sim (e_i^*)$. Certainly, $(x_i, f_i^*) = (z_{n_i}, h_{n_i}^*)$ is a bibasic sequence. If (f_i^*) were equivalent to (x_i^*), then the proof of 4.4 would show that some subsequence of (x_i) would be equivalent to (e_n). Therefore, this subsequence would converge weakly to 0, and this would contradict the initial choice of (y_n). ■

We remark that if the application of the Eberlein–Shmul'yan theorem to the sequence in 4.5 produces an *unconditional* basic sequence (z_n), then there exists no bounded sequence (f_n^*) in X such that f_n^* is a continuous linear extension of x_n^* for each n and $(f_n^*) \sim (x_n^*)$. For suppose that (x_n) is an unconditional basic sequence in the limited subset K of X, and suppose that no subsequence of (x_n) converges weakly to a point of X. Let (x_n^*) be the sequence of coefficient functionals, and for each n let f_n^* be a Hahn–Banach extension of x_n^*.

Now suppose (to the contrary) that $(f_n^*) \sim (x_n^*)$. From the preceding arguments, we know that some subsequence of (f_n^*) is equivalent to (e_i^*). Suppose that $(f_{n_i}^*) \sim (e_i^*)$, and let M be the unconditional basis constant for (x_n). Since the restriction of an isomorphism is an isomorphism, $(f_{n_i}^*) \sim (x_{n_i}^*)$, and $(x_{n_i}^*)$, as a sequence in $[x_n : n \in \mathbb{N}]^*$, is equivalent to (e_i^*). If p is a positive integer and $(\alpha_i)_{i=1}^p$ is a finite sequence of real numbers, then

$$\frac{1}{M}\Big\| \sum_{i=1}^{p} \alpha_i x_{n_i}^* \Big|_{[x_n]} \Big\| \le \Big\| \sum_{i=1}^{p} \alpha_i x_{n_i}^* \Big|_{[x_{n_j} : j \in \mathbb{N}]} \Big\| \le \Big\| \sum \alpha_i x_{n_i}^* \Big|_{[x_n]} \Big\|.$$

Therefore $(x_{n_i}^*)$, as a sequence of functionals in $[x_{n_i}]^*$, is equivalent to (e_i^*). By the proof of 4.4 (an application of Johnson's lemma), some subsequence of (x_{n_i}) is equivalent to (e_n) in c_0, and thus this subsequence must converge weakly to 0. This contradiction shows that (f_n^*) is not equivalent to (e_n^*).

The reader might want to compare this remark with Corollary 1.13, p. 102, of [32].

5. Global properties and duality. In Section 3 and the first part of Section 4 of this paper, we dealt primarily with localized properties, e.g., DP-sets, V-sets, V*-sets and RDP-sets. In this section we study how these localized notions can be used to study more global structure properties. For example, Pełczyński [27] showed that a Banach space X is reflexive iff X has properties V and V*. (The space X has *property V* if every V-subset of X^* is relatively weakly compact, and X has *property V** if every V*-subset of X is relatively weakly compact.) In this same paper, Pełczyński also noted that if X has property V then X^* has V* and if X^* has property V then X has V*.

In [25] Leavelle introduced the *property RDP**: the Banach space X has RDP* provided that every Dunford–Pettis subset of X is relatively weakly compact. Motivated by 3.1(vi) and the preceding observations, we obtain the following characterization of RDP*. If K is a bounded subset of X, we let $\widetilde{K}$ be the weak closure of K and equip $\widetilde{K}$ with the weak topology. Furthermore, $\mathrm{BC}(\widetilde{K})$ will denote the Banach space (with sup norm) of all bounded and continuous real-valued functions on $\widetilde{K}$.

THEOREM 5.1. *The Banach space X has RDP* iff each evaluation map $E : X^* \to \mathrm{BC}(\widetilde{K})$ which is completely continuous is also weakly compact.*

Proof. Suppose that X has RDP*. Suppose also that K is a bounded subset of X so that $E : X^* \to \mathrm{BC}(\widetilde{K})$ is completely continuous. Thus K is a DP-subset of X, and $\mathrm{BC}(\widetilde{K}) = C(\widetilde{K})$. For convenience, denote the unit ball of the dual of $\mathrm{BC}(\widetilde{K})$ by C^*. Of course, E is weakly compact iff E^* is weakly compact iff $E^*(C^*)$ is relatively weakly compact. Now $\mathrm{ext}(C^*) = \{\pm\delta_k : k \in \widetilde{K}\}$. Since $E^*(\delta_k) = k$ and the weakly closed absolutely convex hull of K is weakly compact, it follows that $E^*(C^*)$ is contained in the weakly closed absolutely convex hull of K in X^{**}. However, this is a weakly compact set, and E is weakly compact.

Conversely, suppose that E is weakly compact whenever $E : X^* \to \mathrm{BC}(\widetilde{K})$ is completely continuous. Let K be a Dunford–Pettis subset of X. Therefore $\widetilde{K}$ is Dunford–Pettis, $E : X^* \to \mathrm{BC}(\widetilde{K})$ is completely continuous and consequently weakly compact, and $E^*(C^*)$ is relatively weakly compact. Since $K \subseteq E^*(C^*)$, K is relatively weakly compact. ■

The following proposition demonstrates additional connections that exist among some of the properties that we have discussed.

PROPOSITION 5.2. (i) *If X has property V, then X has RDPP.*

(ii) *If X has property V^*, then X has RDP*.*

(iii) *If X has RDPP, then X^* has RDP*.*

(iv) *If X^* has RDPP, then X has RDP*.*

Moreover, none of these implications can be reversed.

Proof. (i) If $T : X \to Y$ is completely continuous, then $T^*(B_{Y^*})$ is an L-set and $T^*(B_{Y^*})$ is a V-set. Thus T^* and T are weakly compact.

(ii) Since every DP-subset of X is a V*-set and every V*-set is relatively weakly compact, X has RDP*.

(iii) If K is a DP-subset of X^*, then K is an L-subset of X^*, and 3.1(i) and the hypothesis tell us that K is relatively weakly compact.

(iv) If K is a DP-subset of X, then K is an L-subset of X^{**}, and, again by 3.1, K is relatively weakly compact.

To see that none of these implications can be reversed, we begin with a dual space X^* which does not contain ℓ^1. Thus X itself does not contain ℓ^1, and every completely continuous map on X or X^* is even compact. Thus X and X^* both have RDPP, and both X and X^* have RDP* by the preceding argument.

Now let J be the original James space [23]. Since J is separable and 1-codimensional in J^{**}, all duals of J are separable and ℓ^1 fails to embed in any of them. Moreover, none of these spaces can be weakly sequentially complete. Thus J and all of its duals have RDPP and RDP*, and neither J nor any of its duals have property V*. Hence none of these spaces has property V. Thus (i) and (ii) cannot be reversed.

Next let Y be a Banach space so that Y^{**} is separable and contains a complemented copy of ℓ^1; see e.g. James [24]. Let $X = Y^*$. Since ℓ^1 does not embed in Y, Y has RDPP, and X $(= Y^*)$ has RDP*. However, the projection from X^* onto ℓ^1 is completely continuous and certainly is not weakly compact. Therefore X^* does not have RDPP.

To see that (iv) cannot be reversed, we again consider the first Bourgain–Delbaen space [10]. Since X has the Schur property and is plainly not reflexive, the identity map on X is completely continuous and is not weakly compact. Thus X does not have RDPP. However, as was noted earlier, X^* is weakly sequentially complete. Since DP $\subseteq$ BWPC (Theorem 4.2), every DP-subset of X^* is relatively weakly compact, and X^* has RDP*. ∎

In view of Proposition 5.2, the following theorem appears to formally strengthen Proposition 7 of Pełczyński [27].

THEOREM 5.3. *The following are equivalent*:

(i) *X is reflexive.*
(ii) *X has RDPP and V*.*
(iii) *X has V and RDP*.*

Proof. (ii)⇒(i). Since X has RDPP, every bounded set is an RDP-set. Since X has property V*, every V*-set is relatively weakly compact. Therefore by Theorem 4.2 every bounded subset of X is relatively weakly compact, and X is reflexive.

(iii)⇒(i). Since X has RDP* and c_0 contains DP-sets which are not relatively weakly compact, X does not contain c_0. Therefore the identity map I on X is unconditionally converging. By Proposition 1 of Pełczyński [27], I is weakly compact. Thus X is reflexive.

Certainly, (i) implies (ii) and (ii) implies (iii). ∎

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