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WEIGHTED NORM INEQUALITIES AND HOMOGENEOUS CONES

BY

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1. Introduction. In this paper we consider an n-dimensional generalization of the classical Hardy inequality:

Let $1 \le p < \infty$ and $\alpha < 1$. Then for every positive function f defined on the positive half-line we have

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(y)\,dy\right)^p x^{\alpha p}\,dx/x \le C\int_0^\infty f^p(x)x^{\alpha p}\,dx/x. \tag{1}$$

This is Theorem 330 of [5], which shows that the Hardy operator $\mathcal{H}f(x) = \frac{1}{x}\int_0^x f(y)\,dy$ is bounded in the weighted L^p space (with weight $x^{\alpha p}$). The best constant here is $C = 1/(1-\alpha)^p$. We write the inequality in a form slightly different than the one in [5], because it is more suitable for generalization.

We shall use a *homogeneous cone* V in $\mathbb{R}^n$ as a generalization of the half-line $(0,\infty)$, and the *power functions* defined on V in place of the weights $\phi^\gamma(x) = x^\gamma$ (see Section 2 for the definitions). It turns out that the only properties of the Hardy operator needed to obtain the inequality are the homogeneity of the kernel and the fact that $\mathcal{H}\phi^\gamma$ is well defined for $\gamma > -1$. In other words, all operators having these two properties will satisfy the same weighted norm inequality. As examples we consider the Laplace transform, the Riemann–Liouville operator, and the Stieltjes transform.

In the one-dimensional case inequality (1) was generalized in many directions and it was shown that more general functions than the power functions can be taken as weights. Indeed, Muckenhoupt [8] gave a necessary and sufficient condition for the weights. However, the n-dimensional counterpart of this condition is not even sufficient, so it makes sense to find weights which at least satisfy this inequality.

The cones which we consider in this paper are more general than the self-dual cones considered in [9]. Also the weights in [9] were of a more special form—powers of the norm (see (11) below), while in this paper we have n-dimensional power functions (Definition 3). To find these weights

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we must use the theory of homogeneous cones founded by Vinberg [14] and Gindikin [4]. (See also [12] for inequalities with $p \le q$ and [13] for $0 < p < 1$.)

In Section 2 we briefly review the notions we need from the theory of homogeneous cones, as developed by Vinberg [14]. In Section 3 we prove the weighted norm inequality for a class of integral operators with homogeneous kernels. By observing that such integral operators can be written as *convolutions*, the proof of this inequality becomes particularly simple (see also [2] for $n = 1$). In Section 4 we give some examples of homogeneous operators—they are all n-dimensional analogues of some classical operators, and in Section 5 we apply to them the weighted norm inequality. The dual operators of the classical ones give further examples of homogeneous operators (Section 6).

2. The homogeneous cones. A homogeneous cone V in $\mathbb{R}^n$ is a cone endowed with a group operation. These cones were studied in [7, 11, 14, 4]. In this section we review some facts we shall need from the theory of homogeneous cones (see Vinberg [14]). This theory is based on some nonassociative algebras, which are now called Vinberg algebras. There is an analogy with the better known case of symmetric or self-dual cones (which are equal to their dual cone) and where the corresponding algebras are the Jordan algebras (see, for example, [3]). The symmetric cones are fully classified: there are 5 types of them. There are infinitely many types of general homogeneous cones, but their exact classification is not yet known.

DEFINITION 1. Let V be an open convex cone in $\mathbb{R}^n$ which does not contain any straight line. The cone V is said to be *homogeneous* if there is a group G of linear automorphisms (a subgroup of $\mathrm{GL}(n, \mathbb{R})$ which leaves V invariant) which is transitive on V, i.e., such that for any $u, v \in V$ there is an element g of G such that $v = gu$. ∎

A most important property of homogeneous cones is that they always have a *simply transitive* group of automorphisms, i.e. a group G such that for every $u, v \in V$ there is a *unique* $g \in G$ such that $v = gu$. In other words, there is a bijection $\Pi : G \to V$ which, if we fix an element $c \in V$ once for all, assigns to every $v \in V$ a *unique* $g_v \in G$ such that $v = g_v c$. Thus the group operation induces an operation for the elements of the cone

$$v \cdot u = g_v g_u c. \tag{2}$$

We shall call this simply transitive group G the *group of the cone*. This group is triangular (real solvable) and if $\mathfrak{g}$ is the Lie algebra of G, then the exponential mapping is a bijection from the Lie algebra $\mathfrak{g}$ onto G. By taking the derivative of the mapping Π above, we see that the mapping $\pi : \mathfrak{g} \to \mathbb{R}^n$ defined by $X \mapsto Xc$ is a vector space isomorphism. Thus we have bijective mappings between the four sets V, G, $\mathfrak{g}$ and $\mathbb{R}^n$.

If $x \in \mathbb{R}^n$ we write $L(x)$ for the unique element of $\mathfrak{g}$ such that $x = L(x)c$ (i.e. L is the inverse of π; this notation will be justified in Definition 2 below). Then we can define the following operation:

$$x \vartriangle y = L(x)L(y)c$$

for every $x, y \in \mathbb{R}^n$. This bilinear operation introduces into $\mathbb{R}^n$ the structure of a Vinberg algebra. This is the nonassociative algebra described in the next definition.

Also, by the bijectivity of the exponential mapping, every $g \in G$ is of the form $g = \exp L(x)$, for some $L(x) \in \mathfrak{g}$.

Definition 2. A *Vinberg algebra* $\mathcal{B} = (\mathbb{R}, \vartriangle)$ is the vector space $\mathbb{R}^n$ with a bilinear operation $\vartriangle$ such that

(B) $x \vartriangle (y \vartriangle z) - (x \vartriangle y) \vartriangle z = y \vartriangle (x \vartriangle z) - (y \vartriangle x) \vartriangle z$,

(B.1) the operator of left multiplication $L(x) : \mathcal{B} \to \mathcal{B}$, $L(x)y = x \vartriangle y$, has real spectrum,

(B.2) $\operatorname{Tr} L(x \vartriangle x) > 0$, for every $x \in \mathcal{B} \setminus \{0\}$.

The Vinberg algebra $\mathcal{B}$ has an *identity* c if moreover

(B.3) $x \vartriangle c = c \vartriangle x = x$. ■

We write $R(x) : \mathcal{B} \to \mathcal{B}$, $R(x)y = y \vartriangle x$, for the right multiplication in $\mathcal{B}$.

Vinberg's Theorem ([14]). *Let $\mathcal{B}$ be a Vinberg algebra with identity c. Then there is a complete system of orthogonal idempotents $c_1, \ldots, c_m$, with $c_1 + \ldots + c_m = c$; the number m is called the* rank *of $\mathcal{B}$. The algebra $\mathcal{B}$ is decomposed into a direct sum of subspaces*

$$\mathcal{B} = \sum_{i \le j} \mathcal{B}_{ij}, \qquad i, j = 1, \ldots, m, \tag{3}$$

where $\mathcal{B}_{ii} = \mathbb{R}c_i$ and the subspace $\mathcal{B}_{ij}$ is characterized by the fact that the operators $L(c_i)$ and $L(c_j)$ have on it eigenvalue $1/2$, and the operator $R(c_j)$ has eigenvalue 1; all the other $L(c_k)$ and $R(c_k)$ are zero on $\mathcal{B}_{ij}$. ■

This decomposition is analogous to the Pierce decomposition of Jordan algebras. Put $n_{ij} = \dim \mathcal{B}_{ij}$. We have $n_{ii} = 1$, but there is almost no restriction for the numbers n_{ij} (compare with the case of Jordan algebras, where all n_{ij} are necessarily equal, and the only possibilities are 1, 2, 4 or 8).

We shall write x_{ij} for an element of $\mathcal{B}_{ij}$. In particular, $x_{ii} = x_i c_i$, for some $x_i \in \mathbb{R}$. Thus by (3), every $x \in \mathcal{B}$ is of the form

$$x = \sum_{i \le j} x_{ij} = \sum_i x_i c_i + \sum_{i<j} x_{ij}. \tag{4}$$

Consider the Lie algebra $\mathfrak{g}$, which, as every triangular Lie algebra, is decomposed into the sum $\mathfrak{g} = \mathfrak{a} + \mathfrak{n}$ of the abelian subalgebra $\mathfrak{a}$ and the nilpotent subalgebra $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$. This decomposition corresponds to (4) and for $L(x) \in \mathfrak{g}$ we have $\sum_i x_i L(c_i) \in \mathfrak{a}$ and $\sum_{i<j} L(x_{ij}) \in \mathfrak{n}$. The operators $L(c_i)$ constitute a basis of $\mathfrak{a}$—they commute since c_i are orthogonal idempotents; this is easily seen if we note that in terms of the operators of left multiplication condition (B) takes the form $[L(x), L(y)] = L(x \vartriangle y - y \vartriangle x)$.

LEMMA 1 ([14]). *The trace of $L(x)$ equals*

$$\operatorname{Tr} L(x) = \sum_{i=1}^{m} x_i \operatorname{Tr} L(c_i) = \sum_{i=1}^{m} x_i d_i \tag{5}$$

where $d_i = \operatorname{Tr} L(c_i)$ is given by

$$d_i = 1 + \nu_i/2 + \mu_i/2 \quad \textit{with } \nu_i = \sum_{\alpha<i} n_{\alpha i},\ \mu_i = \sum_{\beta>i} n_{i\beta} \tag{6}$$

for $i = 1, \ldots, m$. ∎

This follows from the decomposition (4) by making use of the fact that $\operatorname{Tr} L(x_{ij}) = 0$ and that the eigenvalues of $L(c_i)$ are as in the theorem above.

We shall write $d = (d_1, \ldots, d_m)$, $\nu = (\nu_1, \ldots, \nu_m)$ and $\mu = (\mu_1, \ldots, \mu_m)$ for the three multi-indices in (6), which are characteristic for the cone. We shall need them in Section 4. (See [14, 4 or 10] where the values of these indices are found for different examples of cones.)

Now we consider functions defined on a homogeneous cone $\Phi : V \to \mathbb{R}_+$. By what was said above we can obviously associate with the function Φ the functions

$$\phi : G \to \mathbb{R}_+, \quad \psi : \mathfrak{g} \to \mathbb{R}, \quad \Psi : \mathcal{B} \to \mathbb{R}$$

defined via the following equations:

$$\phi(g) = \Phi(gc), \quad \phi(\exp L(x)) = e^{\psi(L(x))}, \quad \Psi(x) = \psi(L(x)). \tag{7}$$

DEFINITION 3. A function $\Phi : V \to \mathbb{R}_+$ is called a *power function* if the corresponding function $\phi : G \to \mathbb{R}_+$ is a Lie group homomorphism (or, equivalently, $\psi : \mathfrak{g} \to \mathbb{R}$ is a Lie algebra homomorphism). ∎

Thus, by definition, a power function is characterized by any of the following equalities:

$$\Phi(gv) = \phi(g)\Phi(v), \quad \Phi(v \cdot u) = \Phi(v)\Phi(u). \tag{8}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be a multi-index. Let $x \in \mathcal{B}$ be given in its Pierce coordinates (4). Let the function $\Psi^\alpha : \mathcal{B} \to \mathbb{R}$ be defined by

$$\Psi^\alpha(x) = \alpha_1 x_1 + \ldots + \alpha_m x_m \tag{9}$$

and let, for $g = \exp L(x)$ and $v = g_v c$, the associated functions (7) be given by

$$\phi^\alpha(g) = e^{\psi^\alpha(L(x))} = e^{\Psi^\alpha(x)}, \quad \Phi^\alpha(v) = \phi^\alpha(g_v). \tag{10}$$

LEMMA 2. *Let $\Phi : V \to \mathbb{R}_+$ be a power function. Then there exists a multi-index α such that Φ is equal to Φ^α.*

P r o o f. The function ψ associated with Φ in (7) is a homomorphism of the Lie algebra $\mathfrak{g}$ into the abelian Lie algebra $\mathbb{R}$. Then it must be zero on $\mathfrak{n} = [\mathfrak{g}, \mathfrak{g}]$ (since the commutator is 0 in $\mathbb{R}$); thus if we write $L(x) = H + Y$, with $H \in \mathfrak{a}$ and $Y \in \mathfrak{n}$, then $\psi(L(x)) = \psi(H)$. Now *any* linear function is a Lie algebra homomorphism from the abelian subalgebra $\mathfrak{a}$ into $\mathbb{R}$, so that finally, since $H = \sum_i x_i L(c_i)$ (by a remark above $L(c_i)$ are the basis elements), we have $\psi(H) = \alpha_1 x_1 + \ldots + \alpha_m x_m$ with $\alpha_i = \psi(L(c_i))$. This with the notation (9) and (10) proves the lemma. ∎

An important example of a power function is the *norm* of the cone. By definition (see [7]) it is the power function which is associated with the function $\operatorname{Det} g$ (which is obviously a Lie group homomorphism), i.e.

$$\Delta(gv) = \operatorname{Det} g \Delta(v). \tag{11}$$

To find the multi-index of this power function note that $\operatorname{Det}(\exp L(x)) = e^{\operatorname{Tr} L(x)}$. Then we see by (10) that α equals d, where d is the multi-index defined in (5) and (6). Thus

$$\Delta(v) = \Phi^d(v), \qquad \operatorname{Det} g = \phi^d(g). \tag{12}$$

Formula (2) defines a group operation on the elements of V. Denote this group by $(V, \cdot)$ and write v^{-1} for the inverse in this group. This means $v^{-1} = (g_v)^{-1} c$ and equivalently $g_{v^{-1}} = (g_v)^{-1}$.

LEMMA 3 ([4]). *Let V be a homogeneous cone. The measure invariant under the action $v \mapsto gv$ of the group G is given up to a constant by*

$$dm(v) = \frac{dv}{\Delta(v)} = \frac{dv}{\Phi^d(v)} \tag{13}$$

where dv is the Lebesgue measure. Consequently, $dm(v)$ is a left and right Haar measure for the group $(V, \cdot)$, and is thus also invariant under the inversion $v \mapsto v^{-1}$.

P r o o f. Indeed, the first assertion follows from (11) and (12). The second is now obvious from (8) and the third is a well-known property of unimodular groups. ∎

3. Operators with homogeneous kernels. Now we consider integral transforms of functions defined on V and prove that they satisfy a weighted norm inequality similar to Hardy's inequality (1) from the introduction, in other words, that they are bounded operators in weighted L^p spaces.

Let $F : V \to \mathbb{R}_+$ be a positive function defined on V and let dm be the invariant measure (13). For $1 \le p < \infty$ and for a multi-index α we define the weighted L^p norm as

$$\|F\|_{p,\alpha} = \int_V F^p(v)\Phi^{p\alpha}(v)\,dm(v). \tag{14}$$

We shall simply write $\|F\|_p$ when $\alpha = 0$. Let $L^p_\alpha = L^p_\alpha(V)$ be the space of all positive functions F defined on V such that $\|F\|_{p,\alpha} < \infty$.

Put $\check{F}(v) = F(v^{-1})$. Since the measure is inversion invariant (Lemma 3), we have

$$\|F\|_{p,\alpha} = \|\check{F}\|_{p,-\alpha}. \tag{15}$$

Let κ be a multi-index. A function $k : V \times V \to \mathbb{R}_+$ is called a *homogeneous kernel of order* κ if

$$k(gv, gu) = \phi^\kappa(g)k(v, u) \tag{16}$$

for every $g \in G$ and $v, u \in V$. We write $\operatorname{ord}(k) = \kappa$.

Define a *homogeneous integral operator* with kernel k by the formula

$$\mathcal{K}F(v) = \int_V k(v, u)F(u)\,dm(u).$$

We shall also say that κ is the *order* of $\mathcal{K}$.

In this section we prove the boundedness of $\mathcal{K}$ as an operator in weighted L^p spaces. We start with a particular case: in Theorem 1 we establish the inequality without weights for an operator of order 0. The key step in the proof is the fact that a homogeneous operator can be written as a convolution (Lemma 5). Next, in Theorem 2 we give the inequality from L^p_α into L^q_β, with the order κ related to the weights by formula (25). Operators of arbitrary orders will be treated in Section 5 by an easy reduction to these particular cases.

LEMMA 4. *Let k be a homogeneous kernel of order κ. Then*

$$k(v, u) = \Phi^\kappa(v)k(c, v^{-1} \cdot u), \qquad \mathcal{K}\Phi^\alpha(v) = \Phi^{\alpha+\kappa}(v)\mathcal{K}\Phi^\alpha(c). \tag{17}$$

Proof. By using the definition of the product (2) we can write $k(v, u) = k(g_v c, g_v v^{-1} \cdot u)$ and by (16) and (10) this is equal to $\phi^\kappa(g_v)k(c, v^{-1} \cdot u) = \Phi^\kappa(v)k(c, v^{-1} \cdot u)$.

By (17) we have

$$\mathcal{K}\Phi^\alpha(v) = \Phi^\kappa(v) \int_V k(c, v^{-1} \cdot u)\Phi^\alpha(u)\,dm(u).$$

Now change the variable $v^{-1} \cdot u \mapsto u$ in the last integral. By the invariance of dm it is equal to

$$\int_V k(c, u)\Phi^\alpha(u)\,dm(u) = \mathcal{K}\Phi^\alpha(c) \tag{18}$$

multiplied by $\Phi^\alpha(v)$. ∎

Lemma 4 says that a homogeneous operator transforms power functions into power functions, provided the integral (18) is convergent. (See Section 4 for values of α for which the integral is convergent.) Write

$$K(u) = k(c, u); \tag{19}$$

then (18) becomes

$$\mathcal{K}\Phi^{\alpha}(c) = \|K\|_{1,\alpha}. \tag{20}$$

DEFINITION 4. Let F_1 and F_2 be two positive functions defined on V. The *convolution* $F_1 * F_2$ is defined by

$$F_1 * F_2(v) = \int_V F_1(u) F_2(u^{-1} \cdot v)\, dm(u). \ ■$$

LEMMA 5. *Let $\mathcal{K}$ be an integral operator with homogeneous kernel k of order* 0 *and let K be the function defined in* (19). *Then $\mathcal{K}$ is a convolution operator*:

$$\mathcal{K}F = F * \check{K}.$$

Proof. By (17) and (19) we have $k(v, u) = K(v^{-1} \cdot u) = \check{K}(u^{-1} \cdot v)$. ■

In this case, when the operator $\mathcal{K}$ is a convolution, the boundedness of $\mathcal{K}$ follows from the following *Young's inequality* for convolutions.

Let $1 \le p \le q < \infty$ and let r be defined by $1/r = 1/q - 1/p + 1$. Let H be a unimodular locally compact group. If $F_1 \in L^p(H)$ and $F_2 \in L^r(H)$ then

$$\|F_1 * F_2\|_q \le \|F_1\|_p \|F_2\|_r. \tag{21}$$

(See [6], Theorem 20.18, with q and r interchanged, and Theorem 20.14. For $n = 1$ this is Theorem 280 of [5]; see also [2].)

THEOREM 1. *Let $1 \le p \le q < \infty$ and let r be such that $1/r = 1/q - 1/p + 1$. Let $\mathcal{K}$ be an integral operator with homogeneous kernel k of order* 0 *and let K be defined in* (19). *If $\|K\|_r < \infty$, then*

$$\|\mathcal{K}F\|_q \le C\|F\|_p. \tag{22}$$

for every $F \in L^p$. The constant C can be taken to be $\|K\|_r$.

Proof. Indeed, by Lemma 5 we have $\mathcal{K}F = F * \check{K}$. Now by (15) we have $\|K\|_r = \|\check{K}\|_r$ and this is finite by assumption, so that Young's inequality (21) yields $\|F * \check{K}\|_q \le \|F\|_p \|K\|_r$ and this is (22) with $C = \|K\|_r$. ■

REMARK. The best constant C in (22) can actually be smaller than $\|K\|_r$. By applying a result of [1] we see that the best constant for the cone $\mathbb{R}^n_+$ (the set of points with all coordinates positive) is equal to $(A_p A_r A_{q'})^n \|K\|_r$, where $A_m = [(m^{1/m})/(m')^{1/m'}]^{1/2}$ with $1/m + 1/m' = 1$ (so that it is equal to $\|K\|_r$ only in the limiting case when $p = q$ and $r = 1$).

THEOREM 2. *Let* $1 \le p \le q < \infty$ *and let* r *be such that* $1/r = 1/q - 1/p + 1$. *Let* $\mathcal{K}$ *be an integral operator with homogeneous kernel* k *of order* κ *and let* K *be defined in* (19). *Let* α *and* β *be two multi-indices and suppose*

$$\|K\|_{r,-\alpha} < \infty. \tag{23}$$

Then

$$\|\mathcal{K}F\|_{q,\beta} \le C\|F\|_{p,\alpha} \tag{24}$$

for every $F \in L^p_\alpha$ *if and only if*

$$\kappa + \beta - \alpha = 0. \tag{25}$$

The constant C *in* (24) *can be taken to be* $\|K\|_{r,-\alpha}$.

P r o o f. First we show that this is the only possible form of the inequality, i.e. if (24) holds, then the weights satisfy (25). Indeed, consider the function $l_g \circ F(u) := F(gu)$ for $g \in G$. Then it is easily seen that

$$\|l_g \circ F\|_{p,\alpha} = \phi^{-\alpha}(g)\|F\|_{p,\alpha} \tag{26}$$

and

$$\mathcal{K}(l_g \circ F)(v) = \phi^{-\kappa}(g) l_g \circ \mathcal{K}F(v). \tag{27}$$

The following inequality is proved by an application of (27), (26), (24) and again (26):

$$\|\mathcal{K}(l_g \circ F)\|_{q,\beta} \le C\phi^{-\kappa-\beta+\alpha}(g)\|l_g \circ F\|_{p,\alpha},$$

and since the constant should not depend on g we obtain (25).

Now to prove the inequality put

$$h(v,u) = \Phi^\beta(v)k(v,u)\Phi^{-\alpha}(u).$$

Then by (25) the order of h equals $\beta + \kappa - \alpha = 0$. Also, since $H(u) = h(c,u) = K(u)\Phi^{-\alpha}(u)$, we have $\|H\|_r = \|K\|_{r,-\alpha}$ so that the operator $\mathcal{H}$ with kernel h satisfies the conditions of Theorem 1. Thus we have

$$\|\mathcal{H}G\|_q \le C\|G\|_p \tag{28}$$

for every $G \in L^p$. Now

$$\mathcal{H}G(v) = \Phi^\beta(v) \int_V k(v,u)\Phi^{-\alpha}(u)G(u)\,dm(u) = \Phi^\beta(v)\mathcal{K}F(v)$$

where we have put $F = \Phi^{-\alpha}G$. Thus (28) yields

$$\|\Phi^\beta \mathcal{K}F\|_q = \|\mathcal{H}G\|_q \le C\|G\|_p = C\|\Phi^\alpha F\|_p,$$

which with the notation (14) is exactly (24). Since in (28) we had $C \le \|H\|_r$ and this equals $\|K\|_{r,-\alpha}$, the theorem follows. ∎

4. The classical operators. Many of the classical operators can be generalized to n dimensions in such a way that the kernel is homogeneous.

In order to simplify the notation we shall write down the inequalities for these operators in the special case $p = q$. (It is obvious that the general case $p < q$ is treated in a completely analogous way.) For $p = q$ in Theorem 2 we have $r = 1$ and condition (23) reads $\mathcal{K}\Phi^{-\alpha}(c) = \|K\|_{1,-\alpha} < \infty$ (see (20)).

COROLLARY 1. *Let $1 \le p < \infty$. Let $\mathcal{K}$ be an integral operator with homogeneous kernel of order κ and such that $\mathcal{K}\Phi^{-\alpha}(c) < \infty$ for some α. Then*

$$\|\mathcal{K}F\|_{p,\alpha-\kappa} \le C\|F\|_{p,\alpha}. \ \blacksquare \tag{29}$$

Thus in order to apply this corollary we only have to check that $\mathcal{K}\Phi^{-\alpha}(c) < \infty$. This is the only point in the proof of the weighted inequality which really depends on the structure of the homogeneous cone; all the rest is in complete analogy with the one-dimensional case. For the operators we consider in this paper the values of α such that $\mathcal{K}\Phi^{-\alpha}(c) < \infty$ were found by Gindikin [4] (see also [10] for a proof using Vinberg's theory of homogeneous cones). It turns out that for all these operators the values of α are the same; see Lemmas 6–9 below.

Throughout this section we assume that V is a homogeneous cone and that d and ν and μ are the multi-indices characteristic for the cone, defined in (6). When α and β are two multi-indices we shall briefly write $\alpha < \beta$ for $\alpha_i < \beta_i$, $i = 1, \ldots, m$. Also, when 1 stands for a multi-index, it will mean $(1, \ldots, 1)$.

4.1. *Hardy's operator*. The cone V defines a partial order in $\mathbb{R}^n$ in the following way: $v <_V u$ iff $u - v \in V$. Write (a, b) for the "interval" with respect to this order, i.e. for the set of all elements $v \in V$ such that $a <_V v <_V b$.

Hardy's operator is defined by

$$\mathcal{H}F(v) = \int_{(0,v)} F(u)\, dm(u). \tag{30}$$

The kernel is $k(v, u) = \chi_{(0,v)}(u)$, with χ_A the characteristic function of a set A. Since $g \in G$ preserves the cone, it also preserves the order; i.e. $u <_V v$ implies $gu <_V gv$. This shows that the kernel is homogeneous of order 0.

LEMMA 6 ([4]). *The integral*

$$\mathcal{H}\Phi^{\alpha}(c) = \int_{(0,c)} \Phi^{\alpha}(u)\, dm(u)$$

is convergent for $\alpha > \mu/2$. $\blacksquare$

4.2. *The Laplace transform*. Let V be a homogeneous cone and $\mathcal{B}$ its Vinberg algebra as in Section 2. Then we can define an inner product by putting

$$\langle x \mid y\rangle = \operatorname{Tr} L(x \vartriangle y). \tag{31}$$

Condition (B) in Definition 2, written in the form $[L(x), L(y)] = L(x \vartriangle y - y \vartriangle x)$, shows that the bilinear form (31) is symmetric. Condition (B.2) shows that this form is positive-definite.

Let Δ be a norm of V (see (11)). For $v \in V$ define $v^* \in \mathbb{R}^n$ by putting

$$\langle v^* \mid x\rangle = d(\log \Delta)(v)x$$

for every $x \in \mathbb{R}^n$ (see [7 or 11]). In fact, this formula defines a mapping which to every $v \in V$ assigns an element v^* of the dual cone V^*.

Now the Laplace transform is defined, for every $v \in V$, by

$$\mathcal{L}F(v) = \int_V e^{-\langle v^* | u\rangle} F(u)\, dm(u). \tag{32}$$

It is an integral operator with kernel $k(v,u) = e^{-\langle v^* | u\rangle}$ which is homogeneous of order 0. Indeed, the mapping $*$ satisfies $(gv)^* = (g^{-1})^\top v^*$, where $\top$ denotes the transposition with respect to the inner product (31) (see [11]) and thus $\langle (gv)^* \mid gu\rangle = \langle (g^{-1})^\top v^* \mid gu\rangle = \langle v^* \mid u\rangle$, so that finally we have $k(gv, gu) = k(v,u)$, for every $g \in G$.

Lemma 7 ([4]). *The integral*

$$\mathcal{L}\Phi^\alpha(c) = \int_V e^{-\langle c|u\rangle} \Phi^\alpha(u)\, dm(u)$$

is convergent for $\alpha > \mu/2$. ■

The integral in this lemma is called the *Gamma function* of the cone, and was first defined in [7].

4.3. *The Riemann–Liouville operator*. It is defined for a multi-index β by putting

$$\mathcal{R}_\beta F(v) = \int_{(0,v)} \Phi^\beta(v-u) F(u)\, dm(u). \tag{33}$$

The kernel $k(v,u) = \chi_{(0,v)}(u)\Phi^\beta(v-u)$ is homogeneous of order β. Obviously, Hardy's operator is a special case, when $\beta = 0$.

Lemma 8 ([4]). *The integral*

$$\mathcal{R}_\beta \Phi^\alpha(c) = \int_{(0,c)} \Phi^\beta(c-u)\Phi^\alpha(u)\, dm(u)$$

is convergent for $\beta > -1 - \nu/2$ *and* $\alpha > \mu/2$. ■

4.4. *The Stieltjes transform.* It is defined for a multi-index ϱ by putting

$$\mathcal{S}_\varrho \Phi^\alpha(c) = \int_V \Phi^{-\varrho}(v+u) F(u)\, dm(u). \tag{34}$$

The kernel $k(v,u) = \Phi^{-\varrho}(v+u)$ is homogeneous of order $-\varrho$.

LEMMA 9 ([4]). *The integral*

$$\mathcal{S}_\varrho F(v) = \int_V \Phi^{-\varrho}(c+u) \Phi^\alpha(u)\, dm(u)$$

is convergent for $\varrho > \mu$ and $\mu/2 < \alpha < \varrho - \mu/2$. ∎

We have thus checked that all four operators satisfy the two conditions of Corollary 1: they are homogeneous and $\mathcal{K}\Phi^{-\alpha}(c)$ is finite for some α. Then inequality (29) holds for these values of α.

5. Some variants. In this section we obtain some easy corollaries of the preceding. As seen in the following lemma, it is possible to modify the two conditions of Corollary 1 and to change either the order of the operator or the values of α for which $\mathcal{K}\Phi^{-\alpha}(c)$ is finite. In Corollary 2 we make the first change in order to see that all our examples of classical operators satisfy the *same* inequality. In Corollary 3 we make the second change in order to enlarge the domain of these operators; they then take their more usual form.

If k is a homogeneous kernel, put, for some multi-index λ,

$$k_1(v,u) = \Phi^{-\lambda}(v) k(v,u), \tag{35}$$

$$k_2(v,u) = \Phi^{\lambda}(v) k(v,u) \Phi^{-\lambda}(u). \tag{36}$$

Then one can easily change the order of the operator (reducing it, for example, to the case $\operatorname{ord}(k) = 0$), or change the range of α for which $\mathcal{K}\Phi^{-\alpha}(c) < \infty$ holds. More precisely, we have the following lemma, whose proof is quite obvious.

LEMMA 10. *An integral operator $\mathcal{K}$ has order κ and satisfies $\mathcal{K}\Phi^{-\alpha}(c) < \infty$, for some α, if and only if any of the following conditions holds*:

(a) *The operator $\mathcal{K}_1$ with kernel* (35) *has order $\kappa - \lambda$ and $\mathcal{K}_1\Phi^{-\alpha}(c) < \infty$.*

(b) *The operator $\mathcal{K}_2$ with kernel* (36) *has order κ and $\mathcal{K}_2\Phi^{-\alpha+\lambda}(c) < \infty$.* ∎

We note that all the operators considered in the preceding section satisfy the condition $\mathcal{K}\Phi^{-\alpha}(c) < \infty$ for the same values of $\alpha : \alpha < -\mu/2$ (with an additional restriction for α in the case of the Stieltjes transform: $\alpha > -\varrho + \mu/2$). We shall now modify the Riemann–Liouville operator (33) and the Stieltjes transform (34) as in Lemma 10(a), so that their order becomes 0, without changing the condition $\mathcal{K}\Phi^{-\alpha}(c) < \infty$.

Define

$$\mathcal{R}^1_\beta = \Phi^{-\beta}\mathcal{R}_\beta \tag{37}$$

as in (35). Thus by Lemma 10(a), $\mathcal{R}^1_\beta$ is homogeneous of order 0 and $\mathcal{R}^1_\beta\Phi^{-\alpha}(c) = \mathcal{R}_\beta\Phi^{-\alpha}(c)$.

Also, define the modified operator

$$\mathcal{S}^1_\varrho = \Phi^\varrho\mathcal{S}_\varrho \tag{38}$$

which is homogeneous of order 0 and such that $\mathcal{S}^1_\varrho\Phi^{-\alpha}(c) = \mathcal{S}_\varrho\Phi^{-\alpha}(c)$.

Corollary 2. *Let* $1 \le p < \infty$ *and* $\alpha < -\mu/2$. *Let* $\mathcal{K}$ *be any of the following operators*: *Hardy's operator* $\mathcal{H}$ (30), *the Laplace transform* $\mathcal{L}$ (32), *the modified Riemann–Liouville operator* $\mathcal{R}^1_\beta$ (37) (*for* $\beta > -1-\nu/2$), *the modified Stieltjes transform* $\mathcal{S}^1_\varrho$ (38) (*for* $\varrho > \mu$, *with the additional restriction for* α *in this case*: $\alpha > -\varrho + \mu/2$). *Then*

$$\|\mathcal{K}F\|_{p,\alpha} \le C\|F\|_{p,\alpha}. \ \blacksquare$$

Now, usually Hardy's operator is given in the form $\int_{(0,\nu)} F(u)\,du$, since introducing the invariant measure dm in (30) only worsens the singularity at the origin; this accounts for the reduced range of α in Corollary 2: $\alpha < -\mu/2$ (which corresponds to $\alpha < 0$ in dimension 1). We fix this by applying Lemma 10(b). Consider the following operators:

$$\widetilde{\mathcal{H}}F(v) = \Phi^{-d}(v)\int\limits_{(0,\nu)} F(u)\,du, \qquad \widetilde{\mathcal{L}}F(v) = \Phi^{-d}(v)\int\limits_V e^{-\langle v^*|u\rangle}F(u)\,du,$$

$$\widetilde{\mathcal{R}}_\beta F(v) = \Phi^{-\beta-d}(v)\int\limits_{(0,\nu)} \Phi^\beta(v-u)F(u)\,du, \qquad \beta > -1-\nu/2, \tag{39}$$

$$\widetilde{\mathcal{S}}_\varrho F(v) = \Phi^{\varrho-d}(v)\int\limits_V \Phi^{-\varrho}(v+u)F(u)\,du, \qquad \varrho > \mu.$$

Corollary 3. *Let* $1 \le p < \infty$ *and* $\alpha < 1+\nu/2$. *Let* $\widetilde{\mathcal{K}}$ *be any of the operators* (39) (*with the additional restriction* $\alpha > -\varrho + \mu/2 + d$ *for the Stieltjes transform*). *Then*

$$\|\widetilde{\mathcal{K}}F\|_{p,\alpha} \le C\|F\|_{p,\alpha}.$$

Remark. This is exactly the form of the Hardy inequality (1).

Proof (of Corollary 3). All the kernels of the operators (39) are obtained from the corresponding kernels (in Corollary 2) as in (36): $\widetilde{k}(v,u) = \Phi^{-d}(v)k(v,u)\Phi^d(u)$. Thus $\mathrm{ord}(\widetilde{k}) = \mathrm{ord}(k) = 0$. We have to check that $\widetilde{\mathcal{K}}\Phi^{-\alpha}(c) < \infty$ for $\alpha < 1+\nu/2$. By Lemma 10(b) this is equivalent to $\mathcal{K}\Phi^{-\alpha+d}(c) < \infty$ for the original operator. Now if $\alpha < 1+\nu/2$, then $\alpha - d < 1+\nu/2-d = -\mu/2$ (see (6)), and thus the condition holds, as seen from Corollary 2. $\blacksquare$

6. The dual operators. Let $\mathcal{K}$ be a homogeneous operator. The *dual operator* is defined by

$$\mathcal{K}'F(u) = \int_V k(v,u)F(v)\,dm(v).$$

Its kernel $k'(v,u) = k(u,v)$ is obviously homogeneous of the same order.

LEMMA 11. *If* $\mathcal{K}$ *is a homogeneous operator of order* 0, *then* $\mathcal{K}\Phi^{-\alpha}(c) = \mathcal{K}'\Phi^{\alpha}(c)$.

Proof. Indeed, if K is the function defined in (19) and K' the corresponding function for $\mathcal{K}'$, we have $K'(v) = K(v^{-1}) = \check{K}(v)$, by (17). Now by (15) we have $\|K'\|_{1,\alpha} = \|K\|_{1,-\alpha}$ and then (20) yields the lemma. ■

We now consider the dual operators of the operators in Section 4. Write $(v,\infty) = \{u \in V : u >_V v\}$ for infinite intervals. Then the dual of Hardy's operator (30) is

$$\mathcal{H}'F(u) = \int_{(u,\infty)} F(v)\,dm(v).$$

The dual of the Laplace transform (32) is

$$\mathcal{L}'F(u) = \int_V e^{-\langle v^*|u\rangle}F(v)\,dm(v).$$

COROLLARY 4. *Let* $1 \le p < \infty$ *and* $\alpha > \mu/2$. *Then the operators* $\mathcal{H}'$ *and* $\mathcal{L}'$ *satisfy*

$$\|\mathcal{K}'F\|_{p,\alpha} \le C\|F\|_{p,\alpha}.$$

Proof. Indeed, to apply Corollary 1, note that they are both of order 0 and $\mathcal{K}'\Phi^{-\alpha}(c) = \mathcal{K}\Phi^{\alpha}(c)$, by Lemma 11. Now when $\alpha > \mu/2$, we have $\mathcal{K}\Phi^{\alpha}(c) < \infty$ for both these operators, by Lemmas 7 and 8. ■

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