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EXTENDING MONOTONE MAPPINGS

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All spaces are assumed to be Tychonoff. A *monotone* map is a closed continuous surjection with connected fibres. If A and B are subsets of a space X then A is called *locally connected rel* B if for every $a \in A$ and every neighbourhood U of a in X there is a neighbourhood V of a such that $V \subset U$ and $V \cap B$ is connected.

As far as extending monotone maps over compacta the following is known:

PROPOSITION 1. *If $f : X \to Y$ is monotone and C is a compactification of X such that f extends to a continuous $\widetilde{f} : C \to \beta Y$ then $\widetilde{f}$ is monotone.*

PROPOSITION 2. *If $f : X \to Y$ is monotone, D is a compactification of Y such that $D \setminus Y$ is locally connected rel Y, and C is a compactification of X such that f extends to a continuous $\widetilde{f} : C \to D$ then $\widetilde{f}$ is monotone.*

The first proposition is folklore (see Hart [3, Lemma 2.1]) and the second proposition can be found in Dijkstra [1]. The two propositions have the same conclusion but very dissimilar premises: for instance, if Y is metric then its Čech–Stone remainder is never locally connected rel Y. Our first theorem unifies these propositions.

In this paper we will discuss functions $f : X \to Y$ and $\widetilde{f} : C \to D$ such that X and Y are dense subsets of C and D respectively. Unless stated otherwise, if A is a subset of X or Y respectively, then $\overline{A}$ and $\operatorname{int}(A)$ refer to the closure and the interior of A in C or D respectively. Let I be the interval $[0, 1]$. A *zero set* A in a space Y is the preimage of 0 for some continuous $\alpha : Y \to I$. A *perfect* map is a closed continuous surjection with compact fibres.

THEOREM 3. *If D is a compactification of a space Y then the following statements are equivalent*:

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(1) *For every space* X, *every monotone map* $f : X \to Y$, *and every compactification* C *of* X *such that* f *extends to a continuous* $\widetilde{f} : C \to D$, *the map* $\widetilde{f}$ *is monotone.*

(2) *There are a space* X *and a monotone map* $\widetilde{f} : \beta X \to D$ *such that* $\widetilde{f}(X) \subset Y$.

(3) *For any pair of disjoint zero sets* A *and* B *in* Y *we have* $\overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B}) = \emptyset$.

(4) *For any pair of disjoint closed subsets* A *and* B *of* Y *we have* $\overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B}) = \emptyset$.

We obtain Theorem 3 as an immediate corollary of the following more general statement.

THEOREM 4. *If* Y *is a dense subspace of a space* D *then the following statements are equivalent*:

(1) *Let* X *be a dense subspace of a space* C *and let* $\widetilde{f} : C \to D$ *be a closed continuous map such that* $f = \widetilde{f}|X$ *is a monotone map from* X *onto* Y. *If* $\widetilde{f}$ *is perfect or if* C *is normal then* $\widetilde{f}$ *is monotone.*

(2) *There are a space* X, *a space* C *with* $X \subset C \subset \beta X$, *and a monotone map* $\widetilde{f} : C \to D$ *such that* $\widetilde{f}(X) \subset Y$.

(3) *For any pair of disjoint zero sets* A *and* B *in* Y *we have* $\overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B}) = \emptyset$.

(4) *For any pair of disjoint closed subsets* A *and* B *of* Y *we have* $\overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B}) = \emptyset$.

We need an elementary lemma:

LEMMA 5. *If* $f : C \to Y$ *is continuous and* X *is a dense subset of* C *such that* $f|X : X \to Y$ *is closed then for every* $y \in Y$ *we have* $f^{-1}(y) = \overline{f^{-1}(y) \cap X}$.

Proof. Let x be an element of C that is not in $\overline{f^{-1}(y) \cap X}$. To prove $x \notin f^{-1}(y)$, select a closed neighbourhood U of x that is disjoint from $\overline{f^{-1}(y) \cap X}$. Since $f|X$ is closed the set $V = Y \setminus f(U \cap X)$ is an open neighbourhood of y. Note that $f^{-1}(V) \cap \operatorname{int}(U)$ is an open set which is disjoint from X. Since X is dense, $f^{-1}(V)$ and $\operatorname{int}(U)$ are disjoint. Since $x \in \operatorname{int}(U)$ we have $f(x) \neq y$.

Proof of Theorem 4. Statement (2) follows trivially from (1). We shall prove: (2)⇒(3), (3)⇒(4), and (4)⇒(1).

Assume (2) and let A and B be disjoint zero sets in Y such that for some $y \in D$ we have $y \in \overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B})$. Then $y \in D \setminus Y$ and $\widetilde{f}^{-1}(y)$ is a connected subset of $C \setminus X$. If $W = \operatorname{int}(\overline{A \cup B})$ then $\widetilde{f}^{-1}(W) \setminus \overline{\widetilde{f}^{-1}(A \cup B)}$ is an open subset of C that is disjoint from X. Since X is dense in C we have $\widetilde{f}^{-1}(W) \subset \overline{\widetilde{f}^{-1}(A \cup B)}$ and $\widetilde{f}^{-1}(y) \subset \overline{\widetilde{f}^{-1}(A \cup B)}$. Since $\widetilde{f}^{-1}(A)$ and $\widetilde{f}^{-1}(B)$ are disjoint zero sets in $\widetilde{f}^{-1}(Y)$ and $X \subset \widetilde{f}^{-1}(Y) \subset C \subset \beta X$ we see that $\overline{\widetilde{f}^{-1}(A)}$ and $\overline{\widetilde{f}^{-1}(B)}$ are a pair of disjoint closed sets in C that cover $\widetilde{f}^{-1}(y)$. So $\widetilde{f}^{-1}(y)$ is disjoint from one of them, say $\overline{\widetilde{f}^{-1}(A)}$. Then y is not in $\widetilde{f}(\overline{\widetilde{f}^{-1}(A)})$, which contains $\overline{A'}$, because $\widetilde{f}$ is closed and surjective. This is a contradiction.

Assume (3) and let A and B be disjoint closed sets in Y such that for some $y \in D$ we have $y \in \overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B})$. Then $y \in D \setminus Y$. Put $W = \operatorname{int}(\overline{A \cup B})$ and select a continuous $\alpha : D \to [0,1]$ such that $\alpha(y) = 1$ and $\alpha | D \setminus W = 0$. We now define the continuous map $\gamma : Y \to [-1,1]$ as follows:

$$\gamma = (\alpha | A \cup (X \setminus W)) \cup (-\alpha | B \cup (X \setminus W)).$$

Define the zero sets $A' = \gamma^{-1}([1/2,1])$ and $B' = \gamma^{-1}([-1,-1/2])$ in Y. Note that $A' \cup B' = Y \cap \alpha^{-1}([1/2,1])$. Let O stand for the open set $\alpha^{-1}((1/2,1])$ and observe that $O \subset \overline{A' \cup B'}$. So y is in the interior of $\overline{A' \cup B'}$. We show that $y \in \overline{A'}$ (and hence $y \in \overline{B'}$ by symmetry). By assumption, $y \in \overline{A}$ and since O is a neighbourhood of y we have $y \in \overline{A \cap O}$. Note that $A \cap O \subset A'$ and hence $y \in \overline{A'}$.

Assume (4) and let $f : X \to Y$ be a monotone map such that $\widetilde{f} : C \to D$ is a closed continuous extension of f that is not monotone. Assume moreover that $\widetilde{f}$ is perfect or that C is normal. Let y be an element of D with a disconnected fibre. If $y \in Y$ then $\widetilde{f}^{-1}(y) = \overline{f^{-1}(y)}$ by Lemma 5. Since f is monotone this would imply that $\widetilde{f}^{-1}(y)$ is connected and hence we know that $y \in D \setminus Y$. Since $\widetilde{f}^{-1}(y)$ is compact or C is normal we can find a disjoint open cover $\{U, V\}$ of $\widetilde{f}^{-1}(y)$ in C such that both U and V intersect the fibre. Then $F = \widetilde{f}(C \setminus (U \cup V))$ is a closed subset of D that does not contain y. Let W be a closed neighbourhood of y in D that is disjoint from F. Note that $\widetilde{f}^{-1}(W) \subset U \cup V$. Define $A' = U \cap f^{-1}(W) = f^{-1}(W) \setminus V$ and $B' = V \cap f^{-1}(W) = f^{-1}(W) \setminus U$.

Both A' and B' are saturated closed subsets of X. This can be seen as follows: if $b \in Y$ such that $f^{-1}(b)$ intersects for instance A' then $f^{-1}(b) \subset A' \cup B'$ since $A' \cup B' = f^{-1}(W)$ is saturated. Since f is monotone, $f^{-1}(b)$ is connected and hence $f^{-1}(b) \subset A'$. Since f is a closed map we see that $A = f(A')$ and $B = f(B')$ are disjoint closed subsets of Y, whose union is $W \cap Y$. Observe that $\operatorname{int}(W) \subset \overline{A \cup B}$. So y is in the interior of $\overline{A \cup B}$ and by assumption (4), $y \notin \overline{A}$ or $y \notin \overline{B}$. By symmetry we may assume that y is outside $\overline{A}$.

Let x be an element of U such that $\widetilde{f}(x) = y$. Then $U \cap \widetilde{f}^{-1}(W) \setminus \widetilde{f}^{-1}(\overline{A})$ is a neighbourhood of x and hence $P = U \cap \widetilde{f}^{-1}(W) \setminus \overline{A'}$ is a neighbourhood of x. Since $A' = U \cap \widetilde{f}^{-1}(W) \cap X$ we infer that P does not intersect X—a contradiction. So we may conclude that U is disjoint from $\widetilde{f}^{-1}(y)$, which contradicts our assumption that $\{U, V\}$ separates $\widetilde{f}^{-1}(y)$. The proof is complete.

REMARKS. We say that D is a *monotone extension* of Y if Y is a dense subset of D and the pair (Y, D) satisfies the conditions (1)–(4) in Theorem 4. If D is moreover compact then we call it a *monotone compactification* of Y.

Consider Theorem 3. It may be surprising that the criterion expressed by statement (3) does only depend on Y and D and that the domain of the monotone map does not seem to matter. In this context observe that

(5) The extension of the identity $\widetilde{\imath}: \beta Y \to D$ is monotone

is one of many statements that imply (2) and follow from (1).

If we substitute $D = \beta Y$ in Theorem 3 then (3) is obviously satisfied and Proposition 1 follows. If $D \setminus Y$ is locally connected rel Y and A and B are disjoint closed sets in Y such that $y \in \overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B})$ then we can find a neighbourhood $U \subset \operatorname{int}(\overline{A \cup B})$ of y in D such that $U \cap Y$ is connected. Then $U \cap A$ and $U \cap B$ are both nonempty, which means that A and B separate the connected set $U \cap Y$. So D is a monotone extension of Y and Proposition 2 also follows from the theorem.

EXAMPLE 1. As an illustration to Theorem 3 we give a simple example of a monotone compactification that is not covered by Proposition 1 or 2. Let $I = [0, 1]$ and define the following subspaces of $I \times I$:

$$D = (\{0\} \cup \{1/n : n \in \mathbb{N}\}) \times I \quad \text{and} \quad Y = D \setminus \{(0, 0)\}.$$

We verify that D is a monotone compactification of Y and hence Theorem 3 guarantees that for every space X which is the preimage of Y under a perfect monotone map and every compactification C of X the remainder $C \setminus X$ is a continuum.

Let A and B be disjoint closed subsets of Y such that $\overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B}) \neq \emptyset$. Then $\overline{A} \cap \overline{B} \cap \operatorname{int}(\overline{A \cup B}) = \{(0, 0)\}$ and we can find an $\varepsilon > 0$ such that $([0, \varepsilon] \times [0, \varepsilon]) \cap Y \subset A \cup B$. We may assume that $(0, \varepsilon)$ is in A and hence not in B. Since B is closed there is an $N > 1/\varepsilon$ such that $(1/n, \varepsilon) \in A$ for every $n \geq N$. Since for every $n \geq N$, $\{1/n\} \times [0, \varepsilon]$ is a connected subset of $A \cup B$ we have $\{1/n\} \times [0, \varepsilon] \subset A$ for $n \geq N$. Consequently, $([0, 1/N] \times [0, \varepsilon]) \cap Y \subset A$ and hence $(0, 0) \notin \overline{B}$, which is a contradiction.

EXAMPLE 2. Consider condition (1) in Theorem 4. A natural question is whether the mild restriction that $\widetilde{f}$ be perfect or C be normal is really necessary. The following example shows that the answer is yes.

Let L be the "long halfline," i.e. the space $[0, \omega_1) \times [0, 1)$ with the topology generated by the lexicographic order. Let $\alpha L = L \cup \{\omega_1\}$ be the compactification of L. Let $X = Y = L \times [0, 1)$, $C = (\alpha L \times I) \setminus \{(\omega_1, 1)\}$, and let D be the one-point compactification $Y\cup\{\infty\}$ of Y. We take for the monotone map $f : X \to Y$ the identity and $\widetilde{f} : C \to D$ and $\bar{f} : \alpha L \times I \to D$ are the extensions of f. It is obvious that $\{\infty\}$ is locally connected rel Y so D is a monotone compactification of Y. The fibre $\widetilde{f}^{-1}(\infty) = (L \times \{1\}) \cup (\{\omega_1\} \times [0, 1))$ has two components so $\widetilde{f}$ is not monotone.

It remains to show that $\widetilde{f}$ is closed. Let F be a closed subset of C and let G denote the closure of F in $\alpha L\times I$. If $\infty \in \widetilde{f}(F)$ then $\widetilde{f}(F)$ equals $\bar{f}(G)$ and hence is compact. If $\infty \notin \widetilde{f}(F)$ then $\alpha L \times \{1\}$ and $\{\omega_1\} \times [0, 1)$ are disjoint from F. Since $[0, 1)$ is Lindelöf and ω_1 has uncountable cofinality there is a neighbourhood U of ω_1 in αL such that $F \cap (U \times [0, 1)) = \emptyset$. So F is disjoint from $U \times I$ and hence F and $\widetilde{f}(F)$ are compact.

Let D be a compactification of Y. Theorem 3 answers the question when *all* "compactifications" with range D of monotone maps onto Y are monotone. We now turn to the question when we can guarantee the *existence* of monotone "compactifications" onto D of monotone maps onto Y. Before presenting a criterion we discuss an illuminating example.

Example 3. Put $D = I$ and $Y = I \setminus \{1/n : n \in \mathbb{N}\}$. Consider the following closed subspace of $Y \times I$:

$$X = (\{0\} \times I) \cup \bigcup_{n=1}^{\infty} \left(\left(\frac{1}{2n}, \frac{1}{2n-1} \right) \times \{0\} \right) \cup \left(\left(\frac{1}{2n+1}, \frac{1}{2n} \right) \times \{1\} \right).$$

The map $f : X \to Y$ is simply the restriction of the projection. Since X is closed in $Y\times I$ and I is compact we find that the projection f is perfect. Note that every fibre of f is either a singleton or an interval so f is monotone.

Assume now that C is a compactification of X such that f extends to a monotone $\widetilde{f} : C \to D$. Since $\widetilde{f}$ is monotone, $\overline{\widetilde{f}^{-1}((0, 1/n])}$, $n \in \mathbb{N}$, is a decreasing sequence of continua in C. Consequently,

$$K = \bigcap_{n=1}^{\infty} \overline{\widetilde{f}^{-1}((0, 1/n])}$$

is a continuum that is obviously contained in $\widetilde{f}^{-1}(0) = \{0\} \times I$. Note that both $(0, 0)$ and $(0, 1)$ are in K but that $\{0\}\times(0, 1)$ is an open locally compact subspace of X and hence also open in C. This means that K and $\{0\}\times(0, 1)$ are disjoint so the continuum K equals $\{(0, 0), (0, 1)\}$, a contradiction. We may conclude that f does not have a monotone "compactification" whose range is D.

We say that a space Y *has ordered neighbourhood bases* if every $y \in Y$ has a neighbourhood basis that is linearly ordered by the inclusion relation. First countable spaces are obvious examples of such spaces.

THEOREM 6. *If Y is a dense subspace of a space D and Y has ordered neighbourhood bases then the following statements are equivalent*:

(1) *For every space X and every monotone map $f : X \to Y$ there exists a space C such that X is dense in C and f extends to a monotone and perfect $\widetilde{f} : C \to D$.*

(2) *For every closed subspace X of $Y \times I$ such that the projection $f : X \to Y$ is monotone there exists a space C such that X is dense in C and f extends to a monotone $\widetilde{f} : C \to D$.*

(3) *Every $y \in Y$ has a neighbourhood U in D such that U is a monotone extension of $Y \cap U$.*

(4) *There exists an open O in D that is a monotone extension of Y.*

Proof. Statement (2) follows trivially from (1). We shall prove: (4)⇒(1), (3)⇒(4), and ¬(3)⇒¬(2).

Assume (4) and let $f : X \to Y$ be monotone. Extend f to $\bar{f} : \beta X \to \beta D$. Put $U = \bar{f}^{-1}(O)$ and $C' = \bar{f}^{-1}(D)$. Note that $\bar{f}|U$ is a perfect map from U onto O. Since O is a monotone extension of Y we see that $\bar{f}|U$ is monotone. Consider the closed subspace $F = C' \setminus U$ of C' and the closed map $p = \bar{f}|F$ from F onto $G = D \setminus O$. Let C be the adjunction space $C' \cup_p G$ and let $\pi : C' \to C$ be the quotient map. Then we can define a function $\widetilde{f} : C \to D$ such that $\widetilde{f} \circ \pi = \bar{f}|C'$. The map $\widetilde{f}$ obviously extends f and is closed and continuous. If $y \in O$ then $\widetilde{f}^{-1}(y)$ is a fibre of the map $\bar{f}|U$ and hence a continuum. If $y \in D \setminus O = G$ then $\widetilde{f}^{-1}(y)$ is a singleton. So we may conclude that $\widetilde{f}$ is monotone and perfect.

Assume (3). If we define

$$\mathcal{U} = \{U : U \text{ an open subset of } D \text{ such that } U \text{ is a monotone extension of } Y \cap U\}$$

then $O = \bigcup \mathcal{U}$ is an open set in D that contains Y. Let $g : V \to O$ be a perfect extension of the identity on Y such that $Y \subset V \subset \beta Y$. Note that by Theorem 4, $g|g^{-1}(U)$ is monotone for each $U \in \mathcal{U}$ and hence g is monotone. So O is a monotone extension of Y.

Assume that condition (3) is false, i.e. there is a $y \in Y$ such that no neighbourhood U in D is a monotone extension of $Y \cap U$. Let $\{V_\alpha : \alpha < \kappa\}$ be a neighbourhood basis for y in Y where κ is some regular cardinal and $V_\alpha \subset V_\beta$ for $\beta < \alpha < \kappa$. Define $\widetilde{V}_\alpha = D \setminus (\overline{Y \setminus V_\alpha})$ for each α and note that $\{\widetilde{V}_\alpha : \alpha < \kappa\}$ is a neighbourhood basis for y in D because Y is dense and D is regular.

We construct by induction for each $\alpha < \kappa$ an ordinal $\gamma(\alpha) < \kappa$, a point $y_\alpha \in D \setminus Y$, an open subset U_α of D, and disjoint closed subsets A_α and B_α of Y such that

(i) $\gamma(\beta) < \gamma(\alpha)$ for $\beta < \alpha$,
(ii) $y_\alpha \in U_\alpha \cap \overline{A}_\alpha \cap \overline{B}_\alpha$,
(iii) $U_\alpha \subset (\overline{A_\alpha \cup B_\alpha}) \cap \widetilde{V}_{\gamma(\alpha)} \setminus \widetilde{V}_{\gamma(\alpha+1)}$.

Let $\alpha < \kappa$. If α is a successor ordinal then we assume that $\gamma(\alpha)$ has already been selected, if $\alpha = 0$ then we put $\gamma(\alpha) = 0$, and if α is a limit ordinal then we put $\gamma(\alpha) = \sup_{\beta<\alpha} \gamma(\beta)$. We can find a $y_\alpha \in \widetilde{V}_{\gamma(\alpha)}$ and disjoint closed subsets A_α and B_α of Y such that $y_\alpha \in \overline{A}_\alpha \cap \overline{B}_\alpha \cap \operatorname{int}(\overline{A_\alpha \cup B_\alpha})$. Select a $\gamma(\alpha+1) > \gamma(\alpha)$ such that $y_\alpha \notin \overline{\widetilde{V}}_{\gamma(\alpha+1)}$. Then define

$$U_\alpha = \operatorname{int}(\overline{A_\alpha \cup B_\alpha}) \cap \widetilde{V}_{\gamma(\alpha)} \setminus \overline{\widetilde{V}}_{\gamma(\alpha+1)}.$$

Note that the U_α's are pairwise disjoint. Put $O = \bigcup_{\alpha<\kappa} U_\alpha$ and define the subset X of $Y \times I$ by

$$X = ((Y \setminus O) \times I) \cup \bigcup_{\alpha<\kappa} ((A_\alpha \cap U_\alpha) \times \{0\}) \cup ((B_\alpha \cap U_\alpha) \times \{1\}).$$

Let $f : X \to Y$ be the projection. Since $\{A_\alpha \cap U_\alpha, B_\alpha \cap U_\alpha : \alpha < \kappa\}$ is a pairwise disjoint open covering of $O \cap Y$, we see that X is closed in $Y \times I$ and that every fibre of f is a singleton or an interval. Since X is closed we find that f is perfect by the compactness of I. Consequently, f is monotone.

Let C be a space such that X is dense in C and f extends to a monotone $\widetilde{f} : C \to D$. Define the following closed subsets of C:

$$A = \overline{(Y \times \{0\}) \cap X} \quad \text{and} \quad B = \overline{(Y \times \{1\}) \cap X}.$$

We show that $\widetilde{f}^{-1}(O)$ is contained in $A \cup B$. If $x \in \widetilde{f}^{-1}(O)$ and V is a neighbourhood of x that is contained in $\widetilde{f}^{-1}(O)$ then we can find a $z \in V \cap X$. Since $f(z) \in O$ we have $z \in (Y \times \{0,1\}) \cap X$. Hence z is in $A \cup B$ and so is x.

Since $\widetilde{f}$ is closed $\widetilde{f}(A)$ is closed in D. For each $\alpha < \kappa$, $A_\alpha \cap U_\alpha$ is a subset of $\widetilde{f}(A)$. Since $y_\alpha \in \overline{A_\alpha \cap U_\alpha}$ we have $y_\alpha \in \widetilde{f}(A)$. So $\widetilde{f}^{-1}(y_\alpha) \cap A \neq \emptyset$ and by symmetry $\widetilde{f}^{-1}(y_\alpha) \cap B \neq \emptyset$.

Let U and V be disjoint open sets in C such that $(y, 0) \in U$, $(y, 1) \in V$, and $U \cap B = V \cap A = \emptyset$. Since f is perfect we have $\widetilde{f}^{-1}(y) = f^{-1}(y) = \{y\} \times I$. Note that this fact implies that $F = \widetilde{f}(A \setminus U) \cup \widetilde{f}(B \setminus V)$ is a closed subset of D that does not contain y. Since $\sup_{\alpha<\kappa} \gamma(\alpha) = \kappa$ there is an $\alpha < \kappa$ with $\widetilde{V}_{\gamma(\alpha)} \cap F = \emptyset$. So $y_\alpha \in D \setminus F$ by (ii) and (iii). Since $y_\alpha \in U_\alpha \subset O$ we have $\widetilde{f}^{-1}(y_\alpha) \subset A \cup B$. Consequently, $\widetilde{f}^{-1}(y_\alpha) \subset U \cup V$ and since $\widetilde{f}^{-1}(y_\alpha) \cap A \neq \emptyset$ and $\widetilde{f}^{-1}(y_\alpha) \cap B \neq \emptyset$ we have $\widetilde{f}^{-1}(y_\alpha) \cap A \cap U \neq \emptyset$ and $\widetilde{f}^{-1}(y_\alpha) \cap B \cap V \neq \emptyset$. Since U and V are disjoint open sets they separate the fibre $\widetilde{f}^{-1}(y_\alpha)$ and hence $\widetilde{f}$ is not monotone, a contradiction.

Remarks. Note that the condition that Y has ordered neighbourhood bases is only used to prove (2)⇒(3). Without any restrictions on Y we have (3)⇔(4)⇒(1)⇒(2). One can think of other conditions that would make Theorem 6 true. For instance, if Y is an ordered space then the proof can easily be adapted. The question is whether the implications (1)⇒(3) or (2)⇒(3) are true in general.

Proposition 1 implies that every monotone map can be "compactified" to a monotone map by using the Čech–Stone compactifications. For separable metric spaces that result is not very satisfactory, especially since we were motivated to look at monotone maps by a problem formulated in Dijkstra and Mogilski [2], which concerns extendibility of cell-like decompositions of Hilbert space. To address the metric case we have the following

Theorem 7. *If $f : X \to Y$ is a monotone map between separable metric spaces then there exist metric compactifications C and D of X and Y such that f extends to a monotone $\widetilde{f} : C \to D$.*

Proof. The proof uses the Wallman compactification whose definition we now recall. We call a closed basis $\mathfrak{W}$ for the topology of a space X a *Wallman basis* for X if $\mathfrak{W}$ is closed under finite intersections and if $\mathfrak{W}$ is normal (i.e. if A and B are disjoint members of $\mathfrak{W}$ then there are $V, W \in \mathfrak{W}$ such that $V \cup W = X$ and $V \cap B = A \cap W = \emptyset$). If $\mathfrak{W}$ is a Wallman basis for X then the underlying set for the *Wallman compactification* $\omega(\mathfrak{W})$ of X relative $\mathfrak{W}$ is the set of $\mathfrak{W}$-ultrafilters. If $W \in \mathfrak{W}$ then $\overline{W} = \{\mathcal{F} \in \omega(\mathfrak{W}) : W \in \mathcal{F}\}$. The collection $\{\overline{W} : W \in \mathfrak{W}\}$ functions as a closed basis for the topology on $\omega(\mathfrak{W})$. Since $\mathfrak{W}$ is normal $\omega(\mathfrak{W})$ is Hausdorff and if $\mathfrak{W}$ is countable then $\omega(\mathfrak{W})$ is metrizable. We shall use the following well-known fact: if $f : X \to Y$ is a map and $\mathfrak{X}$ and $\mathfrak{Y}$ are Wallman bases on X and Y respectively such that $f^{-1}[\mathfrak{Y}] \subset \mathfrak{X}$ then f extends to a map $\bar{f} : \omega(\mathfrak{X}) \to \omega(\mathfrak{Y})$. See Walker [5] for more information about Wallman compactifications.

Let $f : X \to Y$ be a monotone map between separable metric spaces. Select a countable Wallman basis $\mathfrak{C}_0$ for X. Expand $f[\mathfrak{C}_0]$, which is a countable collection of closed subsets of Y, to a countable Wallman basis $\mathfrak{D}_0$ for Y. Next, expand $f^{-1}[\mathfrak{D}_0] \cup \mathfrak{C}_0$ to a countable Wallman basis $\mathfrak{C}_1$. Continuing this back-and-forth process we find an increasing sequence $(\mathfrak{C}_n)_{n=0}^{\infty}$ of countable Wallman bases for X and an increasing sequence $(\mathfrak{D}_n)_{n=0}^{\infty}$ of countable Wallman bases for Y such that $f[\mathfrak{C}_n] \subset \mathfrak{D}_n$ and $f^{-1}[\mathfrak{D}_n] \subset \mathfrak{C}_{n+1}$ for each $n \ge 0$. So $\mathfrak{C} = \bigcup_{n=0}^{\infty} \mathfrak{C}_n$ and $\mathfrak{D} = \bigcup_{n=0}^{\infty} \mathfrak{D}_n$ are countable Wallman bases for X and Y respectively with the properties $f^{-1}[\mathfrak{D}] \subset \mathfrak{C}$ and $f[\mathfrak{C}] = \mathfrak{D}$. If we define the metric compactifications $C = \omega(\mathfrak{C})$ and $D = \omega(\mathfrak{D})$ then f extends to a continuous $\widetilde{f} : C \to D$.

Let y be an element of D with a disconnected fibre. If $y \in Y$ then $\widetilde{f}^{-1}(y) = \overline{f^{-1}(y)}$ by Lemma 5. Since f is monotone this would imply that

$\widetilde{f}^{-1}(y)$ is connected and hence we may assume that $y \in D \setminus Y$. Write $\widetilde{f}^{-1}(y)$ as a disjoint union of two nonempty compacta A and B. Select from $\mathfrak{C}$ two disjoint elements F and G such that $\overline{F}$ is a neighbourhood of A and $\overline{G}$ is a neighbourhood of B. Then $P = \widetilde{f}(C \setminus \operatorname{int}(\overline{F \cup G}))$ is a closed set that does not contain y. Let W be an element of $\mathfrak{D}$ such that $\overline{W}$ is a neighbourhood of y that is disjoint from P. Note that $f^{-1}(W) \subset F \cup G$. Define $F' = F \cap f^{-1}(W)$ and $G' = G \cap f^{-1}(W)$ and note that both sets are in $\mathfrak{C}$. Also, both F' and G' are saturated subsets of X. This can be seen as follows: if $b \in Y$ such that $f^{-1}(b)$ intersects for instance F' then $f^{-1}(b) \subset F' \cup G'$ because $F' \cup G' = f^{-1}(W)$ is saturated. Since f is monotone, $f^{-1}(b)$ is connected and hence $f^{-1}(b) \subset F'$. Note that $U = f(F')$ and $V = f(G')$ are disjoint elements of $\mathfrak{D}$ and that their union is W. Since $D = \omega(\mathfrak{D})$ we see that $\overline{U}$ and $\overline{V}$ are also disjoint so that one of them does not contain y, say $y \notin \overline{U}$. Then $\widetilde{f}^{-1}(y)$ is disjoint from $\widetilde{f}^{-1}(\overline{U})$ and hence disjoint from $\overline{F'}$. Consequently, $A = \widetilde{f}^{-1}(y) \cap \overline{F'}$ is empty, which is a contradiction.

COROLLARY 8. *Let C and D be separable metric spaces, let $\widetilde{f} : C \to D$ be a closed and continuous map, and let X and Y be dense subsets of C and D respectively such that $f = \widetilde{f}|X$ is a monotone map from X onto Y. Then there is a G_δ-subset G of D such that $Y \subset G$ and $\widetilde{f}|\widetilde{f}^{-1}(G) : \widetilde{f}^{-1}(G) \to G$ is monotone.*

So every extension of a monotone map over metric compactifications restricts to a perfect monotone extension over completions.

Proof. Note that $\widetilde{f}$ is surjective because it is closed and its range contains Y. Let $\widehat{C}$ and $\widehat{D}$ be metric compactifications of C and D such that $\widetilde{f}$ extends to a continuous $\widehat{f} : \widehat{C} \to \widehat{D}$. If we define $\check{X} = \widehat{f}^{-1}(Y)$ then by Lemma 5, $\check{f} = \widehat{f}|\check{X}$ is a perfect monotone map from $\check{X}$ to Y. With Theorem 7 we find metric compactifications X' and Y' of $\check{X}$ and Y respectively such that $\check{f}$ extends to a monotone $f' : X' \to Y'$. According to Lavrentiev [4] there exist G_δ-sets $A \subset \widehat{C}$, $A' \subset X'$, $B \subset \widehat{D}$, $B' \subset Y'$, and homeomorphisms $\alpha : A \to A'$ and $\beta : B \to B'$ such that $\check{X} \subset A$, $\check{X} \subset A'$, $Y \subset B$, $Y \subset B'$, and $\alpha|\check{X}$ and $\beta|Y$ are identity mappings. Let $G' = B' \setminus f'(X' \setminus A')$ and note that G' is a G_δ-set in Y' that contains Y. Define the G_δ-sets $F' = f'^{-1}(G')$, $\widehat{F} = \alpha^{-1}(F')$, and $\widehat{G} = \beta^{-1}(G')$. Since $\widehat{f}|X = \beta^{-1} \circ f' \circ \alpha|X$ and X is dense we have $\widehat{f}|\widehat{F} = \beta^{-1} \circ f' \circ \alpha|\widehat{F}$. Since $f'|F'$ is perfect and monotone so is $\widehat{f}|\widehat{F}$. Put $F = \widehat{F} \cap C$ and $G = \widehat{G} \cap D$. Consider the map $g = \widetilde{f}|F = \widehat{f}|F$ from F to G.

It is obvious that $\widetilde{f}^{-1}(G) = F$ and that g is closed and surjective. It remains to verify that g has connected fibres. If $y \in Y$ then $g^{-1}(y)$ is connected by Lemma 5. Let $y \in G \setminus Y$ and $x \in \widehat{F}$ such that $\widehat{f}(x) = y$. Select a sequence $x_1, x_2, \ldots$ in X that converges to x. If $x \notin C$ then $\{x_n : n \in \mathbb{N}\}$

is closed in C. Since $\widetilde{f} : C \to D$ is closed we see that $\{f(x_n) : n \in \mathbb{N}\}$ is closed in D. This contradicts the fact that $f(x_1), f(x_2), \ldots$ is a sequence in Y that converges to $y \in D \setminus Y$. So we may conclude that if $y \in G \setminus Y$ then $g^{-1}(y) = \widehat{f}^{-1}(y)$. Since $\widehat{f}$ is monotone, g is monotone.

Example 4. In view of Theorems 3 and 7 it is natural to ask the following question: does every separable metric space Y have a metric compactification D with the property that whenever $f : C \to D$ is a map with compact metric domain such that $f|X : X \to Y$ is monotone for some dense $X \subset C$, then f is monotone as well?

The answer is easily seen to be no. Consider a metric compactification D of the natural numbers $\mathbb{N}$ and let $\widetilde{\imath} : \beta\mathbb{N} \to D$ be the extension of the identity. Since $|D \setminus \mathbb{N}| \leq \mathfrak{c}$ and $|\beta\mathbb{N} \setminus \mathbb{N}| = 2^{\mathfrak{c}}$ we can pick a $y \in D \setminus \mathbb{N}$ with nontrivial fibre. Pick a subset A of $\mathbb{N}$ such that both $\overline{A}$ and its complement in $\beta\mathbb{N}$ intersect $\widetilde{\imath}^{-1}(y)$. Let B_1 be the closure of A in D and let B_2 be the closure of $\mathbb{N} \setminus A$ in D. If C is the topological sum of B_1 and B_2 then the natural map from C to D is an extension of the identity that is not monotone.

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