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SUBDIRECT DECOMPOSITIONS OF ALGEBRAS FROM 2-CLONE EXTENSIONS OF VARIETIES

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Let $\tau : F \to \mathbb{N}$ be a type of algebras, where F is a set of fundamental operation symbols and $\mathbb{N}$ is the set of nonnegative integers. We assume that $|F| \geq 2$ and $0 \notin \tau(F)$. For a term φ of type τ we denote by $F(\varphi)$ the set of fundamental operation symbols from F occurring in φ. An identity $\varphi \approx \psi$ of type τ is called clone compatible if φ and ψ are the same variable or $F(\varphi) = F(\psi) \neq \emptyset$. For a variety V of type τ we denote by $V^{\mathrm{c},2}$ the variety of type τ defined by all identities $\varphi \approx \psi$ from $\mathrm{Id}(V)$ which are either clone compatible or $|F(\varphi)|, |F(\psi)| \geq 2$. Under some assumption on terms (condition (0.iii)) we show that an algebra $\mathfrak{A}$ belongs to $V^{\mathrm{c},2}$ iff it is isomorphic to a subdirect product of an algebra from V and of some other algebras of very simple structure. This result is applied to finding subdirectly irreducible algebras in $V^{\mathrm{c},2}$ where V is the variety of distributive lattices or the variety of Boolean algebras.

0. Preliminaries. We consider algebras of a given type $\tau : F \to \mathbb{N}$, where F is a set of fundamental operation symbols and $\mathbb{N}$ is the set of nonnegative integers (cf. [2] and [5]). In this paper we assume that $|F| \geq 2$ and $0 \notin \tau(F)$, i.e. we do not admit nullary fundamental operation symbols.

If φ is a term of type τ we denote by $\mathrm{Var}(\varphi)$ the set of variables occurring in φ, and by $F(\varphi)$ the set of fundamental operation symbols in φ. Writing $\varphi(x_{i_1}, \ldots, x_{i_m})$ instead of φ means that $\mathrm{Var}(\varphi) = \{x_{i_1}, \ldots, x_{i_m}\}$.

In several papers identities of some special structural forms and constructions of algebras connected with them were considered. Let us recall some of them. An identity $\varphi \approx \psi$ of type τ is *regular* if $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$ (see, e.g., [6], [7], [10], [11], [15]). An identity $\varphi \approx \psi$ of type τ is *nontrivializing* or *normal* if it is of the form $x \approx x$ or $F(\varphi) \neq \emptyset \neq F(\psi)$ (see, e.g., [4], [8], [13]). Let P be a partition of F. An identity $\varphi \approx \psi$ of type τ is P-*compatible* if it

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is of the form $x \approx x$ or $F(\varphi) \neq \emptyset \neq F(\psi)$ and the outermost fundamental operation symbols in φ and ψ are in the same block of P (see, e.g., [17]). An identity $\varphi \approx \psi$ of type τ is *biregular* if $\mathrm{Var}(\varphi) = \mathrm{Var}(\psi)$ and $F(\varphi) = F(\psi)$ (see, e.g., [14]–[16]).

In [18] we defined the so-called clone compatible identities as follows: $\varphi \approx \psi$ of type τ is *clone compatible* if it is of the form $x \approx x$ or $F(\varphi) = F(\psi) \neq \emptyset$. If V is a variety of type τ we denote by $\mathrm{Id}(V)$ the set of all identities of type τ satisfied in every algebra from V. For a variety V of type τ we denote by V^{c} the variety of type τ defined by all clone compatible identities from $\mathrm{Id}(V)$. We denote by $V^{\mathrm{c},2}$ the variety of type τ defined by all identities $\varphi \approx \psi$ from $\mathrm{Id}(V)$ satisfying one of the following two conditions:

(0.i) $$F(\varphi) = F(\psi), \quad |F(\varphi)| = 1,$$

(0.ii) $$|F(\varphi)|, |F(\psi)| \geq 2.$$

We call the variety $V^{\mathrm{c},2}$ the 2-*clone extension* of the variety V.

In [18] the variety $V^{\mathrm{c},2}$ was denoted by $\overline{V^{\mathrm{c}}}$. Here we prefer the notation $V^{\mathrm{c},2}$ since it agrees with the notation $V^{\mathrm{c},n}$ from [20] for $n = 2$.

Studying the variety $V^{\mathrm{c},2}$ is very useful if we want to find descriptions of algebras from V^{c}. This is so because in many cases we have $V^{\mathrm{c}} = V^{\mathrm{c},2}$. This is the case if V is a variety of lattices, the variety of Boolean algebras or a variety of groups satisfying $x^n \approx y^n$ for some n (see [18], examples). Moreover, in [18] we found representations of algebras from $\overline{V^{\mathrm{c}}} = V^{\mathrm{c},2}$ by means of so-called clone extensions of algebras from V, where we use the following condition.

(0.iii) For every $f \in F$ there exists a term $q_f(x)$ of type τ such that $F(q_f(x)) = \{f\}$ and the identity $q_f(x) \approx x$ belongs to $\mathrm{Id}(V)$.

Note that this assumption is satisfied in lattices and Boolean algebras since in lattices we have $x + x \approx x \cdot x \approx x$, and in Boolean algebras we have $(x')' \approx x$. This assumption is also satisfied in varieties of groups if they satisfy $x^n \approx y^n$ so $x^{n+1} \approx x$ and $(x^{-1})^{-1} \approx x$.

In [19] we generalize results from [17] and in [18] we deal with free algebras over $V^{\mathrm{c},2}$ and in general over $V^{\mathrm{c},n}$ in some cases. In the present paper under the assumption (0.iii) we give another representation of algebras from $V^{\mathrm{c},2}$. We prove that an algebra $\mathfrak{A}$ belongs to $V^{\mathrm{c},2}$ iff it is isomorphic to a subdirect product of an algebra from V and some algebras easy to describe (see Theorem 1.9).

This subdirect decomposition is useful for finding subdirectly irreducible algebras in $D^{\mathrm{c}} = D^{\mathrm{c},2}$ and $B^{\mathrm{c}} = B^{\mathrm{c},2}$, where D is the variety of distributive lattices and B is the variety of Boolean algebras (Section 2).

If an identity $\varphi \approx \psi$ belongs to $\mathrm{Id}(V)$, we often write $V \models \varphi \approx \psi$. If $\mathfrak{A} = (A; F^{\mathfrak{A}})$ is an algebra from V, $\varphi(x_{i_1}, \ldots, x_{i_m})$ and $\psi(x_{j_1}, \ldots, x_{j_s})$ are terms of type τ, $a_{i_1}, \ldots, a_{i_m}, a_{j_1}, \ldots, a_{j_s} \in A$ and the equality

$$\varphi^{\mathfrak{A}}(x_{i_1}, \ldots, x_{i_m}) = \psi^{\mathfrak{A}}(x_{j_1}, \ldots, x_{j_s})$$

holds in $\mathfrak{A}$ since $V \models \varphi \approx \psi$, then we write

$$\varphi^{\mathfrak{A}}(x_{i_1}, \ldots, x_{i_m}) \stackrel{V}{=} \psi^{\mathfrak{A}}(x_{j_1}, \ldots, x_{j_s}).$$

It should be emphasized that many identities are consequences of (0.iii) and are of the form (0.i) or (0.ii), so they belong to $\mathrm{Id}(V^{\mathrm{c},2})$; for example in $V^{\mathrm{c},2}$ we have

(0.iv) $q_f(q_f(x)) \approx q_f(x)$ for every $f \in F$,
(0.v) $q_f(q_g(x)) \approx q_p(q_s(x))$ for every $f, g, p, s \in F$ with $f \neq g$ and $p \neq s$.

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1. Subdirect decomposition of algebras from $V^{\mathrm{c},2}$. In this section we assume that V is a variety of type τ satisfying (0.iii) and $\mathfrak{A} = (A; F^{\mathfrak{A}})$ is an algebra from $V^{\mathrm{c},2}$. For some distinct $f, g \in F$ and for q_f, q_g satisfying (0.iii) we put $q_h(x) = q_f(q_g(x))$. We define a relation R_h on A putting, for $a, b \in A$,

$$a\, R_h\, b \ \text{ iff } \ q_h^{\mathfrak{A}}(a) = q_h^{\mathfrak{A}}(b).$$

By (0.v) the relation R_h does not depend on the choice of f and g.

LEMMA 1.1. *The relation R_h is a congruence of $\mathfrak{A}$.*

Proof. Obviously R_h is an equivalence. It satisfies the superposition law since for every $s \in F$ and $a_1, \ldots, a_{\tau(s)} \in A$ we have

$$q_h^{\mathfrak{A}}(s^{\mathfrak{A}}(a_1, \ldots, a_{\tau(s)})) \stackrel{V^{\mathrm{c},2}}{=} s^{\mathfrak{A}}(q_h^{\mathfrak{A}}(a_1), \ldots, q_h^{\mathfrak{A}}(a_{\tau(s)})).$$

LEMMA 1.2. *The algebra $\mathfrak{A}/R_h$ belongs to V.*

Proof. If $V \models \varphi \approx \psi$, then by (0.iii), $V \models q_h(\varphi) \approx q_h(\psi)$ and $q_h(\varphi) \approx q_h(\psi)$ is of the form (0.ii). So $V^{\mathrm{c},2} \models q_h(\varphi) \approx q_h(\psi)$. Consequently, $\mathfrak{A}/R_h$ satisfies $\varphi \approx \psi$.

For every $f \in F$ we define a relation R_f on A putting, for $a, b \in A$, $a\, R_f\, b$ iff one of the following two conditions holds:

(1.i) $$q_f^{\mathfrak{A}}(a) = q_f^{\mathfrak{A}}(b),$$

(1.ii) $$q_f^{\mathfrak{A}}(a) = q_h^{\mathfrak{A}}(a) \quad \text{and} \quad q_f^{\mathfrak{A}}(b) = q_h^{\mathfrak{A}}(b).$$

By (0.i) the relation R_f depends on f but not on the choice of q_f.

LEMMA 1.3. *For every $f \in F$ the relation R_f is a congruence of $\mathfrak{A}$.*

Proof. For every $f \in F$ the relation R_f is reflexive and symmetric. Let $a, b, c \in A$. To show the transitivity consider a nontrivial case when $q_f^{\mathfrak{A}}(a) = q_f^{\mathfrak{A}}(b)$, $q_f^{\mathfrak{A}}(b) = q_h^{\mathfrak{A}}(b)$ and $q_f^{\mathfrak{A}}(c) = q_h^{\mathfrak{A}}(c)$. Then by (0.v) we have

$$q_f^{\mathfrak{A}}(a) = q_f^{\mathfrak{A}}(b) = q_h^{\mathfrak{A}}(b) \overset{V^{c,2}}{=} q_g^{\mathfrak{A}}(q_f^{\mathfrak{A}}(b)) = q_g^{\mathfrak{A}}(q_f^{\mathfrak{A}}(a)) = q_h^{\mathfrak{A}}(a)$$

for some $g \neq f$, $g \in F$. Thus $q_f^{\mathfrak{A}}(a) = q_h^{\mathfrak{A}}(a)$ and $q_f^{\mathfrak{A}}(c) = q_h^{\mathfrak{A}}(c)$. The other cases for transitivity are trivial or analogous.

We check the superposition property for R_f. Let $s \in F$ and $a_k \, R_f \, b_k$ for $k \in \{1, \ldots, \tau(s)\}$. If $q_f^{\mathfrak{A}}(a_k) = q_f^{\mathfrak{A}}(b_k)$ for $k = 1, \ldots, \tau(s)$, then

$$q_f^{\mathfrak{A}}(s^{\mathfrak{A}}(a_1, \ldots, a_{\tau(s)})) \overset{V^{c,2}}{=} s^{\mathfrak{A}}(q_f^{\mathfrak{A}}(a_1), \ldots, q_f^{\mathfrak{A}}(a_{\tau(s)}))$$
$$= s^{\mathfrak{A}}(q_f^{\mathfrak{A}}(b_1), \ldots, q_f^{\mathfrak{A}}(b_{\tau(s)})) \overset{V^{c,2}}{=} q_f^{\mathfrak{A}}(s^{\mathfrak{A}}(b_1, \ldots, b_{\tau(s)})).$$

Assume $a_k \, R_f \, b_k$ for $k \in \{1, \ldots, \tau(s)\}$; since $q_f^{\mathfrak{A}}(a_k) = q_h^{\mathfrak{A}}(a_k)$ and $q_f^{\mathfrak{A}}(b_k) = q_h^{\mathfrak{A}}(b_k)$ without loss of generality we can assume $k = 1$. Then

$$q_f^{\mathfrak{A}}(s^{\mathfrak{A}}(a_1, a_2, \ldots, a_{\tau(s)})) \overset{V^{c,2}}{=} s^{\mathfrak{A}}(q_f^{\mathfrak{A}}(a_1), a_2, \ldots, a_{\tau(s)})$$
$$= s^{\mathfrak{A}}(q_h^{\mathfrak{A}}(a_1), a_2, \ldots, a_{\tau(s)}) \overset{V^{c,2}}{=} q_h^{\mathfrak{A}}(s^{\mathfrak{A}}(a_1, a_2, \ldots, a_{\tau(s)})).$$

Similarly

$$q_f^{\mathfrak{A}}(s^{\mathfrak{A}}(b_1, \ldots, a_{\tau(s)})) = q_h^{\mathfrak{A}}(s^{\mathfrak{A}}(b_1, \ldots, b_{\tau(s)})).$$

For $f \in F$ we denote by $V(f)$ the variety of type τ defined by all identities $\varphi \approx \psi$ of type τ satisfying one of the following two conditions:

(1.iii) $$F(\varphi) \setminus \{f\} \neq \emptyset \neq F(\psi) \setminus \{f\},$$

(1.iv) $$V \models \varphi \approx \psi \quad \text{and} \quad F(\varphi) \cup F(\psi) \subseteq \{f\}.$$

LEMMA 1.4. *Obviously* $V(f) \subseteq V^{c,2}$. *Moreover*, *an algebra* $\mathfrak{B} = (B; F^{\mathfrak{B}})$ *belongs to* $V(f)$ *iff it satisfies all identities of the form* (1.iv) *and there exists an element* e_f *in* B *such that the value of every fundamental operation* $g^{\mathfrak{B}}$ *is the constant* e_f *if* $g \in F \setminus \{f\}$, *and the value of* $f^{\mathfrak{B}}$ *is equal to* e_f *if* e_f *occurs among the arguments of* $f^{\mathfrak{B}}$.

LEMMA 1.5. *For every* $f \in F$ *the algebra* $\mathfrak{A}/R_f$ *belongs to* $V(f)$.

Proof. If an identity $\varphi \approx \psi$ is of the form (1.iv), then the identity $q_f(\varphi) \approx q_f(\psi)$ is of the form (0.i), so $\varphi \approx \psi$ holds in $\mathfrak{A}/R_f$. If $\varphi \approx \psi$ is of the form (1.iii), then the identities $q_f(\varphi) \approx q_h(\varphi)$ and $q_f(\psi) \approx q_h(\psi)$ are of the form (0.ii), so $\varphi \approx \psi$ holds in $\mathfrak{A}/R_f$.

We define a relation R_0 on A putting, for $a, b \in A$,

$$a \, R_0 \, b \ \text{ iff } \ a = b \text{ or for some } f_1, f_2 \in F \text{ we have } q_{f_1}^{\mathfrak{A}}(a) = a \text{ and } q_{f_2}^{\mathfrak{A}}(b) = b.$$

Lemma 1.6. *The relation* R_0 *is a congruence of* $\mathfrak{A}$.

P r o o f. Clearly, R_0 is an equivalence. It satisfies the superposition property since for every $s \in F$ and $a_1, \ldots, a_{\tau(s)} \in A$ we have

$$q_s^{\mathfrak{A}}(s^{\mathfrak{A}}(a_1, \ldots, a_{\tau(s)})) \overset{V^{c,2}}{=} s^{\mathfrak{A}}(a_1, \ldots, a_{\tau(s)}).$$

We denote by $V(0)$ the variety of 0-algebras of type τ, i.e. the variety defined by all identities $\varphi \approx \psi$ of type τ with $F(\varphi) \neq \emptyset \neq F(\psi)$ (see [13]). This means that in every algebra from $V(0)$ the value of every fundamental operation and every term function is equal to one fixed constant e_0.

Lemma 1.7. *If* $f \in F$, *then*

$$\begin{aligned} q_f^{\mathfrak{A}}(A) &= \{x : x \in A,\ q_f^{\mathfrak{A}}(x) = x\} \\ &= \Big\{x : \bigvee_{a_1, \ldots, a_{\tau(f)} \in A} f^{\mathfrak{A}}(a_1, \ldots, a_{\tau(f)}) = x\Big\}. \end{aligned}$$

P r o o f. If $a \in q_f^{\mathfrak{A}}(A)$, then there is $b \in A$ with $q_f^{\mathfrak{A}}(b) = a$. So

$$q_f^{\mathfrak{A}}(a) = q_f^{\mathfrak{A}}(q_f^{\mathfrak{A}}(b)) \overset{V^{c,2}}{=} q_f^{\mathfrak{A}}(b) = a$$

by (0.iv). If $q_f^{\mathfrak{A}}(a) = a$, then since q_f is a term different from a variable, the outermost fundamental operation symbol occurring in q_f is f. Thus the last condition of the statement holds.

If $a = f^{\mathfrak{A}}(b_1, \ldots, b_{\tau(f)})$, then

$$q_f^{\mathfrak{A}}(a) = q_f^{\mathfrak{A}}(f^{\mathfrak{A}}(b_1, \ldots, b_{\tau(f)})) \overset{V^{c,2}}{=} f^{\mathfrak{A}}(b_1, \ldots, b_{\tau(f)}) = a,$$

which completes the proof.

We define $\mathbf{0} = \bigcup_{f \in F} q_f^{\mathfrak{A}}(A)$.

Lemma 1.8. *The algebra* $\mathfrak{A}/R_0$ *belongs to* $V(0)$.

P r o o f. This follows from the fact that by Lemma 1.7 one of the congruence classes of R_0 is $\mathbf{0}$ and the remaining classes are singletons.

Lemma 1.9. *The congruence* $R_\cap = R_h \cap \bigcap_{f \in F} R_f \cap R_0$ *coincides with* ω, *the equality in* A.

P r o o f. Let $a, b \in A$. We assume

$$a \neq b. \tag{1.1}$$

We show that one of the congruences R_h, R_f, R_0 separates a and b.

If $a, b \in A \setminus \mathbf{0}$, or $a \in A \setminus \mathbf{0}$ and $b \in \mathbf{0}$, or $a \in \mathbf{0}$ and $b \in A \setminus \mathbf{0}$, then R_0 separates a and b by Lemma 1.7.

Let

$$a, b \in \mathbf{0} \quad \text{and} \quad a, b \in q_f^{\mathfrak{A}}(A) \quad \text{for some } f \in F. \tag{1.2}$$

Then by Lemma 1.7 we have

(1.3) $$q_f^{\mathfrak{A}}(a) = a \quad \text{and} \quad q_f^{\mathfrak{A}}(b) = b.$$

We show that either $\langle a, b\rangle \notin R_f$ or $\langle a, b\rangle \notin R_h$. We cannot have $q_f^{\mathfrak{A}}(a) = q_f^{\mathfrak{A}}(b)$ by (1.1) and (1.3). If $q_f^{\mathfrak{A}}(a) = q_h^{\mathfrak{A}}(a)$ and $q_f^{\mathfrak{A}}(b) = q_h^{\mathfrak{A}}(b)$, then $\langle a, b\rangle \notin R_h$ by (1.3).

Let (1.1) hold and

(1.4) $$a \in q_f^{\mathfrak{A}}(A) \text{ and } b \in q_g^{\mathfrak{A}}(A) \setminus q_f^{\mathfrak{A}}(A) \quad \text{for some distinct } f, g \in F.$$

We show that $\langle a, b\rangle \notin R_g$. We cannot have $q_g^{\mathfrak{A}}(a) = q_g^{\mathfrak{A}}(b)$ since $q_g^{\mathfrak{A}}(b) = b \in q_g^{\mathfrak{A}}(A) \setminus q_f^{\mathfrak{A}}(A)$ by Lemma 1.7 and $q_g^{\mathfrak{A}}(a) = q_g^{\mathfrak{A}}(q_f^{\mathfrak{A}}(a)) = q_f^{\mathfrak{A}}(q_g^{\mathfrak{A}}(a)) \in q_f^{\mathfrak{A}}(A)$. Also, neither $q_g^{\mathfrak{A}}(a) = q_h^{\mathfrak{A}}(a)$ nor $q_g^{\mathfrak{A}}(b) = q_h^{\mathfrak{A}}(b)$ since $q_g^{\mathfrak{A}}(b) = b \in q_g^{\mathfrak{A}}(A) \setminus q_f^{\mathfrak{A}}(A)$ and $q_h^{\mathfrak{A}}(b) = q_f^{\mathfrak{A}}(q_g^{\mathfrak{A}}(b)) \in q_f^{\mathfrak{A}}(A)$. Thus $\langle a, b\rangle \notin R_\cap$, which completes the proof.

In the sequel we adopt the usual notation (see [1], [3]). For two varieties V_1 and V_2 of type τ the notation $V_1 \subseteq V_2$ means that $\mathrm{Id}(V_2) \subseteq \mathrm{Id}(V_1)$. $V_1 \vee V_2$ denotes the join of V_1 and V_2. $\bigvee_{i\in I} V_i$ denotes the join of the family $\{V_i\}_{i\in I}$ of varieties. Finally, $\bigotimes_{i\in I} V_i$ is the class of all algebras isomorphic to a subdirect product of the family $\{\mathfrak{A}_i\}_{i\in I}$ of algebras where $\mathfrak{A}_i$ runs over V_i for every $i \in I$.

For a variety V satisfying (0.iii) we put $V = V(q)$ and let $I = \{q\} \cup F \cup \{0\}$.

THEOREM 1.10. *If a variety V satisfies* (0.iii), *then*

$$\bigvee_{i\in I} V(i) = V^{\mathrm{c},2} = \bigotimes_{i\in I} V(i).$$

P r o o f. It is easy to see that $V \subseteq V^{\mathrm{c},2}$, $V(f) \subseteq V^{\mathrm{c},2}$ for every $f \in F$ and $V_0 \subseteq V^{\mathrm{c},2}$. Thus $\bigvee_{i\in I} V(i) \subseteq V^{\mathrm{c},2}$. By Lemmas 1.1–1.9 and the subdirect decomposition theorem we have $V^{\mathrm{c},2} \subseteq \bigotimes_{i\in I} V(i)$. Then the inclusion $\bigotimes_{i\in I} V(i) \subseteq \bigvee_{i\in I} V(i)$ is obvious.

2. Subdirectly irreducible algebras. An algebra $\mathfrak{A}$ of type τ is said to be *subdirectly irreducible* if for every family $\{R_t\}_{t\in T}$ of congruences of $\mathfrak{A}$ we have:

$$\text{If } \bigcap_{t\in T} R_t = \omega, \text{ then there is } t_0 \in T \text{ with } R_{t_0} = \omega.$$

We shall not consider 1-element algebras to be subdirectly irreducible.

Theorem 1.10 is useful for finding subdirectly irreducible algebras in $V^{\mathrm{c},2}$ since we have

COROLLARY 2.1. *Let V be a variety of type τ satisfying* (0.iii) *and let $\mathfrak{A}$ be a subdirectly irreducible algebra. Then $\mathfrak{A}$ belongs to $V^{\mathrm{c},2}$ iff $\mathfrak{A}$ belongs to one of the varieties V, $V(f)$ for some $f \in F$ or $V(0)$.*

P r o o f. This follows at once from Theorem 1.10.

Before studying subdirectly irreducible algebras we need some properties of the varieties listed in Corollary 2.1.

Corollary 2.2. *Let a variety V of type τ satisfy* (0.iii) *and for some $f \in F$ let V satisfy the semilattice identities*: $f(x,x) \approx x$, $f(x,y) \approx f(y,x)$, $f(f(x,y),z) \approx f(x,f(y,z))$. *Then an algebra $\mathfrak{A}$ belongs to $V(f)$ iff it is a semilattice with respect to $f^{\mathfrak{A}}$, where e_f is* 1 *if $f^{\mathfrak{A}}$ is the join semilattice operation and e_f is* 0 *if $f^{\mathfrak{A}}$ is the meet semilattice operation.*

P r o o f. This follows from Lemma 1.4 since $\mathfrak{A}$ satisfies $f(x,e_f) \approx e_f$ for every x from $\mathfrak{A}$.

Corollary 2.3. *Under the assumptions of Corollary* 2.2, *a nontrivial algebra $\mathfrak{A}$ of type τ belongs to $V(f)$ and is subdirectly irreducible iff $\mathfrak{A}$ is of the form $(\{a,e_f\};F^{\mathfrak{A}})$ where $f^{\mathfrak{A}}(a,a) = a$, $f^{\mathfrak{A}}(x,y) = e_f$ otherwise*; *$s^{\mathfrak{A}}(x_1,\ldots,x_{\tau(s)}) = e_f$ for every $s \in F \setminus \{f\}$ and $x_1,\ldots,x_{\tau(s)} \in \{a,e_f\}$.*

P r o o f. The sufficiency follows from Corollary 2.2 and the fact that a 2-element algebra is always subdirectly irreducible. The necessity follows from Corollary 2.2 where the proof that $\mathfrak{A}$ must be 2-element is analogous to the standard proof for common semilattices.

It was observed by I. Chajda (see [3]) that

Lemma 2.4. *A* 0*-algebra $\mathfrak{A}$ is subdirectly irreducible iff it is* 2*-element.*

P r o o f. If $\mathfrak{A} = (A;F^{\mathfrak{A}})$ is a 0-algebra of type τ with $|A| > 2$, then take three different elements a, b, e_0. Consider two partitions P_1 and P_2 of A where P_1 contains the 2-element block $\{a,e_0\}$ and the remaining blocks are singletons, and P_2 contains the block $\{b,e_0\}$ and the remaining blocks are singletons. Then P_1 and P_2 induce two nontrivial congruences R_1 and R_2 of $\mathfrak{A}$ such that $R_1 \cap R_2 = \omega$. Thus $\mathfrak{A}$ is subdirectly irreducible.

Let $\tau_l : \{+,\cdot\} \to \mathbb{N}$ be a type of algebras with $\tau_l(+) = \tau_l(\cdot) = 2$. Let us consider three algebras $\mathfrak{A}_+$, $\mathfrak{A}_.$ and $\mathfrak{A}_0$ defined as follows:

- $\mathfrak{A}_+ = (\{a,e_+\};+,\cdot)$ where

$$x + y = \begin{cases} x & \text{if } x = y, \\ e_+ & \text{otherwise,} \end{cases} \qquad (2.1)$$

$$x \cdot y = e_+ \quad \text{for } x,y \in \{a,e_+\};$$

• $\mathfrak{A}_{\cdot} = (\{a, e_{\cdot}\}; +, \cdot)$ where

$$x \cdot y = \begin{cases} x & \text{if } x = y, \\ e_{\cdot} & \text{otherwise}, \end{cases} \tag{2.2}$$
$$x + y = e_{\cdot} \quad \text{for } x, y \in \{a, e_{\cdot}\};$$

• $\mathfrak{A}_0 = (\{a, e_0\}; +, \cdot)$ where

$$x + y = x \cdot y = e_0 \quad \text{for } x, y \in \{a, e_0\}. \tag{2.3}$$

THEOREM 2.5. *Let L be a variety of lattices of type τ_l and let $\mathfrak{A}$ be a subdirectly irreducible algebra of type τ_l. Then $\mathfrak{A}$ belongs to $L^{c,2}$ iff $\mathfrak{A}$ belongs to L or $\mathfrak{A}$ is isomorphic to one of the algebras $\mathfrak{A}_+$, $\mathfrak{A}_{\cdot}$ or $\mathfrak{A}_0$.*

Proof. The variety L satisfies (0.iii) since it satisfies

$$x + x \approx x \cdot x \approx x. \tag{2.4}$$

By Corollary 2.1 it is enough to show that the algebras listed in the statement are all subdirectly irreducible algebras from L_+, $L_{\cdot}$, L_0. But thisi follows from Corollary 2.3 and Lemma 2.4, respectively.

COROLLARY 2.6. *Let D be the variety of distributive lattices of type τ_l and let $\mathfrak{A}$ be a subdirectly irreducible algebra of type τ_l. Then $\mathfrak{A}$ belongs to $D^{c,2}$ iff $\mathfrak{A}$ is a 2-element lattice or $\mathfrak{A}$ is isomorphic to one of the algebras $\mathfrak{A}_+$, $\mathfrak{A}_{\cdot}$ or $\mathfrak{A}_0$.*

Proof. This follows from Theorem 2.5 and from the fact that a nontrivial subdirectly irreducible distributive lattice must be 2-element.

For a variety V of type τ we denote by V_{r} the variety of type τ defined by all regular identities from $\mathrm{Id}(V)$. In [6] the notion of a supalgebra of an algebra $\mathfrak{A}$ was defined as follows: let $\mathfrak{A} = (A; F^{\mathfrak{A}})$ be an algebra of type τ and let $b \notin A$. The algebra $\mathfrak{A}^\star = (A \cup \{b\}; F^{\mathfrak{A}^\star})$ is a *supalgebra* of $\mathfrak{A}$ if for every $f \in F$ we have

$$f^{\mathfrak{A}^\star}(a_1, \ldots, a_{\tau(f)}) = \begin{cases} f^{\mathfrak{A}}(a_1, \ldots, a_{\tau(f)}) & \text{if } a_1, \ldots, a_{\tau(f)} \in A, \\ b & \text{otherwise}. \end{cases}$$

In [7] the following was proved.

LEMMA 2.7. *Let V be a variety of type τ such that for some term $\varphi(x, y)$ the identity $\varphi(x, y) \approx x$ belongs to $\mathrm{Id}(V)$. Moreover, let $\mathfrak{A}$ be a subdirectly irreducible algebra of type τ. Then $\mathfrak{A}$ belongs to V_{r} iff $\mathfrak{A}$ belongs to V or $\mathfrak{A}$ is a supalgebra of a 1-element algebra from V, or $\mathfrak{A}$ is a supalgebra of a subdirectly irreducible algebra from V.*

COROLLARY 2.8. *Let $\mathfrak{A}$ be a subdirectly irreducible algebra of type τ_l. Then $\mathfrak{A}$ belongs to $D^{c,2}_{\mathrm{r}}$ iff one of the following cases holds:*

(d$_1$) $\mathfrak{A}$ *is a* 2-*element lattice*,
(d$_2$) $\mathfrak{A}$ *is a supalgebra of a* 1-*element lattice*,
(d$_3$) $\mathfrak{A}$ *is a supalgebra of a* 2-*element lattice*,
(d$_4$) $\mathfrak{A}$ *is isomorphic to one of the algebras* $\mathfrak{A}_+$, $\mathfrak{A}_\cdot$, $\mathfrak{A}_0$.

Proof. In fact, D_{r} satisfies (0.iii) since it satisfies (2.4). Now our corollary follows from Corollary 2.1, Lemma 2.7, Corollary 2.3 and Lemma 2.4 since $D \models x + x \cdot y \approx x$.

Let $\tau_b : \{+, \cdot, '\} \to \mathbb{N}$ be a type of algebras where $\tau_b(+) = \tau_b(\cdot) = 2$ and $\tau_b(') = 1$. Let B be the variety of Boolean algebras of type τ_b.

Let us consider the following two algebras $\mathfrak{B}^1_{\prime}$ and $\mathfrak{B}^2_{\prime}$ of type τ_b:

• $\mathfrak{B}^1_{\prime} = (\{a, b, e_{\prime}\}; +, \cdot, ')$ where $a' = b$, $b' = a$, $(e_{\prime})' = e_{\prime}$, $x + y = x \cdot y = e_{\prime}$ for every $x, y \in \{a, b, e_{\prime}\}$;

• $\mathfrak{B}^2_{\prime} = (\{a, e_{\prime}\}; +, \cdot, ')$ where $a' = a$, $(e_{\prime})' = e_{\prime}$, $x + y = x \cdot y = e_{\prime}$ for every $x, y \in \{a, e_{\prime}\}$.

Lemma 2.9. *Let $\mathfrak{A}$ be a subdirectly irreducible algebra of type τ_b. Then $\mathfrak{A}$ belongs to $B(')$ iff $\mathfrak{A}$ is of the form $\mathfrak{B}^1_{\prime}$ or $\mathfrak{B}^2_{\prime}$.*

Proof. By Lemma 1.4 the variety $B(')$ satisfies $(x')' \approx x$ and the value of the operations $+$ and $\cdot$ in every algebra $\mathfrak{A}$ from $B(')$ is equal to $e_{\prime}$.

Let $\mathfrak{A} = (A; +, \cdot, ')$ be an algebra from $B(')$. The set generated in A by an element $p \in A$ by means of the operation $'$ will be called the $'$-*component generated by* p and denoted by $[p]$. Observe that every $'$-component is 1- or 2-element and $B(')$ satisfies $x \cdot y \approx (x \cdot y)'$ (see (1.iii)). If there are at least three components in A, say $[e_{\prime}]$, C_1 and C_2, then consider two partitions P_1 and P_2 of A where the blocks of P_1 are $C_1 \cup [e_{\prime}]$, and the other blocks are singletons; the blocks of P_2 are $C_2 \cup [e_{\prime}]$, and the other blocks are singletons. Then P_1 and P_2 induce two congruences R_1 and R_2 of $\mathfrak{A}$ which are nontrivial and $R_1 \cap R_2 = \omega$. Thus $\mathfrak{A}$ is subdirectly irreducible. Obviously, $\mathfrak{B}^1_{\prime}$ and $\mathfrak{B}^2_{\prime}$ are subdirectly irreducible and they are the only possible ones up to isomorphism.

Let us consider the following three algebras $\mathfrak{B}_+$, $\mathfrak{B}_\cdot$, $\mathfrak{B}_0$, of type τ_b:

• $\mathfrak{B}_+ = (\{a, e_+\}; +, \cdot, ')$ where $+$ is defined by (2.1) and $x \cdot y = x' = e_+$ for every $x, y \in \{a, e_+\}$;

• $\mathfrak{B}_\cdot = (\{a, e_\cdot\}; +, \cdot, ')$ where $\cdot$ is defined by (2.2) and $x + y = x' = e_\cdot$ for every $x, y \in \{a, e_\cdot\}$;

• $\mathfrak{B}_0 = (\{a, e_0\}; +, \cdot, ')$ where $x + y = x \cdot y = x' = e_0$ for every $x, y \in \{a, e_0\}$.

Theorem 2.10. *Let $\mathfrak{A}$ be a subdirectly irreducible algebra of type τ_b. Then $\mathfrak{A}$ belongs to $B^{\mathrm{c},2}$ iff it is a 2-element Boolean algebra or is of the form $\mathfrak{B}_+$, $\mathfrak{B}_\cdot$, $\mathfrak{B}^1_{\prime}$, $\mathfrak{B}^2_{\prime}$ or $\mathfrak{B}_0$.*

P r o o f. Obviously B satisfies (0.iii) since it satisfies

$$x + x \approx x \cdot x \approx (x')' \approx x. \tag{2.5}$$

Now, the theorem holds by Corollary 2.1, Corollary 2.3, Lemma 2.9 and Corollary 2.4.

COROLLARY 2.11. *Let* $\mathfrak{A}$ *be a subdirectly irreducible algebra of type* τ_b. *Then* $\mathfrak{A}$ *belongs to* $B_{\mathrm{r}}^{c,2}$ *iff one of the following cases holds*:

(k_1) $\mathfrak{A}$ *is a* 2*-element Boolean algebra*,
(k_2) $\mathfrak{A}$ *is a supalgebra of a* 2*-element Boolean algebra*,
(k_3) $\mathfrak{A}$ *is a supalgebra of a* 1*-element algebra of type* τ_b,
(k_4) $\mathfrak{A}$ *is isomorphic to one of the algebras* $\mathfrak{B}_+$, $\mathfrak{B}_\cdot$, $\mathfrak{B}_{\prime}^1$, $\mathfrak{B}_{\prime}^2$, $\mathfrak{B}_0$.

P r o o f. Obviously, B_{r} satisfies (0.iii) since it satisfies (2.5). Now, our theorem follows from Corollary 2.1, Lemma 2.7, Corollaries 2.3 and 2.4, and Lemma 2.9.

Let $\tau_g : \{\cdot, ^{-1}\} \to \mathbb{N}$ be a type of algebras with $\tau_g(\cdot) = 2$ and $\tau_g(^{-1}) = 1$. Let G_n be the variety of groups of type τ_g satisfying $x^n \approx y^n$ for some $n > 2$. We have

LEMMA 2.12. *The variety* $G_n(\cdot)$ *is trivial.*

P r o o f. In $G_n(\cdot)$ we have $x \approx x \cdot x^n \approx x \cdot y^n \approx x \cdot (y^{-1})^n \approx y^{-1}$.

Let us consider the following two algebras $\mathfrak{G}_{-1}^1$ and $\mathfrak{G}_{-1}^2$ of type τ_g:

• $\mathfrak{G}_{-1}^1 = (\{a, b, e_{-1}\}; \cdot, ^{-1})$ where $a^{-1} = b$, $b^{-1} = a$, $(e_{-1})^{-1} = e_{-1}$ and $x \cdot y = e_{-1}$ for every $x, y \in \{a, b, e_{-1}\}$;
• $\mathfrak{G}_{-1}^2 = (\{a, e_{-1}\}; \cdot, ^{-1})$ where $a^{-1} = a$, $(e_{-1})^{-1} = e_{-1}$ and $x \cdot y = e_{-1}$ for every $x, y \in \{a, e_{-1}\}$.

LEMMA 2.13. *Let* $\mathfrak{A}$ *be a subdirectly irreducible algebra of type* τ_g. *Then* $\mathfrak{A}$ *belongs to* $G_n(^{-1})$ *iff* $\mathfrak{A}$ *is isomorphic to* $\mathfrak{G}_{-1}^1$ *or to* $\mathfrak{G}_{-1}^2$.

The proof is quite similar to that of Lemma 2.9.

THEOREM 2.14. *Let* $\mathfrak{A}$ *be a subdirectly irreducible algebra of type* τ_g. *Then* $\mathfrak{A}$ *belongs to* $G_n^{c,2}$ *iff* $\mathfrak{A}$ *belongs to* G_n *or* $\mathfrak{A}$ *is isomorphic to one of the algebras* $\mathfrak{G}_{-1}^1$, $\mathfrak{G}_{-1}^2$, *or* $\mathfrak{A}$ *is a* 2*-element* 0*-algebra of type* τ_g.

P r o o f. G_n satisfies (0.iii) since it satisfies $x^{n+1} \approx (x^{-1})^{-1} \approx x$. Now, the theorem follows from Corollary 2.1, Corollary 2.3, Lemma 2.13 and Corollary 2.4.

By means of subdirectly irreducible algebras of some variety one can describe the lattice of its subvarieties. For $V^{c,2}$ this will be done elsewhere.

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