COLLOQUIUM MATHEMATICUM

VOL. 77 1998 NO. 2

FUNCTIONS CHARACTERIZED BY IMAGES OF SETS

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For non-empty topological spaces X and Y and arbitrary families $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$ we put $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in Y^X : (\forall A \in \mathcal{A})(f[A] \in \mathcal{B})\}$. We examine which classes of functions $\mathcal{F} \subseteq Y^X$ can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$. We are mainly interested in the case when $\mathcal{F} = \mathcal{C}(X,Y)$ is the class of all continuous functions from X into Y. We prove that for a non-discrete Tikhonov space X the class $\mathcal{F} = \mathcal{C}(X,\mathbb{R})$ is not equal to $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ for any $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$. Thus, $\mathcal{C}(X,\mathbb{R})$ cannot be characterized by images of sets. We also show that none of the following classes of real functions can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$: upper (lower) semicontinuous functions, derivatives, approximately continuous functions, Baire class 1 functions, Borel functions, and measurable functions.

1. Basic definitions and facts. Throughout the paper we use the standard definitions and notation. In particular, the family of all functions from a set X into Y is denoted by Y^X. The symbol $|X|$ stands for the cardinality of X and $\mathcal{P}(X)$ for the family of all subsets of X. For a cardinal number κ we write $[X]^\kappa$ to denote the family of all subsets Y of X with $|Y| = \kappa$. (In particular, $[X]^1$ stands for the set of all singletons in X and $[X]^2$ for the family of all doubletons in X.) Similarly we define $[X]^{<\kappa}$, $[X]^{\le\kappa}$ and $[X]^{\ge\kappa}$.

1991 *Mathematics Subject Classification*: Primary 54C05; Secondary 26A15, 18B30, 54C30.

Key words and phrases: Tikhonov space, functionally Hausdorff space, Cook continuum, strongly rigid family of spaces, continuous function, upper or lower semicontinuous function, derivative, approximately continuous function, Baire class 1 function, Borel function, measurable function.

The work of the first two authors was partially supported by the NATO Collaborative Research Grant CRG 950347. Stephen Watson has been supported by Grant No. A8855 of the Natural Sciences and Engineering Research Council of Canada.

We use the symbol $\mathrm{Const}_{X,Y}$ for the class of all constant functions from X into Y, and write just Const when the spaces X and Y are clear from the context. The identity map from X into X is denoted by id_X. For topological spaces X and Y the class of all continuous functions from X into Y is denoted by $\mathcal{C}(X,Y)$.

Following Engelking [4] we say that a space X is *totally disconnected* if all quasi-components of X are singletons. All topological spaces considered in this paper are at least T_0 (distinguish between points) and contain at least two points.

1.1. *Main results*. In order to announce our principal results we also need the following frequently used notation: for non-empty sets X, Y and families $\mathcal{A} \subseteq \mathcal{P}(X)$, $\mathcal{B} \subseteq \mathcal{P}(Y)$,

$$\mathcal{C}_{\mathcal{A},\mathcal{B}} = \{f \in Y^X : (\forall A \in \mathcal{A})(f[A] \in \mathcal{B})\}.$$

Some basic properties of $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ are outlined below in Facts 1.2 and 1.3.

This work is motivated by a paper of Velleman [8] in which it is proved that the class $\mathcal{F} = \mathcal{C}(\mathbb{R},\mathbb{R})$ is not equal to $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ for any $\mathcal{A},\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$. Thus, $\mathcal{C}(\mathbb{R},\mathbb{R})$ cannot be characterized by images of sets. This stays in contrast with the fact that, by definition, the family $\mathcal{C}(X,Y)$ can be characterized by preimages of sets for every pair of topological spaces X, Y:

$$\mathcal{C}(X,Y) = \{f \in Y^X : f^{-1}(V) \text{ is open in } X \text{ for every open } V \subseteq Y\}.$$

This phenomenon justifies the following terminology.

Definition 1.1. Let X and Y be topological spaces. We say that:

- the pair $\langle X,Y\rangle$ of spaces has the *V-property* if there exist $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$ such that $\mathcal{C}(X,Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$;
- X is a *V-space* if $\langle X,X\rangle$ has the V-property.

In these terms Velleman's theorem says that $\mathbb{R}$ is not a V-space. Our aim is to generalize this result to a large class of pairs $\langle X,Y\rangle$ of topological spaces. In Section 3 we characterize the spaces X such that the pair $\langle X,\mathbb{R}\rangle$ has the V-property. These are the spaces X such that every connected component of X is open and admits only constant real-valued functions (Theorem 3.1). In particular, for a non-discrete functionally Hausdorff space (in particular, Tikhonov space) X the pair $\langle X,\mathbb{R}\rangle$ does not have the V-property (Corollary 3.6). The proof is, roughly speaking, based on:

(i) a reduction technique which permits us to consider only connected spaces X (Theorem 2.1);

(ii) a construction, for X such that $\langle X,\mathbb{R}\rangle$ has the V-property and $\mathcal{C}(X,\mathbb{R}) \neq \mathrm{Const}$, of functions $h \in \mathcal{C}(X,\mathbb{R})$ that "detect non-closed sets", i.e., such that h^{-1} is not closed for some nowhere dense $S \subseteq \mathbb{R}$ (Lemma 3.8);

(iii) a construction, for X such that $\mathcal{C}(X,\mathbb{R}) \neq \mathrm{Const}$, of an appropriate discontinuous function $g \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ (Lemma 3.9).

Step (iii) also permits showing in Section 4 that no class of functions from $\mathbb{R}$ to $\mathbb{R}$ between $\mathcal{C}(\mathbb{R},\mathbb{R})$ and the class of measurable functions can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ (Corollary 4.2).

Properties of $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ are given in Section 1.2. In Section 1.3 we give the first examples of non-trivial V-spaces (Cook's continuum) and their permanence properties. More precisely, if the pair $\langle X, Y\rangle$ has the V-property, then so does every pair $\langle X', Y'\rangle$ where X' is a retract of X and Y' is a subspace of Y (Proposition 1.8 and Corollary 1.10). In Section 5.1 step (i) is elaborated further in Theorem 5.1 which permits one to describe the behavior of V-spaces under topological sums (Corollary 5.2). This gives new examples of V-spaces (Corollary 5.6 and Proposition 5.7).

In our main result, Corollary 3.6, $\mathbb{R}$ can be replaced by Sierpiński's dyad S: for a T_0-space X the pair $\langle X, S\rangle$ has the V-property if and only if X is discrete. (See also open question 5.16.) Consequently, if a pair $\langle X, Y\rangle$ has the V-property for T_0-spaces X and Y with $\mathcal{C}(X,Y) \neq Y^X$, then Y is necessarily T_1. Hence among T_0-spaces the finite V-spaces are precisely the discrete ones (Example 5.8(I)). Here we discuss also another class of V-spaces—the spaces with the co-finite (more generally, co-α) topology (Example 5.8(II)).

In Section 5.2 we study stability of the V-property under cartesian products (Proposition 5.9, Corollaries 5.10 and 5.11). We also show that all finite powers of a Cook continuum are V-spaces (Corollary 5.12). We finish Section 5.2 with further examples of V-spaces based on another natural topological construction carried out on Cook's continuum (Example 5.17, Remark 5.18).

1.2. *Properties of* $\mathcal{C}_{\mathcal{A},\mathcal{B}}$. First, note that $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ can be the empty family. This happens, for example, if $\emptyset \in \mathcal{A}$ and $\emptyset \notin \mathcal{B}$. Since this is a trivial case, in what follows we always assume that all classes of functions we consider are non-empty.

Now, if $\mathcal{C}_{\mathcal{A},\mathcal{B}} \neq \emptyset$ it is easy to see that $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}_{\mathcal{A}\setminus\{\emptyset\},\mathcal{B}\setminus\{\emptyset\}}$. Thus, for the remainder of this paper we assume that $\emptyset \notin \mathcal{A}$.

Note also that if $\mathcal{A} = \emptyset$ then $\mathcal{C}_{\mathcal{A},\mathcal{B}} = Y^X$. However, we also have $Y^X = \mathcal{C}_{\mathcal{P}(X),\mathcal{P}(Y)} = \mathcal{C}_{\mathcal{P}(X)\setminus\{\emptyset\},\mathcal{P}(Y)\setminus\{\emptyset\}}$. Thus, we always assume that $\mathcal{A}$ contains a non-empty set.

With this agreement in place we can state the first basic observation that is similar in flavor to that from [2, Thm. 1].

FACT 1.2. (i) *If* $\mathcal{A}^* = \{f[A] : A \in \mathcal{A}\ \&\ f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}\} \subseteq \mathcal{B}$ *then* $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}_{\mathcal{A},\mathcal{A}^*}$.

(ii) $\mathrm{Const} \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ *if and only if* $[Y]^1 \subseteq \mathcal{B}$.

(iii) *If* $[Y]^1 \subseteq \mathcal{B}$ *then* $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}_{\mathcal{A}\setminus[X]^1,\mathcal{B}} = \mathcal{C}_{\mathcal{A}\cup[X]^1,\mathcal{B}}$.

(iv) *If* $[Y]^1 \subseteq \mathcal{B}$ *and there exists* $B \in \mathcal{B} \cap [Y]^2$ *then* $B^X \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

(v) *If* $X = Y$ *then* $\mathrm{id}_X \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ *if and only if* $\mathcal{A} \subseteq \mathcal{B}$.

(vi) *If* $X = Y$ *then* $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ *forms a semigroup with respect to composition iff* $\mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, *where* $\mathcal{A}^*$ *is as in* (i).

Proof. The properties (i)–(v) are obvious, as is the implication "$\Leftarrow$" in (vi). To see the other implication of (vi) notice that, by (i), $\mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*} \subseteq \mathcal{C}_{\mathcal{A},\mathcal{A}^*} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, since, by (v), $\mathcal{A} \subseteq \mathcal{A}^*$. On the other hand, $\mathcal{C}_{\mathcal{A},\mathcal{A}^*} \subseteq \mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*}$, as $\mathcal{C}_{\mathcal{A},\mathcal{A}^*} = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ is closed under composition. ■

In the case when a pair $\langle X, Y\rangle$ has the V-property we can extend the remarks of Fact 1.2 as follows. Note first that if X is discrete (or Y is indiscrete) then $\mathcal{C}(X, Y) = Y^X$, and so $\langle X, Y\rangle$ has the V-property. In fact, any discrete space (and any indiscrete space) is a V-space. Thus, to avoid this trivial case we will try to stay away from the situation when X is discrete.

FACT 1.3. *Let* X *be a non-discrete topological space and* $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}(X, Y)$. *Then*

(i) $\mathrm{Const} \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ *and* $[Y]^1 \subseteq \mathcal{B}$;
(ii) $\mathcal{B} \cap [Y]^2 = \emptyset$;
(iii) $\mathcal{A} \subseteq \mathcal{P}(W) \cup \mathcal{P}(X \setminus W)$ *for every clopen subset* W *of* X;
(iv) *each* $A \in \mathcal{A}$ *is contained in some quasi-component of* X;
(v) X *is not a totally disconnected space*;
(vi) $\mathcal{C}_{\mathcal{A},\mathcal{A}^*} = \mathcal{C}(X, Y)$ *with* $\mathcal{A}^* = \{f[A] : A \in \mathcal{A}\ \&\ f \in \mathcal{C}(X, Y)\} \subseteq \mathcal{B}$;
(vii) *if* $X = Y$ *then* $\mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*} = \mathcal{C}(X, X)$ *where* $\mathcal{A}^*$ *is as in* (vi).

Proof. (i) follows from Fact 1.2(ii).

(ii) follows from (i) and Fact 1.2(iv) since X is not discrete and Y is T_0.

(iii) follows from (ii) since for every $A \in \mathcal{A} \setminus (\mathcal{P}(W) \cup \mathcal{P}(X \setminus W))$ and any distinct $b_0, b_1 \in Y$ the characteristic function $f : X \to Y$ equal to b_1 on W and b_0 on $X \setminus W$ belongs to $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, and so $\{b_0, b_1\} = f[A] \in \mathcal{B}$.

(iv) follows immediately from (iii).

To see (v) note that if X were totally disconnected then, by (iv), $\mathcal{A} \subseteq [X]^1$ and, by Fact 1.2(iii), $\mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}_{\emptyset,\mathcal{B}} = Y^X$, implying that X is discrete. (vi) and (vii) follow immediately from Fact 1.2(i) and (vi), respectively. ■

Note that by Facts 1.2(ii) and 1.3(i) we can assume that $[X]^1 \subseteq \mathcal{A}$ and

$$X = \bigcup \mathcal{A} \tag{1}$$

when considering the problem whether $\langle X, Y\rangle$ has the V-property. Notice also that Fact 1.3(v) implies, in particular, that no non-discrete zero-dimensional space is a V-space.

According to Fact 1.3(vi) if $\langle X, Y\rangle$ is a pair with the V-property for some $\mathcal{A}$ and $\mathcal{B}$, then it is so for $\mathcal{A}$ and $\mathcal{A}^*$, where $\mathcal{A}^*$ consists of all *continuous* images of sets of $\mathcal{A}$. In other words, *the class* $\mathcal{B}$ *is not relevant* once we know that the V-property is available. In particular, for a V-space X we have $\mathcal{C}(X,X) = \mathcal{C}_{\mathcal{A}^*,\mathcal{A}^*}$ for some family $\mathcal{A}^* \subseteq \mathcal{P}(X)$.

1.3. *When the V-property is available*. Below we give some easy examples of pairs with the V-property. The case $\mathcal{C}(X,Y) = Y^X$ was already discussed above. Now we consider the opposite case, i.e., when $\mathcal{C}(X,Y) = \mathrm{Const}$.

LEMMA 1.4. *If* $\mathcal{C}(X,Y)=\mathrm{Const}$, *then the pair* $\langle X,Y\rangle$ *has the V-property.*

Proof. It suffices to note that $\mathcal{C}(X,Y) = \mathcal{C}_{\{X\},[Y]^1}$. ■

A large number of examples of pairs $\langle X,Y\rangle$ with the V-property can be found with the help of the above proposition. We recall that a space X is *irreducible* if every non-empty open subset of X is dense in X (or, equivalently, every open subspace of X is connected).

COROLLARY 1.5. *The pair* $\langle X,Y\rangle$ *has the V-property in either of the following cases.*

- X *is arcwise connected and* Y *does not contain any arc.*
- X *is connected and* Y *is totally disconnected.*
- X *is irreducible and* Y *is Hausdorff.*

Proof. This follows from the fact that in all these cases $\mathcal{C}(X,Y) = \mathrm{Const}$. ■

This idea cannot help to get non-trivial V-spaces X as $\mathrm{id}_X \in \mathcal{C}(X,X)$. To this end we have to take larger $\mathcal{C}(X,X)$.

PROPOSITION 1.6. *If X is a compact topological space such that every continuous function $f : X \to X$ is either constant or a homeomorphism then X is a V-space.*

Proof. Let $\mathcal{A}$ be the family of all closed subsets of X that do not have precisely two elements. We claim that $\mathcal{C}(X,X) = \mathcal{C}_{\mathcal{A},\mathcal{A}}$.

Clearly $\mathcal{C}(X,X)\subset\mathcal{C}_{\mathcal{A},\mathcal{A}}$. To see the other inclusion take $f\in\mathcal{C}_{\mathcal{A},\mathcal{A}}\setminus\mathrm{Const}$. Then f is one-to-one, since otherwise there is a three-element set A (which belongs to $\mathcal{A}$) such that $f[A]$ has two elements, i.e., does not belong to $\mathcal{A}$. Thus, f is continuous, being a closed mapping which is one-to-one and defined on a compact space. ■

In [3] Cook constructed a continuum K such that $\mathcal{C}(K,K) = \mathrm{Const}\cup\{\mathrm{id}_K\}$.

COROLLARY 1.7. *There exists a continuum K (Cook's continuum) which is a V-space.*

Proof. Follows from Proposition 1.6. ■

We finish this section with the following easy but fundamental facts.

PROPOSITION 1.8. *If $\langle X, Z\rangle$ has the V-property and Y is a subspace of Z then $\langle X, Y\rangle$ also has the V-property.*

Proof. Let $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Z)$ be such that $\mathcal{C}(X, Z) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and let $\mathcal{B}' = \mathcal{B} \cap \mathcal{P}(Y)$. It is enough to notice that $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}'}$.

To see this, let $f : X \to Y$. If $f \in \mathcal{C}(X, Y) \subseteq \mathcal{C}(X, Z) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ then $f[A] \in \mathcal{B} \cap \mathcal{P}(Y) = \mathcal{B}'$ for every $A \in \mathcal{A}$, i.e., $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}'}$. Conversely, if $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}'} \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}} = \mathcal{C}(X, Z)$ then $f \in \mathcal{C}(X, Y)$. ∎

Notice that the domain counterpart of Proposition 1.8 strongly fails, in the sense that the V-property of a pair $\langle X, Y\rangle$ is not necessarily inherited even by closed compact subsets of X. (Compare with Corollaries 1.10 and 1.11.) To see this, let K be a continuum which is a V-space (e.g., a Cook continuum) and let S be a converging sequence in K together with its limit point. Then $\langle K, K\rangle$ has the V-property. However, by Fact 1.3(v), $\langle S, K\rangle$ does not have the V-property since S is non-discrete totally disconnected.

Note also that the pair $\langle K, S\rangle$ has the V-property since K is connected and S is totally disconnected (Corollary 1.5). In particular, the property "$\langle X, Y\rangle$ has the V-property" is not symmetric in the sense that there are examples of pairs $\langle X, Y\rangle$ with the V-property such that $\langle Y, Y\rangle$ does not have the V-property. Another example of a "non-symmetric pair" is given by the pairs $\langle \mathbb{R}, K\rangle$ and $\langle K, \mathbb{R}\rangle$. The pair $\langle \mathbb{R}, K\rangle$ has the V-property again by Corollary 1.5 (Cook's continuum K does not contain any arc), while the second pair does not have the V-property by Theorem 3.1.

LEMMA 1.9. *If $\langle X, Y\rangle$ has the V-property and $f : X \to Z$ is a continuous quotient map, then $\langle Z, Y\rangle$ has the V-property.*

Proof. Let $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ with $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$. Then $\mathcal{C}(Z, Y) = \mathcal{C}_{f[\mathcal{A}],\mathcal{B}}$ with $f[\mathcal{A}] = \{f[A] : A \in \mathcal{A}\}$. The inclusion $\mathcal{C}(Z, Y) \subseteq \mathcal{C}_{f[\mathcal{A}],\mathcal{B}}$ is obvious. The other inclusion follows easily from our assumption that f is a quotient map. ∎

Note that in this lemma f being just "continuous surjective" does not suffice. To see this, take any pair $\langle Z, Y\rangle$ that does not have the V-property and take as X the underlying set of Z equipped with the discrete topology. Then $\langle X, Y\rangle$ has the V-property and $\mathrm{id}_Z : X \to Z$ is a continuous bijection.

The above lemma gives a partial domain counterpart of Proposition 1.8.

COROLLARY 1.10. *If $\langle X, Y\rangle$ has the V-property and Z is a retract of X, then also $\langle Z, Y\rangle$ has the V-property. In particular, any retract of a V-space is again a V-space.* ∎

For further use we also give the following particular cases.

COROLLARY 1.11. *If $\langle X, Y\rangle$ has the V-property and Z is a clopen subset of X then $\langle Z, Y\rangle$ also has the V-property. In particular, a clopen subset of a V-space is a V-space.*

Proof. Every clopen subset of a space is its retract. ∎

COROLLARY 1.12. *If $\langle X\times Z, Y\rangle$ has the V-property, then $\langle Z, Y\rangle$ also has the V-property. In particular, if $X\times Z$ is a V-space then so are X and Z.* ∎

2. A reduction theorem. The main goal of this section is to prove the next theorem which partially reduces the question of when the pair $\langle X, Y\rangle$ has the V-property to the case when X is connected. It is a particular case of Theorem 5.1.

THEOREM 2.1. *The pair $\langle X, Y\rangle$ has the V-property if and only if there exists $\mathcal{B}\subseteq\mathcal{P}(Y)$ such that for every component C of X,*

(a) *C is open in X;*
(b) *there exists $\mathcal{A}_C\subseteq\mathcal{P}(C)$ such that $\mathcal{C}(C,Y)=\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$.*

In particular, all pairs $\langle C, Y\rangle$ have the V-property.

In the proof we use the following easy fact.

LEMMA 2.2. *If $\langle X, Y\rangle$ has the V-property then every quasi-component of X is open and connected.*

Proof. Let $\mathcal{C}(X,Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$ and Q be a quasi-component of X. Choose $a\neq b$ in Y and consider the characteristic function $f: X\to\{a,b\}\subseteq Y$ of Q. By Fact 1.3(iii) each $A\in\mathcal{A}$ is either contained in Q or disjoint from Q. In either case $f[A]$ is a singleton, so $f[A]\in\mathcal{B}$ and $f\in\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}(X,Y)$. This yields that Q is clopen. In particular, Q cannot contain proper clopen subsets, hence Q is connected. ∎

Proof of Theorem 2.1. Let $\mathcal{C}(X,Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$. The necessity of (a) follows from Lemma 2.2. To see (b) let C be a component of X and $\mathcal{A}_C=\mathcal{A}\cap\mathcal{P}(C)$. We claim that $\mathcal{C}(C,Y)=\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$.

The inclusion $\mathcal{C}(C,Y)\subset\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ follows from the fact that, by (a), any continuous $f: C\to Y$ can be extended to a continuous function $\widetilde{f}: X\to Y$ and any such function sends sets from $\mathcal{A}_C=\mathcal{A}\cap\mathcal{P}(C)$ into $\mathcal{B}$.

To see the other inclusion take $f: C\to Y$ from $\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ and extend it to $\widetilde{f}: X\to Y$ assigning a constant value on $X\setminus C$. Then, by (a) and Fact 1.3(iii), any $A\in\mathcal{A}$ is either in $\mathcal{A}_C$ or is disjoint from C. In any case $\widetilde{f}[A]\in\mathcal{B}$, i.e., $\widetilde{f}\in\mathcal{C}_{\mathcal{A},\mathcal{B}}=\mathcal{C}(X,Y)$. So $f\in\mathcal{C}(C,Y)$.

To see that the conditions (a) and (b) are sufficient, for every component C of X let $\mathcal{A}_C\subset\mathcal{P}(C)$ be such that $\mathcal{C}(C,Y)=\mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ and define $\mathcal{A}$ as the union of all families $\mathcal{A}_C$. We claim that $\mathcal{C}(X,Y)=\mathcal{C}_{\mathcal{A},\mathcal{B}}$.

Let $f \in \mathcal{C}(X,Y)$ and $A \in \mathcal{A}$. Then there exists a component C of X such that $A \in \mathcal{A}_C$. So, $f[A] = f|_C[A] \in \mathcal{B}$, since $f|_C \in \mathcal{C}(C,Y) = \mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$. Thus, $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

To see the other inclusion take $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Then for every component C of X we have $f \in \mathcal{C}_{\mathcal{A}_C,\mathcal{B}}$ and $f|_C \in \mathcal{C}_{\mathcal{A}_C,\mathcal{B}} = \mathcal{C}(C,Y)$. But all sets C are clopen. So, f is continuous. ∎

Note that according to Theorem 2.1(a), for every connected space C and every space Y the pair $\langle \mathbb{Q} \times C, Y\rangle$ fails to have the V-property. (Here, as elsewhere in the paper, we assume that Y is not indiscrete and $\mathbb{Q}$ denotes the rationals.)

Theorem 2.1 also gives a new proof of Corollary 1.11: if $\langle X,Y\rangle$ has the V-property and Z is a clopen subset of X then $\langle Z,Y\rangle$ also has the V-property. Indeed, let $\mathcal{B} \subset \mathcal{P}(Y)$ be a family satisfying (a) and (b) of Theorem 2.1 for $\langle X,Y\rangle$. Then $\mathcal{B}$ and the same families $\mathcal{A}_C$ satisfy (a) and (b) for $\langle Z,Y\rangle$ since Z is clopen in X.

3. When the pair $\langle X,\mathbb{R}\rangle$ has the V-property. The main goal of this section is to prove the following generalization of Velleman's theorem.

THEOREM 3.1. *Let X be a topological space. The pair $\langle X,\mathbb{R}\rangle$ has the V-property if and only if for every component C of X,*

(i) *C is open in X; and*
(ii) $\mathcal{C}(C,\mathbb{R}) = \mathrm{Const}$.

Before we prove it, let us notice the following corollaries.

COROLLARY 3.2. *Let X be a topological space for which there exists a component C of X such that either C is not open or $\mathcal{C}(C,\mathbb{R}) \neq \mathrm{Const}$. If Y contains an arc then $\langle X,Y\rangle$ does not have the V-property.*

Proof. Follows from Theorem 3.1 and Proposition 1.8. ∎

COROLLARY 3.3. *Let C be a connected topological space. Then the pair $\langle C,\mathbb{R}\rangle$ has the V-property if and only if $\mathcal{C}(C,\mathbb{R}) = \mathrm{Const}$.* ∎

Before we give further corollaries let us see that one can have regular connected topological spaces with the property (ii).

EXAMPLE 3.4. There exists a regular topological space X with $\mathcal{C}(X,\mathbb{R}) = \mathrm{Const}$. (See [4, Sect. 1.5 and Exerc. 2.R] or [5].) In particular, such an X is connected and $\langle X,\mathbb{R}\rangle$ has the V-property.

A topological space X is *functionally Hausdorff* if the functions $f \in \mathcal{C}(X,\mathbb{R})$ separate the points of X. Note that every completely regular space is functionally Hausdorff.

COROLLARY 3.5. *Let X be a non-discrete functionally Hausdorff space. If Y contains an arc then $\langle X, Y\rangle$ does not have the V-property.* ■

COROLLARY 3.6. *Let X be a functionally Hausdorff space. The pair $\langle X, \mathbb{R}\rangle$ has the V-property if and only if X is discrete.* ■

We split the proof of Theorem 3.1 into a sequence of steps. The first one, based on the reduction theorem, reduces the proof to the case of a connected space, i.e., to Corollary 3.3.

Proof of Theorem 3.1. Assume that (i) and (ii) are fulfilled. Then $\mathcal{C}(C, \mathbb{R}) = \mathcal{C}_{\mathcal{P}(C),[\mathbb{R}]^1}$ for every component C of X. So, by Theorem 2.1, the pair $\langle X, \mathbb{R}\rangle$ has the V-property.

On the other hand, assume that $\langle X, \mathbb{R}\rangle$ has the V-property. By Theorem 2.1 every component C of X is open in X, and $\langle C, \mathbb{R}\rangle$ has the V-property. So, Corollary 3.3 yields $\mathcal{C}(C, \mathbb{R}) = \text{Const}$. ■

The proof of Corollary 3.3 is split into the following two steps.

PROPOSITION 3.7. *If X is a topological space for which there exists a continuous function $h : X \to \mathbb{R}$ such that*

$$h^{-1}(S) \text{ is not closed in } X \text{ for some nowhere dense } S \subseteq \mathbb{R} \tag{2}$$

then $\langle X, \mathbb{R}\rangle$ does not have the V-property.

The next lemma ensures the validity of (2) for connected topological spaces with non-constant continuous real-valued functions. The proof of Proposition 3.7 will be given later in this section.

LEMMA 3.8. *Let X be a connected topological space with $\mathcal{C}(X, \mathbb{R}) \neq$ Const. Then there exists a function as in* (2).

Proof. Let $f : X \to \mathbb{R}$ be a non-constant continuous function. We prove first that

$$\text{there exists } T \subseteq \mathbb{R} \text{ such that } f^{-1}(T) \text{ is not closed in } X.$$

Assume otherwise. Since f is non-constant there exists $a \in \mathbb{R}$ such that both $T = [a, \infty)$ and $\mathbb{R} \setminus T$ intersect $f[X]$. This produces a non-trivial partition $f^{-1}(T) \cup f^{-1}(\mathbb{R} \setminus T)$ of X into closed sets, a contradiction. This proves our claim.

Now fix a $T \subseteq \mathbb{R}$ such that $f^{-1}(T)$ is not closed in X. Pick an $x \in \text{cl}(f^{-1}(T)) \setminus f^{-1}(T)$ and define $T^+ = T \cap [f(x), \infty)$ and $T^- = T \cap (-\infty, f(x)]$. Since obviously at least one of the two possibilities

$$x \in \text{cl}(f^{-1}(T^+)) \setminus f^{-1}(T^+) \quad \text{or} \quad x \in \text{cl}(f^{-1}(T^-)) \setminus f^{-1}(T^-)$$

occurs, we can assume without loss of generality that $T = T^-$. Since $f(x) \notin T$, we have $T = T^- \subseteq (-\infty, f(x))$. Next we note that it is not restrictive to assume $T = (-\infty, f(x))$ as $x \in \text{cl}(f^{-1}(-\infty, f(x))) \setminus f^{-1}(-\infty, f(x))$.

Now fix a strictly increasing sequence $\{a_n\}_{n=1}^{\infty}$ in $\mathbb{R}$ converging to $f(x)$ and set

$$A=\bigcup_{n=0}^{\infty} f^{-1}(a_{2n},a_{2n+1}],\qquad B=\bigcup_{n=0}^{\infty} f^{-1}(a_{2n+1},a_{2n+2}]$$

with $a_0=-\infty$. Clearly $f^{-1}(T)=f^{-1}(A\cup B)$, so either $x\in \mathrm{cl}(f^{-1}(A))$ or $x\in \mathrm{cl}(f^{-1}(B))$. Since the proof is similar in both cases assume the first of these. Now define a continuous map $j:\mathbb{R}\to\mathbb{R}$ such that $j(f(x))=0$ and $j[(a_{2n},a_{2n+1}]]=1/(n+1)$. Consider the continuous map $h=j\circ f$ and let S be the set $\{1/n:n\in\omega\}$. Note that $h^{-1}(S)$ contains $f^{-1}(A)$ which has x in its closure but $x\notin h^{-1}(S)$ since $h(x)=j(f(x))=0$. So, h and S satisfy (2). ∎

In the proof of Proposition 3.7 we will use the following lemma. (The "moreover" part will also be used in the next section.)

LEMMA 3.9. *Let X, h and S be as in Proposition* 3.7, $[\mathbb{R}]^1\subseteq\mathcal{B}\subseteq\mathcal{P}(X)$, *$B\in\mathcal{B}$ be infinite, and $\mathcal{A}\subseteq\mathcal{P}(X)$ be such that*

(3) $\quad \mathrm{cl}(h[A])$ *is an interval for every* $A\in\mathcal{A}$.

Assume that there exists a family $\mathcal{J}$ of pairwise disjoint closed subsets of $\mathbb{R}\setminus\mathrm{cl}(S)$ with the property that for every $x<y$,

(4) *either* $[x,y]\subseteq J$ *for some* $J\in\mathcal{J}$ *or* $|\{J\in\mathcal{J}:J\subset(x,y)\}|\ge|B|$

and

(5) $h[A]\cap J\neq\emptyset$ *for every* $A\in\mathcal{A}$ *and* $J\in\mathcal{J}$ *with* $J\subset\mathrm{cl}(h[A])$.

Then there exists $g:\mathbb{R}\to B$ such that $f=g\circ h\in\mathcal{C}_{\mathcal{A},\mathcal{B}}\setminus\mathcal{C}(X,\mathbb{R})$. Moreover, if $\mathrm{cl}(S)$ *has positive Lebesgue measure, then g can be chosen non-measurable.*

Proof. Let $\mathcal{I}$ be the family of all non-empty open intervals with rational endpoints and let $\langle\langle I_\xi,b_\xi\rangle:\xi<|B|\rangle$ be an enumeration of $\mathcal{I}\times B$. By induction on $\xi<|B|$ choose a one-to-one sequence $\langle J_\xi\in\mathcal{J}:\xi<|B|\rangle$ such that

(6) $\quad J_\xi\subseteq I_\xi$ provided $|\{J\in\mathcal{J}:J\subset I_\xi\}|\ge|B|$.

Fix distinct $a,c\in B$ and define $g:\mathbb{R}\to B$ by

$$g(x)=\begin{cases} b_\xi & \text{if } x\in J_\xi \text{ for some } \xi<|B|,\\ a & \text{if } x\in S,\\ c & \text{otherwise.}\end{cases}$$

To see that $f=g\circ h$ belongs to $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ take $A\in\mathcal{A}$. We now show that $f[A]=g[h[A]]\in[\mathbb{R}]^1\cup\{B\}\subseteq\mathcal{B}$.

If $\mathrm{cl}(h[A])$ is a singleton, then so are $h[A]$ and $f[A]=g[h[A]]$. In particular, $f[A]\in[\mathbb{R}]^1\subseteq\mathcal{B}$. So, assume that $\mathrm{cl}(h[A])$ is not a singleton. Then, by (3), there are $x<y$ such that $(x,y)\subseteq\mathrm{cl}(h[A])\subseteq[x,y]$. Consider two cases.

CASE 1: There exists $I \in \mathcal{I}$ such that $I \subseteq (x,y)$ and $|\{J \in \mathcal{J} : J \subset I\}| \geq |B|$. Take $b \in B$. Then there exists $\xi < |B|$ such that $\langle I, b\rangle = \langle I_\xi, b_\xi\rangle$ and, by (6), $J_\xi \subseteq I_\xi = I \subseteq (x,y) \subseteq \mathrm{cl}(h[A])$. In particular, by (5), $h[A] \cap J_\xi \neq \emptyset$ and so $\emptyset \neq g[h[A] \cap J_\xi] \subseteq g[J_\xi] = \{b_\xi\} = \{b\}$. Thus, $b \in g[h[A]]$. Since $b \in B$ was arbitrary, we conclude that $B \subseteq g[h[A]]$. So, $g[h[A]] = B \in \mathcal{B}$.

CASE 2: For every $I \in \mathcal{I}$ if $I \subseteq (x,y)$ then $|\{J \in \mathcal{J} : J \subset I\}| < |B|$. Then, by (4), for every $I \in \mathcal{I}$ with $I \subseteq (x,y)$ there exists $J_I \in \mathcal{J}$ such that $I \subseteq J_I$. Since elements of $\mathcal{J}$ are pairwise disjoint, all J_I must be equal to the same $J_0 \in \mathcal{J}$ and $(x,y) = \bigcup\{I \in \mathcal{I} : I \subseteq (x,y)\} \subseteq J_0$. So, $h[A] \subseteq [x,y] \subseteq \mathrm{cl}(J_0) = J_0$. But g is constant on every $J \in \mathcal{J}$. Thus, $g[J_0]$ is a singleton, implying that $g[h[A]] \in [\mathbb{R}]^1 \subseteq \mathcal{B}$.

To see that $f \notin \mathcal{C}(X,\mathbb{R})$ let $V = h^{-1}(S)$ and $x \in \mathrm{cl}(V) \setminus V$, existing by (2). Then $h(x) \in h[\mathrm{cl}(V)] \subseteq \mathrm{cl}(h[V]) \subset \mathrm{cl}(S)$, while $x \notin V = h^{-1}(S)$, i.e., $h(x) \in \mathrm{cl}(S) \setminus S$. So,

$$c = g(h(x)) = f(x) \in f[\mathrm{cl}(V)]$$

while $c \notin \{a\} = \mathrm{cl}(g[S]) = \mathrm{cl}(g[h[V]]) = \mathrm{cl}(f[V])$, proving that f is discontinuous.

To prove the "moreover" part, take a non-measurable set $E \subseteq \mathrm{cl}(S)$, fix distinct $a, a', c, c' \in B$ and redefine $g : \mathbb{R} \to B$ by

$$g(x) = \begin{cases} b_\xi & \text{if } x \in J_\xi \text{ for some } \xi < |B|, \\ a & \text{if } x \in S \cap E, \\ a' & \text{if } x \in S \setminus E, \\ c & \text{if } x \in E \setminus S, \\ c' & \text{otherwise.} \end{cases}$$

Then $g^{-1}(\{a,c\}) = E$ is non-measurable, so g is not measurable. It is easy to see that for this modification of our original g we still have $f = g \circ h \in \mathcal{C}_{\mathcal{A},\mathcal{B}} \setminus \mathcal{C}(X,\mathbb{R})$. ∎

Proof of Proposition 3.7. By way of contradiction assume that there exist $\mathcal{A} \subseteq \mathcal{P}(X) \setminus \{\emptyset\}$ and $\mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ such that $\mathcal{C}(X,\mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

Note that, by (2), X is not discrete. So, by Fact 1.3, $\mathcal{B}$ contains all singletons and does not contain any doubleton. Moreover, we can assume that

$$\mathcal{B} = \mathcal{A}^* = \{f[A] : A \in \mathcal{A} \ \&\ f \in \mathcal{C}(X,\mathbb{R})\}.$$

Next notice that

(7) $\mathrm{cl}(f[A])$ is an interval for every $A \in \mathcal{A}$ and $f \in \mathcal{C}(X,\mathbb{R})$.

Indeed, otherwise $\mathrm{cl}(f[A])$ is disconnected, so there are two disjoint nonempty closed subsets F_0 and F_1 of $\mathrm{cl}(f[A])$. Then, by normality of $\mathbb{R}$, there exists a continuous function $g : \mathbb{R} \to [0,1]$ with $g[F_0] = \{0\}$ and $g[F_1] = \{1\}$.

Therefore $\bar{f} = g \circ f \in \mathcal{C}(X, \mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and $\{0,1\} = \bar{f}[A] \in \mathcal{B}$, contradicting $\mathcal{B} \cap [\mathbb{R}]^2 = \emptyset$.

Now, $\mathcal{B} \not\subseteq [\mathbb{R}]^1$, since $h \in \mathcal{C}(X, \mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ is not constant. Hence, by (7),

(8) $\mathcal{B}$ contains an infinite set.

Next note that

(9) $\mathcal{B}$ does not contain any infinite countable set.

We apply Lemma 3.9 to show this. So, by way of contradiction assume that there exists a countable infinite $B \in \mathcal{B}$. Note that (7) implies (3). Let $\mathcal{J}$ be a family of non-trivial pairwise disjoint closed subintervals of $\mathbb{R} \setminus \mathrm{cl}(S)$ with the property that between any two distinct intervals from $\mathcal{J}$ there is another interval $J \in \mathcal{J}$, and $\bigcup \mathcal{J}$ is dense in $\mathbb{R}$. It is easy to see that such a $\mathcal{J}$ satisfies (4) and (5). So, Lemma 3.9 leads to a contradiction with our assumption that $\mathcal{C}(X, \mathbb{R}) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

Next note that for every $A \in \mathcal{A}$,

(10) $h[A] \cap P \neq \emptyset$ for every perfect set $P \subset \mathrm{cl}(h[A])$.

Indeed, otherwise there is a continuous "Cantor-like" function g from $\mathbb{R}$ onto $[0,1]$ with $g[\mathrm{cl}(h[A]) \setminus P]$ being countable infinite. Now $g \circ h : X \to \mathbb{R}$ is continuous and $(g \circ h)[A] \subseteq g[\mathrm{cl}(h[A]) \setminus P]$ is infinite countable, contradicting (9).

To finish the proof, take an arbitrary infinite $B \in \mathcal{B}$, which exists by (8), and let $\mathcal{J}$ be a family of pairwise disjoint perfect subsets of $\mathbb{R} \setminus \mathrm{cl}(S)$ such that continuum many of them lie inside any non-degenerate subinterval of $\mathbb{R}$. Then conditions (3)–(5) of Lemma 3.9 are satisfied, implying that $\mathcal{C}(X, \mathbb{R}) \neq \mathcal{C}_{\mathcal{A},\mathcal{B}}$. ∎

4. Families of real functions. Notice that there are non-trivial classes of real functions that are equal to $\mathcal{C}_{\mathcal{A},\mathcal{A}}$ for some $\mathcal{A} \subseteq \mathcal{P}(\mathbb{R})$. For example the class $\mathcal{D}$ of all Darboux functions is defined as the class of functions for which the images of connected sets are connected. Thus, $\mathcal{D} = \mathcal{C}_{\mathcal{A},\mathcal{A}}$, where $\mathcal{A}$ is the family of all connected subsets of $\mathbb{R}$.

The next theorem is a generalization of Theorem 3.1 in the case $X = \mathbb{R}$ and it implies that many classes of real functions cannot be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$.

THEOREM 4.1. *If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$ are such that $\mathcal{C}(\mathbb{R}, \mathbb{R}) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ then there is a non-measurable function $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.*

Proof. The proof is very similar to that of Theorem 3.1. We will use here the identity function id as an h for which any non-closed nowhere dense $S \subseteq \mathbb{R}$ will satisfy (2). We will choose such an S with $\mathrm{cl}(S)$ having positive Lebesgue measure.

Take $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ such that $\mathcal{C}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$. By Fact 1.2(ii) the family $\mathcal{B}$ contains all singletons. Also, by Fact 1.2(iv), if $\mathcal{B}$ contains a doubleton B then $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ contains the characteristic function $h : \mathbb{R} \to B$ of a non-measurable set, i.e., a non-measurable function. So, without loss of generality we can assume that $\mathcal{B}$ does not contain any doubleton. By Fact 1.2(i) we can also assume that

$$\mathcal{B} = \mathcal{A}^* = \{f[A] : A \in \mathcal{A} \ \& \ f \in \mathcal{C}(\mathbb{R},\mathbb{R})\}.$$

Next note that

$$\text{cl}(f[A]) \text{ is an interval for every } A \in \mathcal{A} \text{ and } f \in \mathcal{C}(\mathbb{R},\mathbb{R}), \tag{11}$$

the argument being identical to that for the condition (7) of Theorem 3.1.

Now, $\mathcal{B} \not\subseteq [\mathbb{R}]^1$, since $\text{id} \in \mathcal{C}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Hence, by (11),

$$\mathcal{B} \text{ contains an infinite set.} \tag{12}$$

If $\mathcal{B}$ contains a countable infinite set B then we can apply Lemma 3.9 to the family $\mathcal{J}$ of intervals used to prove the condition (9) of Theorem 3.1, and conclude that there exists a non-measurable function in $\mathcal{C}_{\mathcal{A},\mathcal{B}}$. So assume that $\mathcal{B}$ does not contain a countable infinite subset. Then, as in the case of the proof of condition (10) of Theorem 3.1, we see that

$$A \cap P \neq \emptyset \quad \text{for every } A \in \mathcal{A} \text{ and every perfect set } P \subset \text{cl}(A).$$

To finish the proof, it is enough to apply Lemma 3.9 to the family $\mathcal{J}$ of pairwise disjoint perfect subsets of $\mathbb{R} \setminus \text{cl}(S)$ such that continuum many of them lie inside any non-degenerate subinterval of $\mathbb{R}$. ∎

COROLLARY 4.2. *Neither of the following classes of functions from $\mathbb{R}$ to $\mathbb{R}$ can be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ for any $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(\mathbb{R})$:*

- *the class of upper or lower semicontinuous functions*;
- *the class of derivatives*;
- *the class of approximately continuous functions*;
- *the class of Baire class* 1 *functions*;
- *the class of Borel functions*;
- *the class of measurable functions.*

Proof. If $\mathcal{F}$ is any of the above classes then $\mathcal{C}(\mathbb{R},\mathbb{R}) \subseteq \mathcal{F}$ and every function in $\mathcal{F}$ is measurable. ∎

PROBLEM 4.3. Can the class of all functions $f : \mathbb{R} \to \mathbb{R}$ with the Baire property be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$?

As far as smaller classes of functions are concerned we have the following questions.

PROBLEM 4.4. Can any of the following classes of real functions be represented as $\mathcal{C}_{\mathcal{A},\mathcal{B}}$?

- The class of all linear functions $f(x) = ax + b$.
- The class of all polynomials.
- The class of all real-analytic functions.
- The class C^∞ of infinitely many times differentiable functions.
- The class D^n of n-times differentiable functions, with $1 \le n < \omega$.

5. Further remarks and examples

5.1. *Second reduction theorem.* The next theorem can be considered as a generalization of Theorem 2.1.

THEOREM 5.1. *Let $X = \bigcup_{\alpha\in I} C_\alpha$ and $Y = \bigcup_{\gamma\in J} K_\gamma$ be the partitions of the topological spaces X and Y into connected components. Then $\langle X, Y\rangle$ has the V-property if and only if*

(A) *each C_α is clopen in X; and*

(B) *for every $\alpha \in I$ and $\gamma \in J$ there exist families $\mathcal{A}_\alpha \subseteq \mathcal{P}(C_\alpha)$ and $\mathcal{B}_\gamma \subseteq \mathcal{P}(K_\gamma)$ with the property that*

$$\mathcal{C}(C_\alpha, K_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma} \quad \textit{for every } \alpha \in I \textit{ and } \gamma \in J.$$

Proof. Assume first that $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$ with $\mathcal{B} = \mathcal{A}^*$. Then condition (A) follows from Lemma 2.2.

To see (B) define $\mathcal{A}_\alpha = \mathcal{A} \cap \mathcal{P}(C_\alpha)$ and $\mathcal{B}_\gamma = \mathcal{B} \cap \mathcal{P}(K_\gamma)$. First notice that $\mathcal{A} = \bigcup_{\alpha\in I} \mathcal{A}_\alpha$ follows from Fact 1.3(iv). Also $\mathcal{B} = \bigcup_{\gamma\in J} \mathcal{B}_\gamma$ since continuous functions send connected sets to connected sets. In order to prove that $\mathcal{C}(C_\alpha, K_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma}$ take a continuous map $f : C_\alpha \to K_\gamma$. Extend f to a continuous map $\widetilde{f} : X \to Y$ by choosing an arbitrary point $b \in Y$ and assigning value b to any $x \in X \setminus C_\alpha$. Then $\widetilde{f} \in \mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$. Hence for every $A \in \mathcal{A}_\alpha$ we have $f[A] = \widetilde{f}[A] \in \mathcal{B}$ and $f[A] \in \mathcal{B}_\gamma$ as $f[A] \subseteq K_\gamma$. Thus $f \in \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma}$. The proof of the other inclusion is similar to that for Theorem 2.1.

To prove the other implication first notice that it is true for Y being discrete since we can take $\mathcal{A} = \{C_\alpha : \alpha \in I\}$ and $\mathcal{B} = [Y]^1$. Thus we assume that Y is not discrete.

Define $\mathcal{A} = \bigcup_{\alpha\in I} \mathcal{A}_\alpha$ and $\mathcal{B} = \bigcup_{\gamma\in J} \mathcal{B}_\gamma$. We now prove that $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

First note that each function $f : X \to Y$ which is either continuous or in $\mathcal{C}_{\mathcal{A},\mathcal{B}}$ defines a map $\theta : I \to J$ such that

$$f[C_\alpha] \subseteq K_{\theta(\alpha)}. \tag{13}$$

For continuous f this is obvious. So, let $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and fix $\alpha \in I$ and $x \in C_\alpha$. Let $\mathrm{St}^\omega(x, \mathcal{A}) = \bigcup_n \mathrm{St}^n(x, \mathcal{A})$, where $\mathrm{St}^n(x, \mathcal{A})$ denotes the nth iterated star of the point x with respect to the cover $\mathcal{A}$ of X. (See (1).) It is easy to see that $\mathrm{St}^\omega(x, \mathcal{A}) \subseteq C_\alpha$, and $f[\mathrm{St}^\omega(x, \mathcal{A})]$ is a subset of precisely one K_γ. Thus, it is enough to show that $\mathrm{St}^\omega(x,\mathcal{A}) = C_\alpha$. To this end take a component K_γ of Y with more than one point and consider the characteristic function $f : C_\alpha \to K_\gamma$ of $\mathrm{St}^\omega(x,\mathcal{A})$. It belongs to $\mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma} = \mathcal{C}(C_\alpha, K_\gamma)$, so f is continuous. Hence $\mathrm{St}^\omega(x,\mathcal{A})$ is clopen in C_α. As C_α is connected we conclude $\mathrm{St}^\omega(x,\mathcal{A}) = C_\alpha$.

Now to prove $\mathcal{C}(X,Y)\subseteq\mathcal{C}_{\mathcal{A},\mathcal{B}}$ notice that if $f : X\to Y$ is continuous and θ is as in (13) then $f[A]\in\mathcal{B}_{\theta(\alpha)}$ for every $\alpha\in I$ and $A\in\mathcal{A}_\alpha$. So, $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

To see the other inclusion let $f \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$ and let θ be as in (13). Since the components of X are clopen, it suffices to prove that each restriction $f_\alpha = f|_{C_\alpha}$ is continuous. By the formula (13) we can factorize f_α as the composition of $g_\alpha : C_\alpha \to K_{\theta(\alpha)}$ and the inclusion $K_{\theta(\alpha)} \hookrightarrow Y$, so that the continuity of f_α follows from the continuity of $g_\alpha \in \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_{\theta(\alpha)}} = \mathcal{C}(C_\alpha, K_{\theta(\alpha)})$. ∎

COROLLARY 5.2. *Let $X = \bigoplus_\alpha X_\alpha$ be the topological direct sum of the spaces X_α. Then $\langle X, Y\rangle$ has the V-property if and only if all pairs $\langle X_\alpha, Y\rangle$ have the V-property witnessed by the same $\mathcal{B} \subseteq \mathcal{P}(Y)$.* ∎

COROLLARY 5.3. *Let $X = \bigcup_{\alpha\in I} C_\alpha$ be the partition of X into connected components. Then X is a V-space if and only if*

(A) *each C_α is clopen in X; and*

(B) *for each $\alpha \in I$ there exists a family $\mathcal{A}_\alpha \subseteq \mathcal{P}(C_\alpha)$ such that $\mathcal{C}(C_\alpha, C_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{A}_\gamma}$ for every $\alpha, \gamma \in I$.*

Proof. From the formulation of Theorem 5.1 it follows immediately that for each $\alpha \in I$ there exist families $\mathcal{A}_\alpha, \mathcal{B}_\alpha \subseteq \mathcal{P}(C_\alpha)$ such that $\mathcal{C}(C_\alpha, C_\gamma) = \mathcal{C}_{\mathcal{A}_\alpha,\mathcal{B}_\gamma}$ for every $\alpha, \gamma \in I$. To see that the families $\mathcal{A}_\alpha$ and B_α can be chosen equal it is enough to notice that for a V-space X we can choose $\mathcal{B} = \mathcal{A}$, and then check the definition of $\mathcal{A}_\alpha$ and $\mathcal{B}_\gamma$ in the proof of Theorem 5.1. ∎

COROLLARY 5.4. *Let D be a discrete space. Then $\langle X, D\rangle$ has the V-property if and only if each connected component of X is clopen in X.* ∎

COROLLARY 5.5. *Let D be a discrete space. Then $X \times D$ is a V-space if and only if X is a V-space.*

Proof. The product $X \times D$ is a topological direct sum of $|D|$-many copies of the space X. ∎

COROLLARY 5.6. *If K is a Cook's continuum and D is a discrete space then $X \times D$ is a V-space.* ∎

A family (possibly a proper class) $\{X_\alpha\}_\alpha$ of spaces is *strongly rigid* if the only non-constant maps $X_\alpha \to X_\beta$ are the identities $X_\alpha \to X_\alpha$. A space X is *strongly rigid* if the family $\{X\}$ is strongly rigid. (See [1], [6], [7] for the existence of strongly rigid spaces and families.) Obviously every strongly rigid pair $\{X, Y\}$ of distinct spaces gives rise to two pairs $\langle X, Y\rangle$ and $\langle Y, X\rangle$ having the V-property.

Proposition 5.7. *Let $\{C_\alpha\}_{\alpha\in I}$ be a strongly rigid family of continua. Then the topological direct sum $X=\bigoplus_{\alpha\in I} C_\alpha$ is a V-space.*

Proof. For every $\alpha\in I$ let $\mathcal{A}_\alpha$ be the family of closed subsets of C_α which are not doubletons. Set $\mathcal{A}=\bigcup_{\alpha\in I}\mathcal{A}_\alpha$. We prove that $\mathcal{C}(X,X)=\mathcal{C}_{\mathcal{A},\mathcal{A}}$ by using Corollary 5.3. To this end we must check that $\mathcal{C}(C_\alpha,C_\beta)=\mathcal{C}_{\mathcal{A}_\alpha,\mathcal{A}_\beta}$ for every $\alpha,\beta\in I$.

The case $\alpha=\beta$ was already established in Proposition 1.6. So, assume that $\alpha\neq\beta$. Then $\mathcal{C}(C_\alpha,C_\beta)$ has only constant maps. Suppose $f\in\mathcal{C}_{\mathcal{A}_\alpha,\mathcal{A}_\beta}$ is non-constant. By the choice of $\mathcal{A}_\alpha$ and $\mathcal{A}_\beta$ the map f is injective. Since $C_\alpha\in\mathcal{A}_\alpha$, it follows that $Z=f[C_\alpha]$ is a closed, hence compact, subset of C_β. Moreover, every closed subset of C_α is mapped onto a closed subset of Z. Therefore $f:C_\alpha\to C_\beta$ is a non-constant continuous map, a contradiction. ∎

Example 5.8. (I) In analogy with our main result in Section 3 we discuss here when the pair $\langle X,S\rangle$ has the V-property, where S denotes the Sierpiński dyad. It is easy to see (using Fact 1.2) that for a T_0-space X the pair $\langle X,S\rangle$ has the V-property if and only if X is discrete. Further, using this fact and Proposition 1.8 one can conclude that for T_0-spaces X and Y with $\mathcal{C}(X,Y)\neq Y^X$ (i.e., Y is not indiscrete and X is not discrete) the pair $\langle X,Y\rangle$ may have the V-property only if Y is T_1. Consequently, a finite T_0-space is a V-space if and only if it is discrete.

(II) Now we give examples of V-spaces of arbitrary infinite cardinality which need not be locally compact. (Note that all examples given above were locally compact.) These are non-Hausdorff T_1-spaces. Let X be a set and $\alpha\le|X|$ be a regular cardinal. Consider the co-α topology τ_α on X (having as closed sets: X and all subsets $Y\subseteq X$ with $|Y|<\alpha$). It is easy to see that $f\in\mathcal{C}(X,X)\setminus\mathrm{Const}$ if and only if f has *small fibers* (i.e., $|f^{-1}(x)|<\alpha$ for every $x\in X$). Now with $\mathcal{A}=[X]^1\cup[X]^{\ge\alpha}$ we have $\mathcal{C}(X,X)=\mathcal{C}_{\mathcal{A},\mathcal{A}}$, so that X is a V-space. Note that X is always connected, while τ_α is (locally) compact precisely for $\alpha=\omega$.

5.2. *Behavior under products*. Next we examine when the V-property of a pair $\langle X,Y\rangle$ is preserved under product operations.

Now we prove the counterpart of Corollary 5.2 in the case of products.

Proposition 5.9. *Let X be a space, let $\{Y_\alpha\}_{\alpha\in I}$ be a family of spaces and let $Y=\prod_{\alpha\in I}Y_\alpha$. Then $\langle X,Y\rangle$ has the V-property if and only if all pairs $\langle X,Y_\alpha\rangle$ have the V-property witnessed by the same family $\mathcal{A}\subseteq\mathcal{P}(X)$.*

Proof. The necessity follows from Proposition 1.8. Now assume that all pairs $\langle X,Y_\alpha\rangle$ have the V-property witnessed by the same family $\mathcal{A}\subseteq\mathcal{P}(X)$. According to Fact 1.3(vi),

$$\mathcal{C}(X,Y_\alpha)=\mathcal{C}_{\mathcal{A},\mathcal{B}_\alpha}\quad\text{ for all }\alpha\in I,$$

where $\mathcal{B}_\alpha = \{f_\alpha[A] : A \in \mathcal{A} \ \& \ f_\alpha \in \mathcal{C}(X, Y_\alpha)\}$. For a family $\{f_\alpha : X \to Y_\alpha\}_{\alpha \in I}$ of functions, we denote by $\langle f_\alpha \rangle$ the diagonal map $X \to Y$. We will use the fact that every continuous function $f : X \to Y$ has the form $f = \langle f_\alpha \rangle$, where each $f_\alpha : X \to Y_\alpha$ is continuous. Let $\mathcal{B} = \{\langle f_\alpha \rangle[A] : A \in \mathcal{A} \ \& \ \langle f_\alpha \rangle \in \mathcal{C}(X, Y)\}$. We now show that $\mathcal{C}(X, Y) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$.

So, let $f = \langle f_\alpha \rangle \in \mathcal{C}_{\mathcal{A},\mathcal{B}}$. To prove that $f \in \mathcal{C}(X, Y)$ it is enough to show that $f_\alpha \in \mathcal{C}_{\mathcal{A},\mathcal{B}_\alpha} = \mathcal{C}(X, Y_\alpha)$ for every $\alpha \in I$. So, take $A \in \mathcal{A}$. Then $f[A] \in \mathcal{B}$, i.e., $f[A] = g[A']$ for some $g \in \mathcal{C}(X, Y)$ and $A' \in \mathcal{A}$. Applying the canonical projection $p_\alpha : Y \to Y_\alpha$ to both sides of this equality we get $f_\alpha[A] = g_\alpha[A'] \in \mathcal{B}_\alpha$. So, $f[A] \in \mathcal{C}_{\mathcal{A},\mathcal{B}_\alpha}$.

The inclusion $\mathcal{C}(X, Y) \subseteq \mathcal{C}_{\mathcal{A},\mathcal{B}}$ is a trivial consequence of the definition of $\mathcal{B}$. ∎

COROLLARY 5.10. *Let $\{Y_\alpha\}_{\alpha \in I}$ be a family of spaces. Then $Y = \prod_{\alpha \in I} Y_\alpha$ is a V-space if and only if all pairs $\langle Y, Y_\alpha \rangle$ have the V-property witnessed by the same family $\mathcal{A} \subseteq \mathcal{P}(Y)$.* ∎

In particular, according to Corollary 1.12 every Y_α is a V-space when $\prod_{\alpha \in I} Y_\alpha$ is a V-space.

COROLLARY 5.11. *Let X be a topological space and let α be a cardinal.*

(i) *X is a V-space if and only if $\langle X, X^\alpha \rangle$ has the V-property.*
(ii) *$\langle X^\alpha, X \rangle$ has the V-property if and only if X^α is a V-space.* ∎

Note that by Corollary 1.12 if X^α is a V-space then X is a V-space.

COROLLARY 5.12. *Let K be a Cook continuum and let $n > 0$ be a natural number. Then K^n is a V-space.*

Proof. According to the above corollary it suffices to check that $\langle K^n, K \rangle$ has the V-property.

Let $p_k : K^n \to K$, $1 \le k \le n$, denote the kth projection. We prove by induction on n the following claim:

(I) every non-constant continuous map $f : K^n \to K$ coincides with some projection p_k.

The case $n = 1$ is trivial. Assume that $n > 1$ and that the statement is true for $n - 1$. Fix $a \in K^{n-1}$ and consider a continuous function $f : K^n = K \times K^{n-1} \to K$. Then the function $h_a : K \to X$ defined by $h_a(x) = f(x, a)$ is continuous. Hence, either $h_a = \mathrm{id}_K$, or $h_a \in \mathrm{Const}$. Let $g(a) \in K$ be the value of that constant function in the second case. Put $F = \{a \in K^{n-1} : h_a \equiv g(a)\}$ and $G = \{a \in K^{n-1} : h_a = \mathrm{id}_K\}$. These are disjoint closed subsets of K^{n-1} with $K^{n-1} = F \cup G$. By the connectedness of K^{n-1} we have either $F = K^{n-1}$, or $K^{n-1} = G$.

In the first case we have $h_a \equiv g(a)$ for all $a \in K^{n-1}$. The function $g : K^{n-1} \to K$ obtained in this way is continuous. So, by our inductive hypothesis, g is a projection. (Note that g cannot be constant since f is non-constant and each h_a is constant.) In the second case $h_a = \mathrm{id}_K$ for every a, hence $f = p_1$ is again a projection. This proves our claim.

For a non-empty subset $D \subseteq F = \{1,\ldots,n\}$ denote by $\Delta_D : K \to K^D$ the diagonal map defined by $\Delta_D(x) = \langle x,\ldots,x\rangle \in K^D$. Then it is easy to see that for every continuous map $\varphi : K \to K^n$, $\varphi \neq \Delta_F$, there exists a subset $D \subset F = \{1,\ldots,n\}$ and an element $a \in K^{F\setminus D}$ such that $\varphi : K \to K^n = K^D \times K^{F\setminus D}$ coincides with the map $\langle \Delta_D, g_a\rangle$, where $g_a \in \mathrm{Const}$ is the constant map with value a. Since φ is completely determined by the pair $\langle D, a\rangle \in \mathcal{P}(F) \times K^{F\setminus D}$, we denote this map by $\varphi_{D,a}$.

Now fix $\mathcal{A}$ to be the family of all closed subsets of K which are not doubletons. It follows from the proof of Proposition 1.6 that $\mathcal{C}_{\mathcal{A},\mathcal{A}} = \mathcal{C}_{K,K} = \mathrm{Const} \cup \{\mathrm{id}_K\}$. Set $\mathcal{B} = \{\varphi[A] : \varphi \in \mathcal{C}(K,K^n)\ \&\ A \in \mathcal{A}\}$.

We show that $\mathcal{C}(K^n,K) = \mathcal{C}_{\mathcal{B},\mathcal{A}}$. The inclusion $\mathcal{C}(K^n,K) \subseteq \mathcal{C}_{\mathcal{B},\mathcal{A}}$ is obvious. Assume $f \in C_{\mathcal{B},\mathcal{A}}$. Note that for every $a \in K$ the composition

$$h_a = f \circ \varphi_{\{1,\ldots,n-1\},a} \tag{14}$$

belongs to $\mathcal{C}_{\mathcal{A},\mathcal{A}}$, hence

$$h_a \in \mathrm{Const} \quad \text{or} \quad h_a = \mathrm{id}_K \tag{15}$$

by virtue of the equation $\mathcal{C}_{\mathcal{A},\mathcal{A}} = \mathcal{C}_{K,K}$ and (I). Consider the restriction $d_n = f \circ \Delta_F$ of f to the diagonal of K^n, i.e., $d_n(x) = f(x,\ldots,x)$. The proof of the corollary follows immediately from the next claim which we prove by induction on n.

Claim. (1_n) *If* $d_n \in \mathrm{Const}$ *then* $f \in \mathrm{Const}$.
(2_n) *If* $d_n = \mathrm{id}_K$ *then* $f = p_i$ *for some* $i \in \{1,\ldots,n\}$.

Proof. The case $n = 1$ trivially follows from the equalities $\Delta_F = \mathrm{id}_K$ and $d_n = f$, which are valid for $n = 1$. Assume that $n > 1$ and that the claim is true for $n - 1$.

Case 1: Let $d_n(x) = b \in K$ for every $x \in K$. Fix an arbitrary $a \in K\setminus\{b\}$ and consider h_a as in (14). Then $h_a(a) \neq \mathrm{id}_K$ since $h_a(a) = b \neq a$. Now (15) yields $h_a \in \mathrm{Const}$. Consider the function $f_a : K^{n-1} \to K$ defined by

$$f_a(x_1,\ldots,x_{n-1}) = f(x_1,\ldots,x_{n-1},a). \tag{16}$$

Then $f_a \circ \Delta_{\{1,\ldots,n-1\}} = h_a \in \mathrm{Const}$, so that the inductive hypothesis (1_{n-1}) holds for f_a. Hence $f(x_1,\ldots,x_{n-1},a) = b$ for every $\langle x_1,\ldots,x_{n-1}\rangle \in K^{n-1}$ and $a \in K\setminus\{b\}$. Assume $f \notin \mathrm{Const}$. Then there exists $\langle c_1,\ldots,c_{n-1}\rangle \in K^{n-1}$ such that $f(c_1,\ldots,c_{n-1},b) \neq b$. Now $B = \{\langle c_1,\ldots,c_{n-1}\rangle\} \times K \in \mathcal{B}$ and $|f[B]| = 2$, so that $f[B] \notin \mathcal{A}$, a contradiction. This proves that $f \in \mathrm{Const}$.

Case 2: Let $d_n = \mathrm{id}_K$. For $a \in K$ consider the functions $h_a : K \to K$ as in (14). According to (15) we have two cases.

CASE 2.1: There exists $a \in K$ such that $h_a \in \mathrm{Const}$. From $h_a(a) = d_n(a) = \mathrm{id}_K(a) = a$ we get $h_a(x) = a$ for every $x \in K$. For the function f_a defined as in (16) we have $f_a \circ \Delta_{\{1,\ldots,n-1\}} = h_a \in \mathrm{Const}$, so that the inductive hypothesis (1_{n-1}) holds for f_a. Hence $f_a \in \mathrm{Const}$. This yields $f = p_n$.

CASE 2.2: $h_a = \mathrm{id}_K$ for all $a \in K$. Now for every $a \in K$ the function f_a defined as in (16) satisfies the inductive hypothesis (2_{n-1}), hence there exists $i_a \in \{1, \ldots, n-1\}$ such that $f_a = p_{i_a}$. The proof will be finished if we show that the function $K \to \{1, \ldots, n-1\}$ defined by $a \mapsto i_a$ is constant. Assume the contrary. Then $i_a \neq i_{a'}$ for some $a \neq a'$ from K. Fix $\langle x_1, \ldots, x_{n-1}\rangle \in K^{n-1}$ such that $x_{i_a} \neq x_{i_{a'}}$ and $x_k \in \{x_{i_a}, x_{i_{a'}}\}$ for $k \in \{1, \ldots, n-1\}$. (This is possible since our assumption entails $n > 2$.) Then for the set $B = \{\langle x_1, \ldots, x_{n-1}\rangle\} \times K \in \mathcal{B}$ we have $|f[B]| = 2$, so that $f[B] \notin \mathcal{A}$, a contradiction. ■

We do not know if this result can be extended to all V-spaces:

PROBLEM 5.13. Are finite powers of V-spaces again V-spaces?

In particular, we do not know whether finite powers of the V-spaces defined in Example 5.8(II) are V-spaces. On the other hand, note that infinite powers of a V-space need not be V-spaces. (For example, take any finite discrete non-singleton space.)

In analogy with Proposition 5.7, one could try to extend the validity of Corollary 5.12 to the product of any (finite) strongly rigid family of continua. We offer a partial result here.

PROPOSITION 5.14. *Let $\{X_\alpha\}_{\alpha\in I}$ be a strongly rigid family of continua. Then all pairs $\langle \prod_{\beta\in I} X_\beta, X_\alpha\rangle$ have the V-property.*

Proof. Let $X = \prod_{\beta\in I} X_\beta$. We prove first that $\mathcal{C}(X, X_\alpha) = \mathrm{Const} \cup \{p_\alpha\}$ where $p_\alpha : X \to X_\alpha$ is the canonical projection for $\alpha \in I$.

Fix $\alpha \in I$ and let $X' = \prod\{X_\beta : \beta \in I,\ \beta \neq \alpha\}$. We identify X with $X_\alpha \times X'$.

We show first that $\mathcal{C}(X', X_\alpha) = \mathrm{Const}$. Fix $y = \langle y_\beta\rangle \in X'$ and let

$$X'' = \{x = \langle x_\beta\rangle \in X' : x_\beta \neq y_\beta \text{ for only finitely many } \beta \in I\}.$$

Now fix $f \in \mathcal{C}(X', X_\alpha)$ and set $b = f(y)$. It is easy to see that f takes constant value b on X''. (For $x = \langle x_\beta\rangle \in X''$ argue by induction on the number of $\beta \in I$ with $x_\beta \neq y_\beta$.) Since X_α is Hausdorff and X'' is dense in X' we conclude that f is constant on X'.

Now take $f \in \mathcal{C}(X, X_\alpha)$ and for every $x \in X_\alpha$ consider the restriction of f on $Z = \{x\} \times X'$. By the above claim f has a constant value $\widetilde{f}(x) \in X_\alpha$ on Z. The mapping $x \mapsto \widetilde{f}(x)$ is a continuous function from X_α into X_α.

Hence it is either constant or the identity. Thus f is either constant or equal to p_α.

Let $\mathcal{B}$ be the family of all closed subsets of X_α that are not doubletons and let $\mathcal{A} = \{B \times D \subseteq X : B \in \mathcal{B}\ \&\ D \in [X']^2\}$. Then $\mathcal{C}(X, X_\alpha) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$. ∎

We do not know if it is possible to find a single $\mathcal{A}$ witnessing the property V for all pairs $\langle \prod_{\beta\in I} X_\beta, X_\alpha\rangle$ simultaneously. If this were true, then applying Corollary 5.10 we could conclude that $X = \prod_{\alpha\in I} X_\alpha$ is a V-space.

The above results also leave open the following question regarding subspaces of products. Putting the comment following Corollary 5.11 in negative form we get: *if X is not a V-space, then none of the powers X^α is a V-space*. Hence, by Corollary 5.11, the pair $\langle X^\alpha, X\rangle$ does not have the V-property.

PROBLEM 5.15. Suppose X is not a V-space. Is it true that no pair $\langle Y, X\rangle$ has the V-property where Y is a non-discrete subspace of X^α for some α?

This is true for X equal to $\mathbb{R}$, the Sierpiński dyad S, and the discrete doubleton $\{0, 1\}$. Actually, in these cases the V-property fails for all pairs $\langle Y, X\rangle$ where Y belongs to the larger class $\mathbf{S}(X)$ of spaces that admit a continuous injection into a power of X. (See Corollary 3.5, Example 5.8(I), and Fact 1.3(v). Note that $\mathbf{S}(\mathbb{R})$ are the functionally Hausdorff spaces, $\mathbf{S}(S)$ are the T_0-spaces and $\mathbf{S}(\{0, 1\})$ are the totally disconnected spaces.) We propose the question also in its stronger form:

PROBLEM 5.16. Suppose X is not a V-space. Is it true that for a space $Y \in \mathbf{S}(X)$ the pair $\langle Y, X\rangle$ has the V-property if and only if Y is discrete?

In the semigroup $\mathcal{C}(X, X)$ the largest subgroup $\mathcal{H}(X)$ of all autohomeomorphisms of X has as its smallest natural extension the subsemigroup $\mathcal{H}(X) \cup \mathrm{Const}$. Most of the examples of Hausdorff connected V-spaces we have seen till this point have the property $\mathcal{C}(X, X) = \mathcal{H}(X) \cup \mathrm{Const}$. This suggests the question: does there exist a Hausdorff connected V-space X such that $\mathcal{C}(X, X)$ has non-constant non-injective maps? The powers of Cook's continuum have this property by Corollary 5.12. Here is another example of a V-space with this property.

EXAMPLE 5.17. Let K be a strongly rigid continuum and $a \in K$. Then a is not a cut point of C [6, Theorem 2.2.1]. Let $X = K \vee_a K$ be the adjunction space obtained by gluing two copies of K along the set $\{a\}$. Let $j_i : K \hookrightarrow X$, $i = 1, 2$, be the canonical embeddings of K into X. Then every point of X has the form $j_i(x)$ for some $x \in K$ and $i = 1, 2$. The canonical projection $p : X \to K$ is defined by $p \circ j_1 = p \circ j_2 = \mathrm{id}_K$. The symmetry $s : X \to X$ is defined by $s \circ j_1 = j_2$ and $s \circ j_2 = j_1$. We also have the map $h_1 : X \to X$ with $h_1 \circ j_1 = \mathrm{id}_K$ and $h_1 \circ j_2 : K \to K$ the constant function with value a. The map h_2 is defined analogously. It is easy to see that $\mathcal{C}(X, X) = \mathrm{Const} \cup \{1_X, s, h_1, h_2\}$.

Let $\mathcal{A}$ be the family of closed subsets of K which are not doubletons and $\widetilde{\mathcal{A}} = \{j_i[A] : A \in \mathcal{A}, i = 1,2\}$. Then $\mathcal{C}(X,X) = \mathcal{C}_{\tilde{\mathcal{A}},\tilde{\mathcal{A}}}$. The inclusion $\mathcal{C}(X,X) \subseteq \mathcal{C}_{\tilde{\mathcal{A}},\tilde{\mathcal{A}}}$ is obvious. If $f \in \mathcal{C}_{\tilde{\mathcal{A}},\tilde{\mathcal{A}}}$, then for $i = 1,2$ the restriction of f on $j_i[K]$ is continuous, so that f is continuous as well since $j_1[K]$ and $j_2[K]$ are closed in X.

An alternative proof that $X = K \vee_a K$ is a V-space is given in the following remark.

Remark 5.18. The above example hides several more general facts which we isolate now. For a space Y and a subspace M of Y the adjunction space $X = Y \vee_M Y$ is obtained as above by gluing two copies of Y along M. The maps $j_i : Y \hookrightarrow X$, $i = 1,2$, $s : X \to X$ and $p : X \to Y$ are defined as above. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is *symmetric* if $s(A) \in \mathcal{A}$ for every $A \in \mathcal{A}$.

(a) If $\langle Y, Z\rangle$ has the V-property, then also $\langle X, Z\rangle$ has the V-property witnessed by a symmetric family $\mathcal{A} \subseteq \mathcal{P}(X)$. In particular, if Y is a V-space, then $\langle X, Y\rangle$ has the V-property. (If $\mathcal{C}(Y,Z) = \mathcal{C}_{\mathcal{A},\mathcal{B}}$, then $\widetilde{\mathcal{A}}$ defined as in Example 5.17 is symmetric and $\mathcal{C}(X,Z) = \mathcal{C}_{\tilde{\mathcal{A}},\mathcal{B}}$.)

(b) If $\langle X, Z\rangle$ has the V-property, then it can be witnessed by a symmetric family $\mathcal{A} \subseteq \mathcal{P}(X)$. (Exploit the symmetry s of X.)

(c) If $\langle X, Z\rangle$ has the V-property witnessed by a symmetric family $\mathcal{A} \subseteq \mathcal{P}(X)$ then also $\langle Y, Z\rangle$ has the V-property. In particular, Y is a V-space if and only if $\langle X, Y\rangle$ has the V-property. (Note that Y can be considered as a retract of X via the embeddings j_i.)

(d) If Y is a strongly rigid V-space and M does not cut Y (i.e., $Y \setminus M$ is connected), then X is also a V-space. (It suffices to see that $\langle Y, X\rangle$ has the V-property. If $\mathcal{C}(Y,Y) = \mathcal{C}_{\mathcal{A},\mathcal{A}}$ define $\widetilde{\mathcal{A}}$ as before. To see that $\mathcal{C}(Y,X) \subseteq \mathcal{C}_{\mathcal{A},\tilde{\mathcal{A}}}$ it suffices to note that every $f \in \mathcal{C}(X,Z)$ factorizes either through j_1 or through j_2. For the inverse inclusion one has to prove first that $\mathcal{C}(Y,Y) = \mathcal{C}_{\mathcal{A},\mathcal{A}}$ yields that for the family $\mathcal{A}$ and every $x \in Y$, $Y = \mathrm{St}^{\omega}(x, \mathcal{A})$ as in the proof of Theorem 5.1. This forces the functions of $\mathcal{C}_{\mathcal{A},\tilde{\mathcal{A}}}$ to factorize through either j_1 or j_2.)

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Received 3 December 1997