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HOMEOMORPHIC NEIGHBORHOODS IN μ^{n+1}-MANIFOLDS

BY

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1. Introduction. The notion of n-shape for compact spaces was introduced by Chigogidze [5]. Generalizing n-shape to locally compact spaces, the author [1] introduced the *proper* n-shape, which is defined by using embeddings of spaces into locally compact AR-spaces. In the case of $\dim \le n+1$, the proper n-shape of locally compact spaces can also be defined by using their embeddings into locally compact $(n+1)$-dimensional $LC^n \cap C^n$-spaces (cf. [2]). In this paper, we prove a μ^{n+1}-manifold version of the result of [11], that is, *if X and Y are Z-sets in μ^{n+1}-manifolds M and N respectively, and n-$\mathrm{Sh}_p(X) = n$-$\mathrm{Sh}_p(Y)$, then X and Y have arbitrarily small homeomorphic μ^{n+1}-manifold closed neighborhoods.* As a corollary, if X is a connected Z-set in a μ^{n+1}-manifold and $X \in SUV^n$, then there exists a tree T such that X has arbitrarily small closed neighborhoods homeomorphic ($\approx$) to the Δ_{n+1}-product $T\Delta_{n+1}\mu^{n+1}$ of T and μ^{n+1}. Here, property SUV^n is a noncompact variant of property UV^n, and the Δ_{n+1}-product is defined in [10]; it plays the role of the Cartesian product in the category of μ^{n+1}-manifolds. For a locally finite polyhedron P, $P\Delta_{n+1}\mu^{n+1}$ is the μ^{n+1}-manifold having the same proper n-homotopy type of P.

2. Preliminaries. In this paper, spaces are separable metrizable and maps are continuous. The $(n+1)$-dimensional universal Menger compactum is denoted by μ^{n+1} and a manifold modeled on μ^{n+1} is called a μ^{n+1}-*manifold*. We define $\mu^{n+1}_\infty = \mu^{n+1} \setminus \{*\}$, where $* \in \mu^{n+1}$. Recall that two proper maps $f, g : X \to Y$ are *properly n-homotopic* (written $f \simeq^n_p g$) if, for any proper map $\alpha : Z \to X$ from a space Z with $\dim Z \le n$ into X, the compositions $f\alpha$ and $g\alpha$ are properly homotopic in the usual sense ($f\alpha \simeq_p g\alpha$). A μ^{n+1}-manifold M lying in a μ^{n+1}-manifold N is said to be *n-clean* in N (cf. [8]) if M is closed in N and there exists a closed μ^{n+1}-manifold $\delta(M)$ in M such that

(i) $(N \setminus M) \cup \delta(M)$ is a μ^{n+1}-manifold;

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(ii) $\delta(M)$ is a Z-set in both M and $(N \setminus M) \cup \delta(M)$; and
(iii) $M \setminus \delta(M)$ is open in N.

REMARK 2.1Let P be a PL-manifold and L a submanifold in P such that $\operatorname{Bd} L = \operatorname{Bd}(P\setminus L)$. By [7, Theorem 1.6], there exists a proper UV^n-surjection $f : N \to P$ from a μ^{n+1}-manifold N satisfying the following conditions:

(a) $f^{-1}(K)$ is a μ^{n+1}-manifold for any closed subpolyhedron K of P; and
(b) $f^{-1}(Z)$ is a Z-set in $f^{-1}(K)$ for any Z-set Z in a closed subpolyhedron K of P.

Then it is easy to see that $M = f^{-1}(L)$ is an n-clean submanifold of N with $\delta(M) = f^{-1}(\operatorname{Bd} L)$.

LEMMA 2.2. *Let Y be closed in a locally compact $C^n \cap LC^n$-space N. Assume that $r : V_0 \to Y$ is a proper retraction of a closed neighborhood V_0 of Y in N. Then for each closed neighborhood V of Y in N there exists a closed neighborhood V' of Y in N such that $V' \subset V \cap V_0$ and $\mathrm{id}_{V'} \simeq^n_p r|_{V'}$ in V.*

Proof. Let $\mathcal{W}$ be an open cover of $V \cap V_0$ such that if one of any two $\mathcal{W}$-close maps from an arbitrary locally compact space is proper, then the other is also proper. Since $\operatorname{int}(V \cap V_0)$ is LC^n, there exists an open cover $\mathcal{U}$ of $\operatorname{int}(V \cap V_0)$ such that any two $\mathcal{U}$-close maps from a space with $\dim \le n$ to $\operatorname{int}(V \cap V_0)$ are $\mathcal{W}$-homotopic. By the continuity of r, for any $U \in \mathcal{U} \cap Y = \{U \in \mathcal{U} \mid U \cap Y \neq \emptyset\}$ and $x \in U \cap Y$ there exists a closed neighborhood $\overline{V}_x$ of x in N such that $\overline{V}_x \subset U$ and $r(\overline{V}_x) \subset U \cap Y$. Since Y is locally compact, $\{\overline{V}_x \mid x \in Y\}$ has a locally finite refinement $\mathcal{V}'$. Then $V' = \bigcup \mathcal{V}'$ is the desired neighborhood. ■

Let X and Y be closed sets in locally compact $C^n \cap LC^n$-spaces M and N respectively. Recall that a proper n-fundamental net $\mathbf{f} = \{f_\lambda \mid \lambda \in \Lambda\} : X \to Y$ in (M, N) is *generated by* a proper map $f : X \to Y$ (or f *generates* $\mathbf{f}$) provided $f = f_\lambda|_X$ for all $\lambda \in \Lambda$. The proper n-homotopy class $[\mathbf{f}]$ of $\mathbf{f}$ is *generated by* f if f generates some $\mathbf{f}' \in [\mathbf{f}]$.

PROPOSITION 2.3. *If Y is a locally compact LC^n-space, then the proper n-homotopy class of $\mathbf{f} : X \to Y$ in (M, N) is generated by a proper map $f : X \to Y$.*

Proof. Since Y is LC^n, there exist a closed neighborhood V_0 of Y in N and a proper retraction $r : V_0 \to Y$ by [3, Lemma 3.2]. By Lemma 2.2, for each closed neighborhood V of Y in N there exists a closed neighborhood V' of Y in N such that $r|_{V'} \simeq^n_p \mathrm{id}_{V'}$ in V. Then there exist a closed neighborhood U' of X in M and $\lambda_0 \in \Lambda$ such that $f_\lambda|_{U'} \simeq^n_p f_{\lambda_0}|_{U'}$ in V' for all $\lambda \ge \lambda_0$. Let $r' : N \to N$ be an extension of r and $f'_\lambda = r' f_{\lambda_0}$.

Note that $\mathbf{f}' = \{f'_\lambda\}$ is generated by $f = rf_{\lambda_0}|_X$, i.e., $f'_\lambda|_X = f$. Since $f'_\lambda|_{U'} = r'f_{\lambda_0}|_{U'} \simeq^n_p f_{\lambda_0}|_{U'} \simeq^n_p f_\lambda|_{U'}$ in V for all $\lambda \geq \lambda_0$, we have $\mathbf{f}' \simeq^n_p \mathbf{f}$. ∎

3. Homeomorphic neighborhoods in μ^{n+1}-manifolds

LEMMA 3.1. *Let X be a Z-set in a μ^{n+1}-manifold M. Then there exists a closed embedding $F : M \to \mu^{n+1}_\infty$ such that $F(M)$ is a neighborhood of $F(X)$ in μ^{n+1}_∞ and $F(M)$ is n-clean in μ^{n+1}_∞ with $\delta(F(M)) \approx M$.*

Proof. By [6, Theorem 9], there exists a proper $(n+1)$-invertible UV^n-surjection $f : M \to P$ from M to a locally finite $(n+1)$-dimensional polyhedron P. We can assume that P is a closed subpolyhedron in $(I^{2(n+1)+1} \times \{0\}) \setminus \{*\}$, where $* = (0, \ldots, 0) \in I^{2(n+1)+2}$. Then there exists a proper $(n+1)$-invertible UV^n-surjection $f : N \to I^{2(n+1)+2} \setminus \{*\}$ from a μ^{n+1}-manifold N as in Remark 2.1. Since $I^{2(n+1)+2} \setminus \{*\} \simeq^n_p \mu^{n+1}_\infty$ and f is proper UV^n, we have $N \approx \mu^{n+1}_\infty$ (cf. [7, Theorem 1.3]).

Let $N(P)$ be a regular neighborhood of P. By Remark 2.1, $M' = f^{-1}(N(P))$ and $M'' = f^{-1}(\operatorname{Bd} N(P))$ are μ^{n+1}-manifolds in N and M' is n-clean with $\delta(M') = M''$. Since P is a Z-set in $N(P)$, it follows that $N(P) \simeq^n_p \operatorname{Bd} N(P)$. By [7, Theorem 1.4], there exist homeomorphisms $g : M \to M'$ and $g' : M \to M''$. By the Z-set unknotting theorem [4], there exists a homeomorphism $h : M' \to M'$ such that $h(g(X)) \cap M'' = \emptyset$. Then $F = hg$ is the desired closed embedding. ∎

THEOREM 3.2. *Let X and Y be Z-sets in μ^{n+1}-manifolds M and N respectively, such that $n\text{-}\mathrm{Sh}_p(X) \leq n\text{-}\mathrm{Sh}_p(Y)$. Then, for each neighborhood U of X in M and each neighborhood V of Y in N, there exists an open neighborhood V' of Y such that for every μ^{n+1}-manifold closed neighborhood S of Y in $V' \cap V$, there exists a μ^{n+1}-manifold closed neighborhood R of X in U which is homeomorphic to S.*

Proof. By Lemma 3.1, we can assume that $M = N = \mu^{n+1}_\infty$ and U is n-clean. Let $\mathbf{f} : X \to Y$ in $(\mu^{n+1}_\infty, \mu^{n+1}_\infty)$ and $\mathbf{g} = \{g_\delta \mid \delta \in \Delta\} : Y \to X$ in $(\mu^{n+1}_\infty, \mu^{n+1}_\infty)$ be proper n-fundamental nets such that $\mathbf{gf} \simeq^n_p \mathbf{i}_X$. Let U' be a closed neighborhood of X such that $U' \subset \operatorname{int} U$. Then there exist $\delta_0 \in \Delta$, $\lambda_0 \in \Lambda$ and a closed neighborhood W of Y with $W \subset V$ such that $g_\delta f_\lambda|_X \simeq^n_p \mathrm{id}_X$ in U', $g_\delta|_W \simeq^n_p g_{\delta_0}|_W$ in U' for each $\delta \geq \delta_0$, $\lambda \geq \lambda_0$. By the Z-set approximation theorem [4], there exists a Z-embedding $g'_{\delta_0} : W \to \operatorname{int} U$ approximating g_{δ_0}. Note that $g'_{\delta_0}|_Y$ is properly n-homotopic to the inclusion in μ^{n+1}_∞. By the Z-set unknotting theorem [4], there exists a homeomorphism $h : \mu^{n+1}_\infty \to \mu^{n+1}_\infty$ such that $hg'_{\delta_0}|_Y = \mathrm{id}_Y$.

Let $V' = h(\operatorname{int} U)$ and $S \subset V \cap V'$ be a closed μ^{n+1}-manifold neighborhood of Y. Then $S' = h^{-1}(S)$ is a μ^{n+1}-manifold closed neighborhood of

$g'_{\delta_0}(Y)$ lying in $\operatorname{int} U$. Let W' be a closed neighborhood of Y lying in $\operatorname{int} S$ so that $g'_{\delta_0}(W') \subset \operatorname{int} S'$. Then there exists $\lambda \geq \lambda_0$ such that $f_\lambda(X) \subset W'$. By the Z-set approximation theorem, we can assume that f_λ is a Z-embedding. Note that $g'_{\delta_0} f_\lambda(X) \subset g'_{\delta_0}(W') \subset \operatorname{int} S'$ and $g'_{\delta_0} f_\lambda|_X \simeq^n_p g_{\delta_0} f_\lambda|_X \simeq^n_p \mathrm{id}_X$ in $U' \subset \operatorname{int} U$. By the Z-set unknotting theorem, there exists a homeomorphism $h' : \mu^{n+1}_\infty \to \mu^{n+1}_\infty$ such that $h' g'_{\delta_0} f_\lambda|_X = \mathrm{id}_X$ and $h'|_{\mu^{n+1}_\infty \setminus \operatorname{int} U} = \mathrm{id}_{\mu^{n+1}_\infty \setminus \operatorname{int} U}$. Then $R = h'(S')$ is the desired neighborhood. ∎

For the Δ_{n+1}-product, refer to [10].

LEMMA 3.3. *Let P be a locally finite polyhedron embedded in μ^{n+1}_∞ as a closed set and U a neighborhood of P in μ^{n+1}_∞. Then there exists a μ^{n+1}-manifold closed neighborhood V of P such that $V \subset U$, V is n-clean in μ^{n+1}_∞ and $V \approx \delta(V) \approx P\Delta_{n+1}\mu^{n+1}$.*

Proof. We can assume that $P \subset (I^{2(n+1)+1} \times \{0\}) \setminus \{*\} \subset I^{2(n+1)+2} \setminus \{*\} = M$ as a closed subpolyhedron and μ^{n+1}_∞ is obtained from M by the Lefschetz construction [9, 2.1, II]. Let $\mathcal{L}$ be a combinatorial triangulation of M and $\widetilde{U}$ be a neighborhood of P in M such that $U = \mu^{n+1}_\infty \cap \widetilde{U}$. By Whitehead's theorem [12], there exists a subdivision $\mathcal{L}'$ of $\mathcal{L}$ such that $\mathcal{L}'$ refines $\{M \setminus P\} \cup \{\widetilde{U}\}$.

Let $N(P, \operatorname{sd} \mathcal{L}')$ be a regular neighborhood of P obtained from the barycentric subdivision $\operatorname{sd} \mathcal{L}'$ of $\mathcal{L}'$ and let V be a μ^{n+1}-manifold obtained from $N(P, \operatorname{sd} \mathcal{L}')$ by the Lefschetz construction. Then V is n-clean in μ^{n+1}_∞ and such that $\delta(V) = \mu^{n+1}_\infty \cap \operatorname{Bd} N(P, \operatorname{sd} \mathcal{L}')$ and $V \setminus \delta(V) = \mu^{n+1}_\infty \cap \operatorname{int} N(P, \operatorname{sd} \mathcal{L}')$. Now there exists a proper deformation retraction $r : N(P, \operatorname{sd} \mathcal{L}') \to P$, and we have a proper UV^n-retraction $r|_V : V \to P$. Since there exists a proper UV^n-surjection $P\Delta_{n+1}\mu^{n+1} \to P$ (see [10]), and by [7, Theorem 1.4], V and $P\Delta_{n+1}\mu^{n+1}$ are homeomorphic. Since P is a Z-set in $N(P, \operatorname{sd} \mathcal{L}')$, we have $N(P, \operatorname{sd} \mathcal{L}') \simeq^n_p \operatorname{Bd} N(P, \operatorname{sd} \mathcal{L}')$, which implies $\delta(V) \approx V$ by [7, Theorem 1.4] again. ∎

THEOREM 3.4. *Let X be a Z-set in a μ^{n+1}-manifold M and P an $(n+1)$-dimensional locally finite polyhedron such that n-$\mathrm{Sh}_p(X) \leq n$-$\mathrm{Sh}_p(P)$. Then X has arbitrarily small closed neighborhoods U_α, $\alpha \in A$, such that*

(1) *each U_α is n-clean in M;*

(2) *$U_\alpha \approx \delta(U_\alpha) \approx P\Delta_{n+1}\mu^{n+1}$; and*

(3) *for each $\alpha, \beta \in A$ there exists a homeomorphism $h : U_\alpha \to U_\beta$ fixing X.*

Proof. By Lemma 3.1, we can assume that X and P are closed sets in μ^{n+1}_∞. Let $\mathbf{f} : X \to P$ and $\mathbf{g} : P \to X$ be proper n-fundamental nets in

$(\mu_\infty^{n+1}, \mu_\infty^{n+1})$ such that $\mathbf{gf} \simeq_p^n \mathbf{i}_X$. By Proposition 2.3, $\mathbf{f}$ is generated by a proper map $f : X \to P$. Let $A = \{\alpha \mid \alpha$ is a closed neighborhood of X in $\mu_\infty^{n+1}\}$. For each $\alpha \in A$, there exist $\delta_\alpha \in \Delta$ and a closed neighborhood W of P which is homeomorphic to $P\Delta_{n+1}\mu^{n+1}$, such that $g_\delta|_W \simeq_p^n g_{\delta_\alpha}|_W$ and $g_\delta f \simeq_p^n \mathrm{id}_X$ in α for all $\delta \geq \delta_\alpha$ by Lemma 3.3. By the same argument as in Theorem 3.2, we may assume that $g_{\delta_\alpha}|_W$ is a Z-embedding of $P\Delta_{n+1}\mu^{n+1}$ into α and $X \subset g_{\delta_\alpha}(W)$. Then $g_{\delta_\alpha}^{-1}|_X \simeq_p^n g_{\delta_\alpha}^{-1} g_{\delta_\alpha} f|_X \simeq_p^n f$ in $P\Delta_{n+1}\mu^{n+1}$.

Let $\alpha, \beta \in A$. Since $g_{\delta_\alpha}^{-1}|_X \simeq_p^n f \simeq_p^n g_{\delta_\beta}^{-1}|_X$ in $P\Delta_{n+1}\mu^{n+1}$, by the Z-set unknotting theorem, there exists a homeomorphism $G : P\Delta_{n+1}\mu^{n+1} \to P\Delta_{n+1}\mu^{n+1}$ such that $Gg_{\delta_\alpha}^{-1}|_X = g_{\delta_\beta}^{-1}|_X$. Then $h = g_{\delta_\beta} G g_{\delta_\alpha}^{-1}$ is the desired homeomorphism. ■

In [1], it is proved that if X is connected, then $X \in SUV^n$ if and only if $n\text{-Sh}_p(X) = n\text{-Sh}_p(T)$ for some tree T. So we have the following:

COROLLARY 3.5. *Let X be a connected Z-set in a μ^{n+1}-manifold and $X \in SUV^n$. Then X has an arbitrarily small closed μ^{n+1}-manifold neighborhood V such that $V \approx T\Delta_{n+1}\mu^{n+1}$ for some tree T.*

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