A COUNTEREXAMPLE TO A CONJECTURE OF BASS, CONNELL AND WRIGHT

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Let \( F = X - H : k^n \to k^n \) be a polynomial map with \( H \) homogeneous of degree 3 and nilpotent Jacobian matrix \( J(H) \). Let \( G = (G_1, \ldots, G_n) \) be the formal inverse of \( F \). Bass, Connell and Wright proved in [1] that the homogeneous component of \( G_i \) of degree \( 2d + 1 \) can be expressed as \( G_i^{(d)} = \sum_T \alpha(T)^{-1} \sigma_i(T) \), where \( T \) varies over rooted trees with \( d \) vertices, \( \alpha(T) = \text{Card Aut}(T) \) and \( \sigma_i(T) \) is a polynomial defined by (1) below. The Jacobian Conjecture states that, in our situation, \( F \) is an automorphism or, equivalently, \( G_i^{(d)} \) is zero for sufficiently large \( d \). Bass, Connell and Wright conjecture that not only \( G_i^{(d)} \) but also the polynomials \( \sigma_i(T) \) are zero for large \( d \).

The aim of the paper is to show that for the polynomial automorphism (4) and rooted trees (3), the polynomial \( \sigma_2(T_s) \) is non-zero for any index \( s \) (Proposition 4), yielding a counterexample to the above conjecture (see Theorem 5).

1. Preliminaries. Throughout the paper \( k \) is a field of characteristic zero. A polynomial map from \( k^n \) to \( k^n \) is called a polynomial automorphism if it has an inverse that is also a polynomial map. The sequence \( X = (X_1, \ldots, X_n) \) denotes the identity automorphism and \( J(F) \) denotes the Jacobian matrix of \( F \).

Conjecture 1 (Jacobian Conjecture). If \( F = (F_1, \ldots, F_n) : k^n \to k^n \) is a polynomial map and \( \det J(F) \in k \setminus \{0\} \), then \( F \) is a polynomial automorphism.

For a historical survey and detailed introduction to the subject see [1]. The Jacobian Conjecture is still open for all \( n \geq 2 \).

Yagzhev [4] and Bass, Connell and Wright in [1] proved that it suffices to prove the Jacobian Conjecture for all \( n \geq 2 \) and polynomial maps of the
form $F_i = X_i - H_i$, where for $i = 1, \ldots, n$ the polynomial $H_i$ is homogeneous of degree $3$.

Note that if $F = X - H$, where $H_1, \ldots, H_n$ are homogeneous of degree $\geq 2$, then the condition $\det J(F) \in k \setminus \{0\}$ is equivalent to the nilpotency of $J(H)$ ([1, Lemma 4.1]).

2. The tree expansion of the formal inverse. We recall some definitions and facts from [1] (see also [3]).

Let $F : k^n \to k^n$ be a polynomial map of the form $F_i = X_i - H_i$, where each $H_i$ is homogeneous of degree $\delta \geq 2$ ($i = 1, \ldots, n$). It is well known ([1, Chapter III]) that for $F$ there exist unique formal power series $G_1, \ldots, G_n \in k[[X_1, \ldots, X_n]]$ defined by the conditions $G_i(F_1, \ldots, F_n) = X_i$ for $i = 1, \ldots, n$. We call $G = (G_1, \ldots, G_n)$ the formal inverse of $F$.

One can write $G_i = \sum_{d \geq 0} G_i^{(d)}$, where the component $G_i^{(d)}$ is a homogeneous polynomial of degree $d(\delta - 1) + 1$.

It is obvious that the Jacobian Conjecture is true if and only if $G_i$ is a polynomial for $i = 1, \ldots, n$.

If $T$ is a non-directed tree, then $V(T)$ denotes its set of vertices and (the symmetric subset) $E(V) \subseteq V(T) \times V(T)$ is the set of edges. A rooted tree $T$ is defined as a tree with a distinguished vertex $rt_T \in V(T)$ called a root.

We define, by induction on $j$, the sets $V_j(T)$ of vertices of height $j$. Let $V_0(T) = \{rt_T\}$ and for $j > 0$ let $v \in V_j(T)$ iff there exists $w \in V_{j-1}(T)$ such that $(w, v) \in E(T)$ and $v \not\in V_i(T)$ for $i < j$.

For $v \in V_j(T)$ we set

$$v^+ = \{w \in V_{j+1}(T) : (w, v) \in E(T)\}.$$

Rooted trees form a category in which a morphism $T \to T'$ is a map $f : V(T) \to V(T')$ such that $f(rt_T) = rt_{T'}$ and $(f \times f)(E(T)) \subseteq E(T')$. For a rooted tree $T$ we denote by $\text{Aut}(T)$ the group of all automorphisms of $T$, and $\alpha(T) = \text{Card Aut}(T)$. Moreover, $T_d$ denotes the set of representatives of isomorphism classes of rooted trees with $d$ vertices.

Suppose now that $H = (H_1, \ldots, H_n)$ and $H_1, \ldots, H_n \in k[X_1, \ldots, X_n]$ are homogeneous of degree $\delta \geq 2$. For a particular $i \in \{1, \ldots, n\}$, a rooted tree $T$ and an $i$-rooted labeling $f$ of $T$ (that is, by definition, a function $f : V(T) \to \{1, \ldots, n\}$ such that $f(rt_T) = i$) we define polynomials

$$P_{T,f} = \prod_{v \in V(T)} \left( \prod_{w \in v^+} D_{f(w)} H_{f(v)} \right)$$

and

$$\sigma_i(T) = \sum_f P_{T,f}$$

($f$ varies over all $i$-rooted labelings of $T$).
Using the above assumptions and definitions we can quote the following theorem ([1, Ch. III, Theorem 4.1], [3, Theorem 4.3]).

**Theorem 2** (Bass, Connell, Wright). If the matrix $J(H)$ is nilpotent, then $G^{(0)}_i = X_i$, and for $d \geq 1$, 

$$G^{(d)}_i = \sum_{T \in T_d} \frac{1}{\alpha(T)} \sigma_i(T).$$

Let $[J(H)^e]$ denote the differential ideal of $k[X_1, \ldots, X_n]$ generated by all entries of $J(H)^e$, that is, the ideal generated by elements of the form $D^{p_1}_1 \ldots D^{p_n}_n f$ for any $(p_1, \ldots, p_n) \in \mathbb{N}^n$ and any entry $f$ of $J(H)^e$.

Let us formulate the following conjecture which is the main object of interest in our paper ([1, Ch. III, Conjecture 5.1], [4, 5.2]).

**Conjecture 3** (Bass, Connell, Wright). If $e \geq 1$, then there is an integer $d(e)$ such that for all $d \geq d(e)$, $T \in T_d$ and $i = 1, \ldots, n$ we have $\sigma_i(T) \in [J(H)^e]$.

If Conjecture 3 is true for $e = 3$, then the Jacobian Conjecture is also true. Indeed, if $F = X - H : k^n \to k^n$, det $J(H) = 1$ and $H_i$ are homogeneous of degree 3, then the matrix $J(H)$ is nilpotent. Hence $J(H)^n = 0$ and, by Conjecture 3, for all $T \in T_d$, $d \geq d(n)$ and $i = 1, \ldots, n$, we have $\sigma_i(T) = 0$. Substituting this into (2) we get $G^{(d)}_i = 0$ for $d \geq d(n)$, so $G_i$ are polynomials and $F$ is an automorphism.

### 3. A counterexample

Let us define the following sequence of rooted trees:

$$T_0 = \quad \in T_4$$

$$T_s = \quad \in T_{2s+4} \quad \text{for } s \geq 1,$$

where always the lowest vertex is a root.

**Proposition 4.** For the polynomial endomorphism $F : k^4 \to k^4$ defined by

$$F = (X_1 + X_4(X_1 X_3 + X_2 X_4),$$

$$X_2 - X_3(X_1 X_3 + X_2 X_4), X_3 + X_4^3, X_4)$$
and rooted trees $T_s$, $s \geq 0$, defined by (3), we have

$$\sigma_1(T_s) = 0, \quad \sigma_2(T_s) = (-1)^{s+1} \cdot 6 \cdot X_4^{4s+7}(X_1X_3 + X_2X_4),$$

$$\sigma_3(T_s) = 0, \quad \sigma_4(T_s) = 0.$$

**Proof.** The endomorphism $F$ has the form $X - H$, where

$$H_1 = -X_1X_3X_4 - X_2X_4^2, \quad H_2 = X_1X_3^2 + X_2X_3X_4,$$

$$H_3 = -X_4^3, \quad H_4 = 0.$$

We proceed by induction on $s$.

Let $s = 0$. Let $V(T_0) = \{\text{rt}T_0 = 0, 1, 2, 3\}$. Then, for $i = 1, 2, 3, 4$,

$$\sigma_i(T_0) = \sum_{f: V(T_0) \to \{1, 2, 3, 4\}} \prod_{v \in V(T_0)} \left( \prod_{w \in v^+} D_f(w) \right)H_{f(v)}$$

$$= \sum_{f: \{1, 2, 3\} \to \{1, 2, 3, 4\}} D_{f(1)}D_{f(2)}D_{f(3)}H_{f(1)} \cdot H_{f(2)} \cdot H_{f(3)}.$$

It is obvious that $D_{a_1}D_{a_2}D_{a_3}X_{b_1}X_{b_2}X_{b_3}$ can be non-zero only if the sequences $(a_1, a_2, a_3)$ and $(b_1, b_2, b_3)$ have the same elements up to order. Hence, by (5), we have

$$\sigma_1(T_0) = 6 \cdot D_1D_3D_4H_1 \cdot H_1H_3H_4 + 3 \cdot D_2D_4D_4H_1 \cdot H_2H_3^2 = 0,$$

$$\sigma_2(T_0) = 3 \cdot D_1D_3D_3H_2 \cdot H_1H_3^2 + 6 \cdot D_2D_3D_4H_2 \cdot H_2H_3H_4$$

$$= -6 \cdot X_4(X_1X_3 + X_2X_4) \cdot (-X_4^3)^2$$

$$= (-1)^1 \cdot 6 \cdot X_4^7(X_1X_3 + X_2X_4),$$

$$\sigma_3(T_0) = D_4D_4D_4H_3 \cdot H_3^3 = 0,$$

$$\sigma_4(T_0) = 0.$$

Let $s \geq 0$ and assume that the statement of the proposition holds for $s$. Then (it is a particular case of “tree surgery”; see [1] or [3])

$$\sigma_i(T_{s+1}) = \sum_{a=1}^{4} \left( \sum_{j=1}^{4} D_jD_aH_i \cdot H_j \right) \cdot \sigma_a(T_s).$$

By assumption, $\sigma_a(T_s) = 0$ for $a \neq 2$. Therefore

$$\sigma_i(T_{s+1}) = \left( \sum_{j=1}^{4} D_jD_2H_i \cdot H_j \right) \cdot \sigma_2(T_s)$$
and hence, by (5) and the assumption,
\[
\begin{align*}
\sigma_1(T_{s+1}) &= D_4 D_2 H_1 \cdot H_4 \cdot \sigma_2(T_s) = 0, \\
\sigma_2(T_{s+1}) &= (D_3 D_2 H_2 \cdot H_3 + D_4 D_2 H_2 \cdot H_4) \cdot \sigma_2(T_s) \\
&= X_4 \cdot (-X_4^3) \cdot (-1)^{s+1} \cdot 6 \cdot X_4^{4s+7} (X_1 X_3 + X_2 X_4) \\
&= (-1)^{(s+1)+1} \cdot 6 \cdot X_4^{4(s+1)+7} (X_1 X_3 + X_2 X_4), \\
\sigma_3(T_{s+1}) &= 0, \\
\sigma_4(T_{s+1}) &= 0,
\end{align*}
\]
which completes the proof. ■

**Remark.** A. van den Essen [2] proved that the endomorphism \(F : \mathbb{C}^4 \to \mathbb{C}^4\) defined by (4) is a counterexample to a conjecture of Meisters.

**Theorem 5.** Conjecture 3 is false for \(\delta = 3\) and \(e \geq 4\).

**Proof.** Let \(F\) be the endomorphism defined by (4). Then \(F = X - H\), where \(H\) is homogeneous of degree \(\delta = 3\). One can verify that \(F\) is an automorphism and its inverse is
\[
F^{-1} = G = X + H + G^{(2)} + G^{(3)},
\]
where
\[
G^{(2)} = (X_1 X_4, -X_4^3 (2X_1 X_3 + X_2 X_4), 0, 0), \quad G^{(3)} = (0, X_1 X_4^6, 0, 0).
\]
Therefore \(G^{(d)} = 0\) for \(d \geq 4\).

Moreover, \(J(H)^3 \neq 0\) and \(J(H)^4 = 0\). Hence \([J(H)^e] = 0\) for \(e \geq 4\).

On the other hand, by Proposition 4, we have \(\sigma_2(T_s) \neq 0\) for \(s \geq 0\). Therefore \(\sigma_2(T_s) \notin [J(H)^e]\) for \(s \geq 0\) and \(e \geq 4\).

Since \(T_s \in T_{2s+4}\) and \(\lim_{s \to \infty} (2s + 4) = \infty\), for \(e \geq 4\) there is no \(d(e)\) as in Conjecture 3. ■

**4. Final remarks.** In [1, Proposition 5.3] it was shown that Conjecture 3 is true for \(e = 1\) with \(d(1) = 1\) and for \(e = 2\) with \(d(2) = 2\). We have proved in Theorem 5 that Conjecture 3 is false for \(e \geq 4\). The case \(e = 3\) remains open but the author’s computer calculations show that the following conjecture is plausible.

**Conjecture 6.** There is an integer \(d(3)\) with the following property. If \(H = (H_1, \ldots, H_n)\), the polynomials \(H_1, \ldots, H_n \in k[X_1, \ldots, X_n]\) are homogeneous of degree 3, and \(J(H)^3 = 0\), then for \(d \geq d(3)\), a rooted tree \(T \in T_d\) and all \(i = 1, \ldots, n\), the polynomial \(\sigma_i(T)\) equals zero.

It is evident that for \(e = 3\) Conjecture 3 implies Conjecture 6.

Computer calculations show that \(d(3) \geq 19\).
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