COLLOQUIUM MATHEMATICUM

VOL. 77 1998 NO. 2

ON THE UNIFORMIZATION OF HARTOGS DOMAINS IN $\mathbb{C}^2$ AND THEIR ENVELOPES OF HOLOMORPHY

BY

EWA LIGOCKA (WARSZAWA)

1. Introduction and statement of results. Let Ω be a domain in $\mathbb{C}^2$. We say that Ω is a *Hartogs domain* iff for every $(z,w)\in\Omega$ and $\theta\in\mathbb{R}$ we have $(z,e^{i\theta}w)\in\Omega$. We denote by Ω_z the set $\{w\in\mathbb{C}:(z,w)\in\Omega\}$ and call it the *vertical section* of Ω at z. We say that Ω is a *Hartogs domain over D* if $\Pi_1(\Omega)=D$, where $\Pi_1(z,w)=z$.

There are numerous papers devoted to Hartogs domains and their envelopes of holomorphy, e.g. [10], [8], [2]. In these papers the Hartogs domains with connected vertical sections were studied. Diederich and Fornæss [3] introduced an important class of Hartogs domains with disconnected vertical sections, the so-called "worm domains".

Barrett and Fornæss [1] gave a simple geometric construction of a Riemann surface $R(\Omega)$ associated with a C^1-smooth pseudoconvex, bounded Hartogs domain Ω in $\mathbb{C}^2$ such that Ω is biholomorphically equivalent to a Hartogs domain in $R(\Omega)\times\mathbb{C}$ and over $R(\Omega)$ with connected vertical sections. Unfortunately, for nonpseudoconvex Hartogs domains this nice construction leads to non-Hausdorff spaces.

In the present note we give another construction, based on Malgrange's construction of envelopes of holomorphy (via sheaves of holomorphic functions) [5].

This construction will permit us to associate with every Hartogs domain Ω in $\mathbb{C}^2$ an open Riemann surface $R(\Omega)$ (a Riemann domain over $\mathbb{C}$) and a biholomorphic embedding $\Psi:\Omega\to R(\Omega)\times\mathbb{C}$ with the following properties:

(a) If Ω is pseudoconvex then $\Psi(\Omega)$ is a Hartogs domain in $R(\Omega)\times\mathbb{C}$ with connected vertical sections, and $\Pi_1(\Psi(\Omega))=R(\Omega)$.

(b) If Ω is nonpseudoconvex then its envelope of holomorphy $E(\Omega)$ can be represented as a Hartogs domain in $R(\Omega)\times\mathbb{C}$ with connected vertical sections and such that $\Pi_1(E(\Omega))=R(\Omega)$.

(b) is a generalization of Corollary 2.5 of [2].

1991 *Mathematics Subject Classification*: 32E, 32H.

We now use the Koebe–Poincaré uniformization theorem for open Riemann surfaces (see [6]) and find a holomorphic covering map $\Psi : \Delta \overset{\text{onto}}{\longrightarrow} R(\Omega)$ (or $\Psi : \mathbb{C} \overset{\text{onto}}{\longrightarrow} R(\Omega)$ in the nonhyperbolic case); Δ denotes here, as usual, the unit disc in $\mathbb{C}$.

We consider the mapping $\widetilde{\Psi} : \Delta \times \mathbb{C} \overset{\text{onto}}{\longrightarrow} R(\Omega) \times \mathbb{C}$ (or $\widetilde{\Psi} : \mathbb{C} \times \mathbb{C} \overset{\text{onto}}{\longrightarrow} R(\Omega)$). It is of course a holomorphic covering map.

The set $\widetilde{\Omega} := \widetilde{\Psi}^{-1}(E(\Omega))$ is hence (by Stein's theorem [9]) a pseudoconvex Hartogs domain in $\Delta \times \mathbb{C}$ (or $\mathbb{C} \times \mathbb{C}$) and over Δ (or $\mathbb{C}$) with connected vertical sections, such that $\Pi_1(\widetilde{\Psi}^{-1}(E(\Omega))) = \Delta$ (or $\Pi_1(\widetilde{\Psi}^{-1}(E(\Omega))) = \mathbb{C}$). Finally, we get our

MAIN THEOREM. *For every Hartogs domain in* $\mathbb{C}^2$ *there exists a pseudoconvex Hartogs domain* $\widetilde{\Omega}$ *over the unit disc* Δ (*or over* $\mathbb{C}$) *with connected vertical sections and a holomorphic covering map*

$$\widetilde{\Psi} : \widetilde{\Omega} \overset{\text{onto}}{\longrightarrow} E(\Omega)$$

where $E(\Omega)$ *denotes*, *as before*, *the envelope of holomorphy of* Ω.

COROLLARY 1. *Every pseudoconvex Hartogs domain* Ω *in* $\mathbb{C}^2$ *can be holomorphically covered by a pseudoconvex Hartogs domain over* Δ (*or* $\mathbb{C}$) *with connected vertical sections.*

COROLLARY 2. *If* Ω *is a pseudoconvex bounded Hartogs domain in* $\mathbb{C}^2$ *not intersecting the complex line* $\{w = 0\}$ *and the homotopy group of* Ω *has exactly one generator then* Ω *is biholomorphically equivalent to a Hartogs domain over* Δ *with connected vertical sections.*

Corollary 2 is a generalization of the same statement for worm domains [1].

2. The construction of $R(\Omega)$**.** Let F denote the family of all holomorphic functions f on Ω for which $\partial f/\partial w \equiv 0$ on Ω.

These functions depend locally only on z. Fix some point $(z_0, w_0) \in \Omega$. Consider the sheaf of holomorphic functions with values in $\mathbb{C}^F$ over $\mathbb{C}$.

Let R be a component of the above sheaf space containing the point $[F]_{z_0}$, the germ of the family F at z_0 ($[F]_{z_0} = \{[f]_{z_0}\}_{f \in F}$). Then R is a Riemann domain over $\mathbb{C}$.

We can now define a biholomorphic embedding of Ω into $R \times \mathbb{C}$ as $\Phi(z, w) = ([F]_z, w)$ and define $R(\Omega) = \Pi_1(\Phi(\Omega))$. Then $R(\Omega)$ is an open subset of a sheaf space and therefore a well defined Riemann surface (a Riemann domain over $\mathbb{C}$).

3. Proofs. Every function f holomorphic on the Hartogs domain Ω can be written in the form

$$f = \sum_{j=-\infty}^{\infty} f_j w^j, \quad \frac{\partial f_j}{\partial w} \equiv 0 \quad \text{on } \Omega \text{ for each } j \in \mathbb{Z}.$$

This implies that if $g \in H(\Phi(\Omega))$ then $g = \sum_{j=-\infty}^{\infty} g_j w^j$, $\partial g_j/\partial w \equiv 0$ on $\Phi(\Omega)$ for each $j \in \mathbb{Z}$.

The construction of $R(\Omega)$ also implies that if $(\xi_0, w_1), (\xi_0, w_2) \in \Phi(\Omega)$ and $|w_1| < |w_2|$ then every function g holomorphic on $\Phi(\Omega)$ extends to a function $\widetilde{g}$ holomorphic on an open neighborhood of the set $\{(\xi, w) \in R(\Omega) \times \mathbb{C} : |w_1| \le |w| \le |w_2|,\ \xi = \xi_0\}$. Moreover, if $(\xi_0, w) \in \Phi(\Omega)$ and $|w_1| < |w| < |w_2|$ then $\widetilde{g}(\xi_0, w) = g(\xi_0)$.

Thus there exists a Hartogs domain $\widetilde{D}$ in $R(\Omega) \times \mathbb{C}$ with connected vertical sections such that $\widetilde{D} \supset \Phi(\Omega)$ and every holomorphic function on $\Phi(\Omega)$ extends holomorphically to $\widetilde{D}$.

Analogously to the case of Hartogs domains in $\mathbb{C}^2$ (see [2]), $(\widetilde{D}, R(\Omega) \times \mathbb{C})$ is a Runge pair if $\widetilde{D} \cap \{(\xi, w) : w = 0\} \neq \emptyset$, and $(\widetilde{D}, R(\Omega) \times (\mathbb{C} \setminus \{0\}))$ is a Runge pair otherwise.

Hence the envelope of holomorphy $E(\Omega) \approx E(\Phi(\Omega)) = E(\widetilde{D})$ is a Hartogs domain in $R(\Omega) \times \mathbb{C}$ with connected vertical sections. This last statement can be proved in exactly the same way as an analogous fact in [2]. However, we can obtain an easier proof if we use the representation of $E(\widetilde{D})$ as the set of linear multiplicative functionals on $H(\widetilde{D})$ (the space of holomorphic functions on $\widetilde{D}$) (see [4]). The fact that $(\widetilde{D}, R(\Omega) \times \mathbb{C})$ or $(\widetilde{D}, R(\Omega) \times (\mathbb{C} \setminus \{0\})$ forms a Runge pair and $R(\Omega) \times \mathbb{C}$ and $R(\Omega) \times (\mathbb{C} \setminus \{0\})$ are Stein manifolds implies $E(\widetilde{D}) \subset R(\Omega) \times \mathbb{C}$. Now $E(\widetilde{D})$ must be a Hartogs domain, because the action of the group $\{r^{i\theta}\}_{\theta \in \mathbb{R}} = T$ extends to an envelope of holomorphy in an obvious way and must agree with $(\xi, w) \to (\xi, e^{i\theta} w)$ on $H(R(\Omega) \times \mathbb{C})$ (or $H(R(\Omega) \times \mathbb{C} \setminus \{0\})$). Moreover, $E(\widetilde{D})$ must have connected vertical sections by the first part of our proof.

Thus (a) and (b) are proved and the Main Theorem follows.

Corollary 1 is an immediate consequence of the Main Theorem.

If the assumptions of Corollary 2 are satisfied then $R(\Omega)$ must be simply connected, and therefore it is conformally equivalent to the unit disc by the Riemann mapping theorem.

4. Admissible families of holomorphic functions. Let Ω be a pseudoconvex Hartogs domain in $\mathbb{C}^2$. Let F denote, as before, the family of holomorphic functions f on Ω for which $\partial f/\partial w \equiv 0$ on Ω.

Let $F_1 \subset F$ be a subfamily of F. We can repeat the construction from Section 2 taking the family F_1 instead of F. As a result we get a Riemann surface $R_1(\Omega)$ and a biholomorphic imbedding $\Phi_1 : \Omega \to R_1(\Omega) \times \mathbb{C}$ such that $\Pi_1(\Phi_1(\Omega)) = R_1(\Omega)$.

We say that the family F_1 is *admissible* if $\Phi_1(\Omega)$ has connected vertical sections.

We have the following

PROPOSITION 1. *If F_1 is an admissible family then $R_1(\Omega)$ is conformally equivalent to $R(\Omega)$.*

Proof. Let $\xi \in R(\Omega)$. Take w such that $(\xi, w) \in \Phi(\Omega)$. Define $\Psi(\xi) = \Pi_1\Phi_1\Phi^{-1}(\xi, w)$. Since Ω is pseudoconvex, $R(\Omega)$ has connected vertical sections and $\Psi(\xi)$ is well defined. Since $R_1(\Omega)$ has connected vertical sections we have $\Psi^{-1}(\xi) = \Pi_1\Phi\Phi_1^{-1}(\xi, w)$ for $(\xi, w) \in \Phi_1(\Omega)$.

Let us give the following two examples of admissible families:

(i) Let Ω be a bounded pseudoconvex Hartogs domain in $\mathbb{C}^2$ with C^1-smooth boundary.

Let $K((z, w), (t, s))$ be the Bergman kernel function of Ω. It can be written in the form

$$K((z, w), (t, s)) = \sum_{j=-\infty}^{\infty} k_j((z, w), (t, s)) w^j \overline{s}^j,$$

where for each $j \in \mathbb{Z}$, $\partial k_j/\partial w = \partial k_j/\partial \overline{s} = 0$ on $\Omega \times \Omega$.

Take $F_1 = \{k_j((z, w), (t, s))\}_{(t,s)\in\Omega,\, j\in\mathbb{Z}}$. If F_1 is not admissible then there exists a larger domain $\widetilde{\Omega}$ such that $K((\cdot, \cdot), (t, s))$ extends holomorphically to $\widetilde{\Omega}$ for all $(t, s) \in \Omega$. Since $K((z, w), (t, s)) = \overline{K((t, s), (z, w))}$, there exists $\widetilde{\Omega}_1$ with $\Omega \subset \widetilde{\Omega} \subset \widetilde{\Omega}_1$ such that $K((z, w), (z, w))$ extends to a real-analytic function on $\widetilde{\Omega}_1$.

However, Ohsawa [7] proved that if Ω is bounded, pseudoconvex with C^1-smooth boundary then $K((z, w), (z, w)) \to \infty$ as $(z, w) \to \partial\Omega$.

Hence F_1 must be admissible.

(ii) Let Ω be a worm domain (see [3] or [1]). It was shown in [1] that $F = \{x^{1/p}\}$ is admissible for p sufficiently large (depending on Ω).

PROBLEM 1. Which pseudoconvex Hartogs domains in $\mathbb{C}^2$ admit finite admissible families of holomorphic functions?

5. Planar Hartogs domains in $\mathbb{C}^2$. An open Riemann surface R is called *planar* if it is conformally equivalent to an open domain in $\mathbb{C}$. An open Riemann surface R is planar iff every Jordan curve in R dissects R (see [6]).

A Hartogs domain Ω in $\mathbb{C}^2$ will be called *planar* iff $R(\Omega)$ is a planar Riemann surface. We have the following

PROPOSITION 2. *If Ω is a planar Hartogs domain in $\mathbb{C}^2$ then its envelope of holomorphy $E(\Omega)$ is biholomorphically equivalent to a Hartogs domain in $\mathbb{C}^2$ with connected vertical sections.*

This is an immediate consequence of (a) and (b).

There exist nonplanar Hartogs domains (see [1], §5, Example).

PROBLEM 2. Does there exist a pseudoconvex, bounded Hartogs domain in $\mathbb{C}^2$ with C^1-smooth boundary, which is not planar? (Worm domains are planar!)

REFERENCES

[1] D. E. Barrett and J. E. Fornæss, *Uniform approximation of holomorphic functions on bounded Hartogs domains in* $\mathbb{C}^2$, Math. Z. 191 (1986), 61–72.

[2] E. Casadio Tarabusi and S. Trapani, *Envelopes of holomorphy of Hartogs and circular domains*, Pacific J. Math. 149 (1991), 231–249.

[3] K. Diederich and J. E. Fornæss, *Pseudoconvex domains: an example with nontrivial Nebenhülle*, Math. Ann. 225 (1977), 275–292.

[4] R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Englewood Cliffs, N.J., 1965.

[5] B. Malgrange, *Lectures on the Theory of Functions of Several Complex Variables*, Tata Institute, Bombay, 1958 (reissued 1965).

[6] R. Nevanlinna, *Uniformisierung*, Springer, 1953.

[7] T. Ohsawa, *A remark on the completeness of the Bergman metric*, Proc. Japan Acad. Ser. A Math. Sci. 57 (1981), 238–240.

[8] M. Shirinbekov, *Construction of envelopes of holomorphy for multiple Hartogs domains*, Mat. Zametki 27 (1980), 77–87 (in Russian).

[9] K. Stein, *Überlagerungen holomorph-vollständiger komplexer Räume*, Arch. Math. (Basel) 7 (1956), 354–361.

[10] V. S. Vladimirov and M. Shirinbekov, *On the construction of envelopes of holomorphy for Hartogs domains*, Ukrain. Mat. Zh. 15 (1963), 189–192 (in Russian).

Department of Mathematics, Informatics and Mechanics
Warsaw University
Banacha 2
02-097 Warszawa, Poland
Email: elig@hydra.mimuw.edu.pl

Received 14 October 1997;
revised 29 December 1997