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SOME REMARKS ON RATIONAL MÜNTZ APPROXIMATION ON $[0,\infty)$

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The following result is proved in the present paper: Let $\{\lambda_n\}$ be an increasing sequence of distinct real numbers which approaches a finite limit λ as n goes to infinity and for which

$$\limsup_{n\to\infty}(\lambda-\lambda_n)\sqrt[3]{n}=\infty.$$

Then the rational combinations of $\{x^{\lambda_n}\}$ form a dense set in $C_{[0,\infty]}$. One could note that the method used in this paper is probably more interesting than the result itself.

1. Introduction. Let $C_{[0,1]}$ be the continuous function space on $[0,1]$. As a generalization of the Weierstrass theorem, Müntz considered the use of combinations of $\{x^{\lambda_n}\}$ to approximate functions in $C_{[0,1]}$. The well-known Müntz theorem (cf. Cheney [2]) states that the linear combinations of $\{x^{\lambda_n}\}$ for

$$0=\lambda_0<\lambda_1<\lambda_2<\ldots$$

are dense in $C_{[0,1]}$ if and only if

$$\sum_{n=1}^{\infty}\frac{1}{\lambda_n}=\infty.$$

Since there are many great differences between rational and polynomial approximations, in the early seventies, Newman conjectured that the completeness question for rational functions from Müntz systems had a completely different answer. He asked: "What is the condition on the λ_n which makes the rational combinations of $\{x^{\lambda_n}\}$ (in symbols $R(\Lambda)$) dense in $C_{[0,1]}$? The correct necessary and sufficient condition is not simply that $\sum_{n=1}^{\infty}1/\lambda_n = \infty$; what is it?"

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This question stood unsolved for some years. In 1976, Somorjai [3] first proved a surprising result that for any sequence $\{\lambda_n\}$ of distinct nonnegative increasing numbers, $R(\Lambda)$ are always dense in $C_{[0,1]}$. In 1978 Bak and Newman [1] generalized Somorjai's conclusion to include sequences of distinct positive numbers, and recently, in [5] we showed that the same result also holds when $\{\lambda_n\}$ is a sequence of distinct negative numbers.

To consider this problem more generally (since one benefit of rational approximation appears to be the approximation to functions in an unbounded interval, which usual polynomial approximation cannot be applied to), we proved the following theorem in [4], which generalizes all the above results.

THEOREM 1. *Let $C^*_{[0,\infty]}$ be the space of all continuous functions on $[0,\infty)$ with*

$$\lim_{x\to\infty} f(x) = f(0),$$

and $\{\lambda_n\}_{n=1}^\infty$ a sequence of nonnegative numbers with infinitely many distinct elements. Then $R(\Lambda)$ are dense in $C^*_{[0,\infty]}$.

Actually, in case $\{\lambda_n\}$ has an infinite cluster point, the following complete result was proved in [4].

THEOREM 2. *Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of nonnegative numbers with infinitely many distinct elements which has an infinite cluster point. Then $R(\Lambda)$ are dense in $C_{[0,\infty]}$, the space of all continuous functions on $[0,\infty)$ for which $\lim_{x\to\infty} f(x)$ exists and is finite.*

Therefore a natural question was raised in [4]:

PROBLEM. *Let $\{\lambda_n\}_{n=1}^\infty$ be a sequence of nonnegative numbers with infinitely many distinct elements which has no infinity cluster point. Do $R(\Lambda)$ form a dense set in $C_{[0,\infty]}$?*

Since Müntz rational combinations are not closed under addition, which excludes almost all standard methods except direct constructions, this problem appears hard. In the present paper, we will prove that the problem has an affirmative answer for some sequences of nonnegative numbers. Although we still do not know if this result is sharp or not, the method used does reflect some new ideas, which we hope could open a new way in this direction.

2. Results and proof

THEOREM 3. *Let $\{\lambda_n\}$ be an increasing sequence of distinct real numbers which approaches a finite limit λ as n goes to infinity and for which*

$$\limsup_{n\to\infty} (\lambda - \lambda_n)\sqrt[3]{n} = \infty.$$

Then the rational combinations of $\{x^{\lambda_n}\}$ form a dense set in $C_{[0,\infty]}$.

We also can state Theorem 3 in the following form, which gives a partial affirmative answer to the above-mentioned problem.

THEOREM 3′. *Let $\{\lambda_n\}$ be a monotone sequence of distinct positive numbers which approaches a finite limit λ as n goes to infinity and for which*

$$\limsup_{n\to\infty} |\lambda-\lambda_n|\sqrt[3]{n}=\infty.$$

Then the rational combinations of $\{x^{\lambda_n}\}$ form a dense set in $C_{[0,\infty]}$.

For instance, we have the following application.

COROLLARY. *Let $\lambda_n = n^{-\gamma}$, $n=1,2,\ldots,$ $\gamma<1/3$. Then the rational combinations of $\{x^{\lambda_n}\}$ form a dense set in $C_{[0,\infty]}$.*

We first establish the following lemma.

LEMMA. *Given sufficiently large N, let*

$$x_j=\begin{cases} -\dfrac{N}{j+2}, & j=0,1,\ldots,N-2,\\ \dfrac{j-2N+2}{N}, & j=N-1,\ldots,3N-2,\\ \dfrac{N}{4N-j-2}, & j=3N-1,\ldots,4N-4,\end{cases}$$

and

$$Q_j(x)=\begin{cases} 1, & j=0,\\ (x+N)^{2(j+N)^3}\displaystyle\prod_{l=1}^{j}(x_l+N)^{-\Delta(2(l+N)^3)}, & j=1,\ldots,4N-5,\\ \left(\dfrac{4x}{N}\right)^{1296N^3}, & j=4N-4,\end{cases}$$

where

$$\Delta a_1=a_1,\qquad \Delta a_n=a_n-a_{n-1},\quad n\ge 2.$$

Then there is an absolute constant $A>0$ such that for $x_k\le x\le x_{k+1}$, $k=0,1,\ldots,4N-4$, and $j\in\{0,1,\ldots,4N-5\}\setminus\{k,k+1\}$ we have

$$0\le \frac{Q_j(x)}{Q_k(x)}\le e^{-A|j-k|}. \tag{1}$$

For $x_k\le x\le x_{k+1}$, $k=6,\ldots,4N-10$,

$$0\le \frac{Q_{4N-4}(x)}{Q_k(x)}\le e^{-N}e^{-A(4N-k-4)}, \tag{2}$$

and for $|x| \geq N/2$ *and* $j \leq 4N-5$,

$$0 \leq \frac{Q_j(x)}{Q_{4N-4}(x)} \leq e^{-N} e^{-A(4N-j-4)}. \tag{3}$$

P r o o f. We start from the proof of (1). Assume first that $x_k \leq x \leq x_{k+1}$, $k = 1, \ldots, 4N-5$, and $j < k$. We divide the proof into the following cases. Denote by A_i, $i = 1, 2, \ldots$, some absolute positive constants throughout.

CASE 1: $1 \leq k \leq N-2$. We check that

$$\begin{aligned}\frac{Q_j(x)}{Q_k(x)} &= \prod_{l=j+1}^{k} \left(\frac{x_l + N}{x + N}\right)^{\Delta(2(l+N)^3)} \leq \prod_{l=j+1}^{k} \left(\frac{x_l + N}{x_k + N}\right)^{\Delta(2(l+N)^3)} \\ &= \prod_{l=j+1}^{k} \left(1 - \frac{k-l}{(l+2)(k+1)}\right)^{\Delta(2(l+N)^3)}.\end{aligned}$$

Since $0 < l+2 \leq N$, $0 < k+1 \leq N$ and $\Delta(2(l+N)^3) \geq 6N^2$ we see that

$$\prod_{l=j+1}^{k} \left(1 - \frac{k-l}{(l+2)(k+1)}\right)^{\Delta(2(l+N)^3)} \leq e^{-A_1(k-j-1)},$$

so that

$$\frac{Q_j(x)}{Q_k(x)} \leq e^{-A_1(k-j-1)} \leq e^{-A_2(k-j)}.$$

CASE 2: $N-2 \leq j < k \leq 3N-2$. Now similarly

$$\begin{aligned}\frac{Q_j(x)}{Q_k(x)} &\leq \prod_{l=j+1}^{k} \left(\frac{x_l + N}{x_k + N}\right)^{\Delta(2(l+N)^3)} \\ &= \prod_{l=j+1}^{k} \left(1 - \frac{k-l}{k - 2N + 2 + N^2}\right)^{\Delta(2(l+N)^3)} \leq e^{-A_3(k-j)}.\end{aligned}$$

CASE 3: $3N-2 \leq j < k \leq 4N-5$. In this case we have

$$\begin{aligned}\frac{Q_j(x)}{Q_k(x)} &\leq \prod_{l=j+1}^{k} \left(\frac{x_l + N}{x_k + N}\right)^{\Delta(2(l+N)^3)} \\ &= \prod_{l=j+1}^{k} \left(1 - \frac{k-l}{(4N-k-1)(4N-l-2)}\right)^{\Delta(2(l+N)^3)} \leq e^{-A_4(k-j)}.\end{aligned}$$

CASE 4: $0 \leq j \leq N-2$ and $N-1 \leq k \leq 3N-2$. Applying the known results of Cases 1 and 2 we have

$$\frac{Q_j(x)}{Q_k(x)} = \frac{Q_j(x)}{Q_{N-1}(x)} \frac{Q_{N-1}(x)}{Q_k(x)} \leq e^{-A_2(N-j-1)} e^{-A_3(k-N+1)} \leq e^{-A_5(k-j)}.$$

CASE 5: $0 \le j \le N-2$ and $3N-1 \le k \le 4N-5$.

CASE 6: $N-1 \le j \le 3N-2$ and $3N-1 \le k \le 4N-5$.

In both these cases, in a similar way to Case 4 we can prove

$$\frac{Q_j(x)}{Q_k(x)} \le e^{-A_6(k-j)}.$$

Combining all cases together we have proved (1) for $x_k \le x \le x_{k+1}$, $k = 1, \ldots, 4N-5$, and $j < k$.

When $x_k \le x \le x_{k+1}$, $k = 1, \ldots, 4N-5$, and $k+1 < j < 4N-4$, by applying a similar argument we can obtain

$$\frac{Q_j(x)}{Q_k(x)} \le e^{-A_7(j-k)}.$$

Combination of all the above results yields (1).

To prove (3), we need to verify that for $|x| \ge N/2$,

$$\begin{aligned}\frac{Q_j(x)}{Q_{4N-4}(x)} \\ &= \prod_{l=1}^{j}\left(\frac{N}{4(x_l+N)}\right)^{\Delta(2(l+N)^3)}\left(\frac{x+N}{x}\right)^{2(j+N)^3}\left(\frac{4x}{N}\right)^{-1296N^3+2(j+N)^3} \\ &\le e^{-(5N-j-4)}.\end{aligned}$$

Indeed, since $|x| \ge N/2$,

$$\frac{N}{4(x_l+N)} \le 1, \qquad \frac{x+N}{x} \le 3,$$

and $4x/N \ge 2$, we get

$$\begin{aligned}\frac{Q_j(x)}{Q_{4N-4}(x)} &\le 3^{2(j+N)^3}2^{-1296N^3+2(j+N)^3} \le 2^{-1296N^3+6(j+N)^3} \\ &\le e^{-216N^3+(j+N)^3} \le e^{-(5N-j)^3} \le e^{-N}e^{-(4N-j-4)},\end{aligned}$$

which is the required result.

The proof of (2) is similar. ∎

Proof of Theorem 3. Suppose that there is a sequence $\{n_l\}$ of natural numbers such that

$$\lim_{l\to\infty}(\lambda - \lambda_{n_l})\sqrt[3]{n_l} = \infty.$$

For convenience we still write n_l as n. Let

$$N = \left[\sqrt[3]{\frac{n}{2592}}\right]$$

for sufficiently large n, and let $P_k(x, a_0, a_1, \ldots, a_k)$ denote the kth divided difference of x^α at $\alpha = a_0, a_1, \ldots, a_k$ with respect to α, that is,

$$P_0(x, a_0) = x^{a_0},$$
$$P_k(x, a_0, a_1, \dots, a_k) = \frac{P_{k-1}(x, a_0, a_1, \dots, a_{k-1}) - P_{k-1}(x, a_1, a_2, \dots, a_k)}{a_0 - a_k}.$$

Write

$$P^*_{4N-4}(x) = P_{1296N^3}(x, \lambda_{n-1296N^3}, \lambda_{n-1296N^3+1}, \dots, \lambda_n).$$

Since $\lambda_n < \lambda_{n+1}$ and $\lim_{n\to\infty} \lambda_n = \lambda$, we select n large enough to make

(4) $$\lambda - \lambda_{[n/2]} < 1/2,$$

and then choose sufficiently large $m \geq n+1$ such that

(5) $$\frac{1}{2}(\lambda - \lambda_n) \leq \lambda_m - \lambda_n < \lambda - \lambda_n,$$

and

(6) $$0 < \lambda - \lambda_m < N^{-1}.$$

Then let

$$P_0(x) = P_0(e^N x, \lambda_m),$$

for $j = 1, \dots, 4N-5$,

$$P_j(x) = P_{2(j+N)^3}(e^N x, \lambda_m, \lambda_{m+1}, \dots, \lambda_{m+2(j+N)^3}),$$

and

$$P_{4N-4}(x) = P_{1296N^3}(x, \lambda_{m+250N^3}, \lambda_{m+250N^3+1}, \dots, \lambda_{m+1546N^3}).$$

By the mean value theorem,

(7) $$P_j(x) = \frac{(e^N x)^{\eta_j} \log^{2(j+N)^3}(e^N x)}{(2(j+N)^3)!},$$

$$\lambda_m \leq \eta_j \leq \lambda_{m+2(j+N)^3}, \qquad j = 1, \dots, 4N-5,$$

(8) $$P_{4N-4}(x) = \frac{x^{\eta_{4N-4}} \log^{1296N^3} x}{(1296N^3)!},$$

$$\lambda_{m+250N^3} \leq \eta_{4N-4} \leq \lambda_{m+1546N^3},$$

and

(9) $$P^*_{4N-4}(x) = \frac{x^{\eta^*_{4N-4}} \log^{1296N^3} x}{(1296N^3)!},$$

$$\lambda_{[n/2]} \leq \lambda_{n-1296N^3} \leq \eta^*_{4N-4} \leq \lambda_n.$$

Now let $f \in C_{[0,\infty]}$ and set

$$g(x) = f(x) - f(0).$$

Then $g(0)=0$. Write

$$S_0(x)=P_0(x),\qquad S_j(x)=(2(j+N)^3)!\prod_{l=1}^{j}(x_l+N)^{-\Delta(2(j+N)^3)}P_j(x)$$

for $1\le j\le 4N-5$, and

$$S_{4N-4}(x)=(1296N^3)!\left(\frac{4}{N}\right)^{1296N^3}P_{4N-4}(x),$$

$$S^*_{4N-4}(x)=(1296N^3)!\left(\frac{4}{N}\right)^{1296N^3}P^*_{4N-4}(x),$$

where x_j, $j=0,1,\dots,4N-4$, are the numbers defined in the Lemma. Define

$$R(x)=\frac{\sum_{j=0}^{4N-4}g(t_j)S_j(x)}{\sum_{j=0}^{4N-4}S_j(x)+S^*_{4N-4}(x)},$$

where $t_j=e^{x_j}$. Then $R(x)\in R(\Lambda)$. By (7)–(9),

$$\begin{aligned}R(x)\\ &=\frac{\sum_{j=0}^{4N-5}g(t_j)(e^Nx)^{\eta_j}Q_j(\log x)+g(t_{4N-4})x^{\eta_{4N-4}}Q_{4N-4}(\log x)}{\sum_{j=0}^{4N-5}(e^Nx)^{\eta_j}Q_j(\log x)+x^{\eta_{4N-4}}Q_{4N-4}(\log x)+x^{\eta^*_{4N-4}}Q_{4N-4}(\log x)}\\ &=\frac{\sum_{j=0}^{4N-5}g(t_j)(e^Nx)^{\alpha_j}Q_j(\log x)+g(t_{4N-4})x^{\alpha_{4N-4}}Q_{4N-4}(\log x)}{\sum_{j=0}^{4N-5}(e^Nx)^{\alpha_j}Q_j(\log x)+x^{\alpha_{4N-4}}Q_{4N-4}(\log x)+Q_{4N-4}(\log x)},\end{aligned}$$

where $\eta_0=\lambda_m$ and for $j=0,1,\dots,4N-5$,

$$0<\lambda_m-\lambda_n<\alpha_j=\eta_j-\eta^*_{4N-4}<\eta_{4N-4}-\eta^*_{4N-4}=\alpha_{4N-4}.$$

From (5), (6), we also see that

$$\alpha_j>\tfrac{1}{2}(\lambda-\lambda_n)\tag{10}$$

for all $j=0,1,\dots,4N-4$, and

$$|\alpha_j-\alpha_k|\le 2N^{-1}.\tag{11}$$

By (4), for all $j=0,1,\dots,4N-4$,

$$\alpha_j<1/2.\tag{12}$$

We estimate $g(x)-R(x)$. Write

$$
\begin{aligned}
g(x)-R(x)&\\
=&\frac{g(x)Q_{4N-4}(\log x)}{\sum_{j=0}^{4N-5}(e^Nx)^{\alpha_j}Q_j(\log x)+x^{\alpha_{4N-4}}Q_{4N-4}(\log x)+Q_{4N-4}(\log x)}\\
&+\Big(\frac{\sum_{j=0}^{4N-5}(g(x)-g(t_j))(e^Nx)^{\alpha_j}Q_j(\log x)}{\sum_{j=0}^{4N-5}(e^Nx)^{\alpha_j}Q_j(\log x)+x^{\alpha_{4N-4}}Q_{4N-4}(\log x)+Q_{4N-4}(\log x)}\\
&+\frac{(g(x)-g(t_{4N-4}))x^{\alpha_{4N-4}}Q_{4N-4}(\log x)}{\sum_{j=0}^{4N-5}(e^Nx)^{\alpha_j}Q_j(\log x)+x^{\alpha_{4N-4}}Q_{4N-4}(\log x)+Q_{4N-4}(\log x)}\Big)\\
=:&\,I_1(x)+I_2(x).
\end{aligned}
$$

Let $x\in[t_k,t_{k+1}]$, $k=6,\dots,4N-10$. Since $Q_j(x)\ge 0$ for all $j=0,1,\dots,4N-4$ and all $x\in(-\infty,\infty)$, we have

$$
\begin{aligned}
|I_2(x)|\le{}& |g(x)-g(t_k)|+|g(x)-g(t_{k+1})|\\
&+\frac{(\sum_{j=0}^{k-1}+\sum_{k+1}^{4N-5})|g(x)-g(t_j)|(e^Nx)^{\alpha_j}Q_j(\log x)}{(e^Nx)^{\alpha_k}Q_k(\log x)}\\
&+\frac{|g(x)-g(t_{4N-4})|x^{\alpha_{4N-4}}Q_{4N-4}(\log x)}{\sum_{j=0}^{4N-5}(e^Nx)^{\alpha_j}Q_j(\log x)+x^{\alpha_{4N-4}}Q_{4N-4}(\log x)+Q_{4N-4}(\log x)}\\
=:{}&J_1(x)+J_2(x)+J_3(x)+J_4(x).
\end{aligned}
$$

Evidently,

$$
\begin{aligned}
J_1(x)&=O(\omega(g,N^{-1})+\omega(g^*,N^{-1})+\omega(g^{**},N^{-1})),\\
J_2(x)&=O(\omega(g,N^{-1})+\omega(g^*,N^{-1})+\omega(g^{**},N^{-1})),
\end{aligned}
$$

where $\omega(f,t)$ is the usual modulus of continuity of f on the interval $[0,1]$, and

$$
g^*(t)=\begin{cases} g(e^{1/t}), & 0<t\le 1,\\ \lim_{x\to\infty}g(x), & t=0,\end{cases}\qquad g^{**}(t)=\begin{cases} g(e^{-1/t}), & 0<t\le 1,\\ g(0), & t=0.\end{cases}
$$

From (1),

$$
\begin{aligned}
J_3(x)&\le\Big(\sum_{j=1}^{k-1}+\sum_{j=k+1}^{4N-5}\Big)|g(x)-g(t_j)|e^{-A|k-j|}(e^Nx)^{\alpha_j-\alpha_k}\\
&\le(\omega(g,N^{-1})+\omega(g^*,N^{-1})+\omega(g^{**},N^{-1}))O\Big(\sum_{j=0}^{\infty}je^{-Aj}\Big)\\
&=O(\omega(g,N^{-1})+\omega(g^*,N^{-1})+\omega(g^{**},N^{-1})),
\end{aligned}
$$

where for the estimate

$$
(e^Nx)^{\alpha_j-\alpha_k}=O(1)
$$

we have applied (11) together with the fact that $e^{-N} \le x \le e^N$ in this case. Similarly, by (2) and (11), we obtain

$$J_4(x) \le \frac{|g(x) - g(t_{4N-4})| x^{\alpha_{4N-4}} Q_{4N-4}(\log x)}{(e^N x)^{\alpha_k} Q_k(\log x)} \le \|g\| O(e^{-N})$$

and

$$|I_1(x)| \le \|g\| O(e^{-N}),$$

where

$$\|g\| = \max_{0 \le x < \infty} |g(x)|.$$

All the above estimates together yield

$$(13) \quad |g(x) - R(x)| = O(\omega(g, N^{-1}) + \omega(g^*, N^{-1}) + \omega(g^{**}, N^{-1})) + \|g\| O(e^{-N})$$

for $x \in [t_k, t_{k+1}]$, $k = 6, \dots, 4N-10$.

Suppose now $x \in [t_k, t_{k+1}]$, $k = 0, 1, \dots, 5$. In a similar manner to the above discussions, by (1) (we still use the symbols $J_i(x)$, $i = 1,2,3,4$, as above), for $i = 1,2,3$,

$$J_i(x) = O(\omega(g, N^{-1}) + \omega(g^*, N^{-1}) + \omega(g^{**}, N^{-1})).$$

As for $J_4(x)$, we note that from (10),

$$x^{\alpha_{4N-4}} \le \exp(-C_1 N(\lambda - \lambda_n)) \le \exp(-C \sqrt[3]{n} (\lambda - \lambda_n)) =: \varepsilon_n,$$

where C and C_i, $i=1,2,\dots$, are some absolute positive constants. Therefore

$$J_4(x) = \varepsilon_n O(\|g\|).$$

It is clear that

$$|I_1(x)| \le \max_{0 \le x \le t_6} |g(x)| =: \sigma_n.$$

Altogether we have

$$(14) \quad |g(x) - R(x)| = O(\omega(g, N^{-1}) + \omega(g^*, N^{-1}) + \omega(g^{**}, N^{-1})) + O(\varepsilon_n \|g\| + \sigma_n)$$

for $x \in [t_k, t_{k+1}]$, $k = 0, 1, \dots, 5$.

Let $x \in [t_k, t_{k+1}]$, $k = 4N-9, \dots, 4N-5$. The estimates for $J_i(x)$, $i = 1,2,3$, are the same as in the above case. We readily check that

$$J_4(x) \le \max_{t_{4N-9} \le x, x' < \infty} |g(x) - g(x')| = O(\omega(g^*, N^{-1})),$$

and

$$|I_1(x)| \le \varepsilon_n \|g\|$$

due to the same argument as above. Hence

$$(15) \quad |g(x) - R(x)| = O(\omega(g, N^{-1}) + \omega(g^*, N^{-1}) + \omega(g^{**}, N^{-1})) + O(\varepsilon_n \|g\|)$$

for $x \in [t_k, t_{k+1}]$, $k = 4N-9, \dots, 4N-5$.

In case $x \le e^{-N/2}$ we rewrite $I_2(x)$ as follows:

$$|I_2(x)| \le \frac{\sum_{j=0}^{4N-5} |g(x)-g(t_j)|(e^N x)^{\alpha_j} Q_j(\log x)}{Q_{4N-4}(\log x)} + \frac{|g(x)-g(t_{4N-4})| x^{\alpha_{4N-4}} Q_{4N-4}(\log x)}{Q_{4N-4}(\log x)} =: K_1(x) + K_2(x).$$

We estimate that

$$K_1(x) \le \|g\| O\Big(e^{-N(1-\alpha_j)} \sum_{j=0}^{\infty} e^{-Aj}\Big) = O(e^{-N/2}\|g\|)$$

by (3) and (12), and

$$K_2(x) \le 2\|g\| \exp(-C_2 N(\lambda - \lambda_n)) = O(\varepsilon_n \|g\|).$$

On the other hand,

$$|I_1(x)| \le \max_{0 \le x \le t_0} |g(x)|.$$

So for $x \le e^{-N/2}$ we have

$$|g(x) - R(x)| = O(e^{-N/2}\|g\|) + O(\varepsilon_n \|g\| + \sigma_n). \tag{16}$$

Finally, when $x \ge e^{N/2}$, a similar argument applied to $I_1(x)$ leads to

$$|I_1(x)| \le \frac{\|g\| Q_{4N-4}(\log x)}{x^{\alpha_{4N-4}} Q_{4N-4}(\log x)} = O(\varepsilon_n \|g\|).$$

At the same time, by (3),

$$\begin{aligned} |I_2(x)| &\le \frac{\sum_{j=0}^{4N-5} |g(x)-g(t_j)|(e^N x)^{\alpha_j} Q_j(\log x)}{x^{\alpha_{4N-4}} Q_{4N-4}(\log x)} \\ &\quad + \frac{|g(x)-g(t_{4N-4})| x^{\alpha_{4N-4}} Q_{4N-4}(\log x)}{x^{\alpha_{4N-4}} Q_{4N-4}(\log x)} \\ &\le (\omega(g, N^{-1}) + \omega(g^*, N^{-1}) + \omega(g^{**}, N^{-1})) \\ &\quad \times \Big(1 + \sum_{j=0}^{4N-5} e^{N\alpha_j} e^{-N}(4N-j-4) e^{-(4N-j-4)}\Big) \\ &= O(\omega(g, N^{-1}) + \omega(g^*, N^{-1}) + \omega(g^{**}, N^{-1})). \end{aligned}$$

That is, for $x \ge e^{N/2}$,

$$\begin{aligned} &|g(x) - R(x)| \\ &\quad = O(\omega(g, N^{-1}) + \omega(g^*, N^{-1}) + \omega(g^{**}, N^{-1})) + O(\varepsilon_n \|g\|). \end{aligned} \tag{17}$$

If we note that $\lim_{n\to\infty} \sigma_n = 0$ since $g(0) = 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$ because of the assumption of Theorem 3 as well as other conditions, we now see that a combination of (13)–(17) yields that for $g \in C_{[0,\infty]}$ with $g(0) = 0$ and any given $\varepsilon > 0$, there is an $R(x) \in R(\Lambda)$ such that

$$\|g - R\| \le \varepsilon,$$

or for $f \in C_{[0,\infty]}$,

$$\|f(x) - f(0) - R(x)\| \le \varepsilon,$$

and $R(x) + f(0) \in R(\Lambda)$. Theorem 3 is proved. ∎

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