COLLOQUIUM MATHEMATICUM

VOL. 77 1998 NO. 2

GENERAL LOCAL COHOMOLOGY MODULES AND KOSZUL HOMOLOGY MODULES

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Introduction. Throughout this paper, A denotes a commutative Noetherian ring (with non-zero identity), and M a finitely generated A-module. Moreover, $\mathcal{C}(A)$ denotes the category of all A-modules and A-homomorphisms.

The interaction between local cohomology theory and torsion theories has been verified in [1], where it is shown that, over commutative Noetherian rings, the study of torsion theories is equivalent to the study of local cohomology in the general sense. The main purpose of this note is to establish a connection between general local cohomology modules and the homology modules of Koszul complexes.

Let us review the main concepts concerning the theory of general local cohomology modules. A *system of ideals* of A (see [1]) is a non-empty set Φ of ideals of A such that, whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}b$. Such a system Φ determines the Φ-torsion functor $\Gamma_\Phi : \mathcal{C}(A) \to \mathcal{C}(A)$. This is the subfunctor of the identity functor on $\mathcal{C}(A)$ for which

$$\Gamma_\Phi(G) = \{g \in G : \mathfrak{a}g = 0 \text{ for some } \mathfrak{a} \in \Phi\}$$

for each A-module G. Note that in [1], Γ_Φ is denoted by L_Φ and called the "general local cohomology functor with respect to Φ". For each $i \geq 0$, the ith right derived functor of Γ_Φ is denoted by H^i_Φ. Moreover, when Φ consists of only the powers of an ideal $\mathfrak{a}$, Γ_Φ is just the usual local cohomology functor with respect to $\mathfrak{a}$, and H^i_Φ is naturally equivalent to the functor $H^i_\mathfrak{a}$ for all $i \geq 0$.

For a rectangular set U (of order n) over A (see [5]), let $\Phi(U)$ be the set of all ideals which are generated by an element of U. Then $\Phi(U)$ is a system of ideals of A. One of the main results of [5] is Statement 2, which shows that, for each $i = 0, 1, \ldots, n-1$, $H^i_{\Phi(U)}(M) \cong \varinjlim_{x \in U} H^i(x, M)$, where $H^i(x, M)$ denotes the ith Koszul homology module of M with respect to x.

1991 *Mathematics Subject Classification*: 13D45, 13D25, 13B30.

In this paper we show that, for an arbitrary system Φ of ideals of A and every positive integer n, there exists a triangular subset $U_n(\Phi, M)$ of A^n such that, for all $i = 0, 1, \ldots, n-1$, $H^i_\Phi(M)$ can be realized as the direct limit of the system of the homology modules of the Koszul complexes $\{K^\bullet(x, M) : x \in U_n(\Phi, M)\}$, where $K^\bullet(x, M)$ denotes the Koszul complex of M with respect to x. Also, we show that, for all $i \geq 0$, the general local cohomology module $H^i_\Phi(M)$ may be viewed as the ith homology module of a certain complex of A-modules and A-homomorphisms which involves modules of generalized fractions derived from the family $(U_n(\Phi, M))_{n\in\mathbb{N}}$ of triangular subsets. Note that the latter result was obtained in [3] for a certain system Φ of ideals which are derived from a triangular subset of A^n.

1. Preliminaries. Throughout this section Φ is a system of ideals of A and n is a positive integer. We use $\mathbb{N}$ (respectively $\mathbb{N}_0$) to denote the set of positive (respectively non-negative) integers and $D_n(A)$ to denote the set of $n \times n$ lower triangular matrices over A. For $H \in D_n(A)$, $|H|$ denotes the determinant of H.

A *triangular subset* of A^n (see [11]) is a non-empty subset U of A^n such that (i) whenever $(x_1, \ldots, x_n) \in U$, then $(x_1^{t_1}, \ldots, x_n^{t_n}) \in U$ for all choices of positive integers $t_1, \ldots, t_n$; and (ii) whenever $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in U$ then there exist $(z_1, \ldots, z_n) \in U$ and $H, K \in D_n(A)$ such that $H[x_1 \ldots x_n]^T = [z_1 \ldots z_n]^T = K[y_1 \ldots y_n]^T$, where ^{T} denotes the matrix transpose.

Whenever we can do so without ambiguity, we denote $(x_1, \ldots, x_n) \in A^n$ by x and $[x_1 \ldots x_n]^T$ by x^T.

Now we recall the following definition and proposition.

1.1. Definition (see [10]). Let $\mathfrak{a}$ be an ideal of A. Let $x_1, \ldots, x_n$ be a sequence of elements of $\mathfrak{a}$. Then the sequence $x_1, \ldots, x_n$ is called an $\mathfrak{a}$-*filter regular* M-*sequence* if

$$x_i \notin \mathfrak{p} \quad \text{for all } \mathfrak{p} \in \operatorname{Ass}_A\Big(M/\Big(\sum_{r=1}^{i-1} Ax_r\Big)M\Big)\backslash V(\mathfrak{a})$$

for all $i = 1, \ldots, n$, where $V(\mathfrak{a})$ denotes the set of prime ideals of A containing $\mathfrak{a}$. It is easy to see that $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular M-sequence if and only if

$$\operatorname{Supp}((x_1, \ldots, x_i)M :_M x_{i+1}/(x_1, \ldots, x_i)M) \subseteq V(\mathfrak{a})$$

for all $i = 0, 1, \ldots, n-1$. Note that $x_1, \ldots, x_n$ is a poor M-sequence if and only if it is an A-filter regular M-sequence. It is easy to see that the analogue of [13, Appendix 2(ii)] holds whenever A is Noetherian, M is finitely generated and $\mathfrak{m}$ is replaced by $\mathfrak{a}$; so that, if $x_1, \ldots, x_n$ is an $\mathfrak{a}$-filter regular M-sequence, then there is an element $y \in \mathfrak{a}$ such that $x_1, \ldots, x_n, y$ is an

$\mathfrak{a}$-filter regular M-sequence. Thus, for every positive integer n, there exists an $\mathfrak{a}$-filter regular M-sequence of length n.

1.2. PROPOSITION (see [7]). *Let $\mathfrak{a}$ be an ideal of A and $x_1, \ldots, x_n$ be a sequence of elements of A. Then the following conditions are equivalent*:

(i) $x_1, \ldots, x_n$ *is an $\mathfrak{a}$-filter regular M-sequence*;

(ii) $\frac{x_1}{1}, \ldots, \frac{x_i}{1}$ *is a poor $M_{\mathfrak{p}}$-sequence in $A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \operatorname{Supp}(M) \setminus V(\mathfrak{a})$ and $i = 1, \ldots, n$*;

(iii) $x_1^{t_1}, \ldots, x_n^{t_n}$ *is an $\mathfrak{a}$-filter regular M-sequence for all $t_1, \ldots, t_n \in \mathbb{N}$.*

We shall need the following result.

1.3. LEMMA (see [14, Chapter II, 3.13 and 3.14]). *Let n be a positive integer and suppose that $x_1, \ldots, x_n$ and $y_1, \ldots, y_{n+1}$ are poor M-sequences. Then $(\sum_{r=1}^{n+1} Ay_r) \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A(M/(\sum_{r=1}^{n} Ax_r)M)$.*

Proof. Assume the contrary. Let $\mathfrak{p} \in \operatorname{Ass}_A(M/(\sum_{r=1}^{n} Ax_r)M)$ be such that $(\sum_{r=1}^{n+1} Ay_r) \subseteq \mathfrak{p}$.

We claim that, for all $j = 0, 1, \ldots, n$, $H_{\mathfrak{p}}^i(M/(\sum_{r=1}^{j} Ax_r)M) = 0$ for every $i = 0, 1, \ldots, n-j$. We prove this by induction on j. The case $j = 0$ is clear, since $y_1, \ldots, y_{n+1} \in \mathfrak{p}$ is an M-sequence. Assume, inductively, that j is an integer with $0 \le j < n$ and we have proved that $H_{\mathfrak{p}}^i(M/(\sum_{r=1}^{j} Ax_r)M) = 0$ for all $i = 0, 1, \ldots, n-j$. Consider the exact sequence

$$0 \to M/\Big(\sum_{r=1}^{j} Ax_r\Big)M \xrightarrow{x_{j+1}} M/\Big(\sum_{r=1}^{j} Ax_r\Big)M \to M/\Big(\sum_{r=1}^{j+1} Ax_r\Big)M \to 0.$$

It induces the long exact sequence

$$\begin{aligned} 0 &\to H_{\mathfrak{p}}^0\Big(M/\Big(\sum_{r=1}^{j} Ax_r\Big)M\Big) \xrightarrow{x_{j+1}} H_{\mathfrak{p}}^0\Big(M/\Big(\sum_{r=1}^{j} Ax_r\Big)M\Big) \\ &\to H_{\mathfrak{p}}^0\Big(M/\Big(\sum_{r=1}^{j+1} Ax_r\Big)M\Big) \to \ldots \to H_{\mathfrak{p}}^i\Big(M/\Big(\sum_{r=1}^{j} Ax_r\Big)M\Big) \\ &\xrightarrow{x_{j+1}} H_{\mathfrak{p}}^i\Big(M/\Big(\sum_{r=1}^{j} Ax_r\Big)M\Big) \to H_{\mathfrak{p}}^i\Big(M/\Big(\sum_{r=1}^{j+1} Ax_r\Big)M\Big) \to \ldots \end{aligned}$$

By the inductive hypothesis, $H_{\mathfrak{p}}^i(M/(\sum_{r=1}^{j+1} Ax_r)M) = 0$ for each $i = 0, 1, \ldots, n-(j+1)$. The claim now follows by induction. Therefore, in particular, $H_{\mathfrak{p}}^0(M/(\sum_{r=1}^{n} Ax_r)M) = 0$. Hence $\mathfrak{p} \notin \operatorname{Ass}_A(M/(\sum_{r=1}^{n} Ax_r)M)$, which is the required contradiction.

1.4. Proposition. *Let Φ be a system of ideals of A and n be a positive integer. Set*

$$U_n(\Phi, M) := \{(x_1, \ldots, x_n) \in A^n : x_1, \ldots, x_n \text{ is an } \mathfrak{a}\text{-filter regular } M\text{-sequence for some } \mathfrak{a} \in \Phi\}.$$

Then $U_n(\Phi, M)$ is a triangular subset of A^n.

Proof. As we mentioned earlier, for all $\mathfrak{a} \in \Phi$, there exists an $\mathfrak{a}$-filter regular M-sequence of length n. Hence $U_n(\Phi, M)$ is non-empty. Also, by 1.2, $(x_1^{t_1}, \ldots, x_n^{t_n}) \in U_n(\Phi, M)$ whenever $(x_1, \ldots, x_n) \in U_n(\Phi, M)$ and $t_1, \ldots, t_n \in \mathbb{N}$. Let $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in U_n(\Phi, M)$. Then there exist $\mathfrak{a}, \mathfrak{b} \in \Phi$ such that $x_1, \ldots, x_n$ (respectively $y_1, \ldots, y_n$) form an $\mathfrak{a}$-filter (respectively $\mathfrak{b}$-filter) regular M-sequence. Also, since Φ is a system of ideals of A, there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$.

We show that there exist $z_1, \ldots, z_n \in A$ which form a $\mathfrak{c}$-filter regular M-sequence with $z_i \in (\sum_{r=1}^{i} Ax_r) \cap (\sum_{r=1}^{i} Ay_r)$ for all $i = 1, \ldots, n$. To do this, let $\mathfrak{p} \in \operatorname{Ass}_A(M) \setminus V(\mathfrak{c})$. Then $\mathfrak{p} \in (\operatorname{Ass}_A(M) \setminus V(\mathfrak{a})) \cap (\operatorname{Ass}_A(M) \setminus V(\mathfrak{b}))$. So $\mathfrak{c} \cap Ax_1 \cap Ay_1 \not\subseteq \bigcup_{\mathfrak{p} \in \operatorname{Ass}_A(M) \setminus V(\mathfrak{c})} \mathfrak{p}$. Therefore there exists $z_1 \in \mathfrak{c} \cap Ax_1 \cap Ay_1$ such that $z_1 \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A(M) \setminus V(\mathfrak{c})$. Now suppose, inductively, that there exists a $\mathfrak{c}$-filter regular M-sequence $z_1, \ldots, z_l$ $(1 \le l < n)$ with $z_i \in (\sum_{r=1}^{i} Ax_r) \cap (\sum_{r=1}^{i} Ay_r)$ for all $i = 1, \ldots, l$. Let $\mathfrak{p} \in \operatorname{Ass}_A(M/(\sum_{r=1}^{l} Az_r)M) \setminus V(\mathfrak{c})$. Then $\mathfrak{p}A_{\mathfrak{p}} \in \operatorname{Ass}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}/(\sum_{r=1}^{l} A_{\mathfrak{p}} \frac{z_r}{1})M_{\mathfrak{p}})$. Also, by 1.2, $\frac{x_1}{1}, \ldots, \frac{x_{l+1}}{1}$ and $\frac{z_1}{1}, \ldots, \frac{z_l}{1}$ are poor $M_{\mathfrak{p}}$-sequences. Hence, by 1.3, $(\sum_{r=1}^{l+1} Ax_r) \not\subseteq \mathfrak{p}$. Similarly, $(\sum_{r=1}^{l+1} Ay_r) \not\subseteq \mathfrak{p}$. Thus there exists $z_{l+1} \in \mathfrak{c} \cap (\sum_{r=1}^{l+1} Ax_r) \cap (\sum_{r=1}^{l+1} Ay_r)$ such that $z_{l+1} \notin \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Ass}_A(M/(\sum_{r=1}^{l} Az_r)M) \setminus V(\mathfrak{c})$. Therefore the inductive step is complete.

2. The results. Let us, firstly, review the construction of the Koszul complex and establish our notations.

Let $x = x_1, \ldots, x_n$ be a sequence of elements of A. For $j \in \mathbb{N}$ with $1 \le j \le n$, we write

$$I(j, n) = \{(\alpha_1, \ldots, \alpha_j) \in \mathbb{N}^j : 1 \le \alpha_1 < \ldots < \alpha_j \le n\}.$$

We denote by e_α the exterior product $e_{\alpha_1} \wedge \ldots \wedge e_{\alpha_j}$ for all $\alpha = (\alpha_1, \ldots, \alpha_j) \in I(j, n)$. The (ascending) *Koszul complex of* A with respect to x has the form

$$K^\bullet(x) : 0 \to K^0(x) \xrightarrow{d^0(x)} K^1(x) \xrightarrow{d^1(x)} \ldots \xrightarrow{d^{n-1}(x)} K^n(x) \to 0,$$

where, for $0 \le j \le n$, $K^j(x) = \bigwedge^j A^n$, with basis e_α, $\alpha \in I(j, n)$, and the homomorphisms $d^j(x)$ (for $0 \le j \le n-1$) given by $d^j(x)(e_\alpha) = (\sum_{k=1}^{n} x_k e_k) \wedge e_\alpha$ for all $\alpha \in I(j, n)$. After tensoring this complex with M we get the complex $K^\bullet(x, M)$, with differentials $d^j(x, M) = d^j(x) \otimes \mathrm{id}_M$, which is called the *Koszul complex of* M with respect to x.

Let U be a triangular subset of A^n. Define a relation $\leq$ on U as follows: for x and y in U, we set $x \leq y$ if and only if there exists $H \in D_n(A)$ such that $[y_1 \dots y_n]^T = H[x_1 \dots x_n]^T$. Then $\leq$ is a quasi-order on U and $(U, \leq)$ is directed set.

Consider $x, y \in U$, where $y^T = Hx^T$ for some $H = (h_{ij}) \in D_n(A)$. One can define, for any j with $0 \leq j \leq n$, a map $\bigwedge^j H : \bigwedge^j A^n \to \bigwedge^j A^n$ given by $e_\alpha \mapsto He_{\alpha_1} \wedge \dots \wedge He_{\alpha_j}$ or, equivalently, $e_\alpha \mapsto \sum_{\beta \in I(j,n)} \Delta_{\beta,\alpha} e_\beta$ for $\alpha \in I(j,n)$. Here

$$\Delta_{\beta,\alpha} := \begin{vmatrix} h_{\beta_1\alpha_1} & \dots & h_{\beta_1\alpha_j} \\ \vdots & & \vdots \\ h_{\beta_j\alpha_1} & \dots & h_{\beta_j\alpha_j} \end{vmatrix}$$

is the $j \times j$-minor of H determined by β and α (see for example [5]). Set $\bigwedge^j(H, M) := \bigwedge^j H \otimes \mathrm{id}_M$. By [4, 1.6.8] the map $\bigwedge^\bullet(H, M)$ is a map of Koszul complexes from $K^\bullet(x, M)$ to $K^\bullet(y, M)$. Let $\mathcal{K} = \{K^\bullet(x, M) : x \in U\}$. Then $\mathcal{K}$ forms a directed system for the maps $\bigwedge^\bullet(H, M)$.

We shall require the following result.

2.1. Lemma (see E. S. Golod [5, Lemma 1]). *Let $x, y \in U$ and let $H, K \in D_n(A)$ with $y^T = Hx^T = Kx^T$. Set $D = \mathrm{diag}(y_1, \dots, y_n)$. Then the homologies of $\bigwedge^\bullet(DH, M)$ and $\bigwedge^\bullet(DK, M)$ are the same.*

It follows from Lemma 2.1 that the induced direct system of homology modules of $\mathcal{K}$ has the usual properties of standard direct limit systems where there is only one morphism between comparable objects (see [9, Theorem 2.17 and its proof], for example).

2.2. Remark. Let Φ be a system of ideals of A. Recall that an A-module G is *Φ-torsion* if each element of G is annihilated by an ideal belonging to Φ. If G is a Φ-torsion A-module, then each $\mathfrak{p} \in \mathrm{Ass}_A(G)$ contains an ideal belonging to Φ; it is easy to deduce from this and the Matlis–Gabriel decomposition theory of injective A-modules that every term in the minimal injective resolution of G is also Φ-torsion, so that $H^i_\Phi(G) = 0$ for all $i > 0$.

The following proposition plays a key role in this paper.

2.3. Proposition. *Let I be the set of non-negative integers or $I = \{0, 1, \dots, n\}$ for some $n \in \mathbb{N}_0$. Let Φ be a system of ideals of A. Let*

$$0 \xrightarrow{d^{-1}} K^0 \xrightarrow{d^0} K^1 \to \dots \to K^n \xrightarrow{d^n} K^{n+1} \to \dots$$

be a complex $K^\bullet$ of A-modules and A-homomorphisms such that

(a) *$K^0 = M$ and, for all $i \in I$, $H^j_\Phi(K^{i+1}) = 0$ for every $j \geq 0$,*

(b) *$\mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$ is Φ-torsion for all $i \in I$.*

Then $H^i_\Phi(M) \cong \mathrm{Ker}\, d^i / \mathrm{Im}\, d^{i-1}$ for all $i \in I$.

Proof. Let $i \in I$. Consider the exact sequences

$$0 \to \operatorname{Ker} d^{i+1} \to K^{i+1} \to \operatorname{Im} d^{i+1} \to 0$$

and

$$0 \to \operatorname{Im} d^{i} \to K^{i+1} \to \operatorname{Coker} d^{i} \to 0.$$

Then

$$H^0_{\Phi}(\operatorname{Ker} d^{i+1}) = 0 = H^0_{\Phi}(\operatorname{Im} d^{i}) \tag{1}$$

and

$$H^j_{\Phi}(\operatorname{Im} d^{i+1}) \cong H^{j+1}_{\Phi}(\operatorname{Ker} d^{i+1}) \quad \text{for all } j \in \mathbb{N}_0. \tag{2}$$

In particular,

$$H^1_{\Phi}(\operatorname{Ker} d^{i}) = 0 \quad \text{for all } i \in I \setminus \{0\}. \tag{3}$$

Next, the exact sequence $0 \to \operatorname{Ker} d^0 \to M \to \operatorname{Im} d^0 \to 0$ induces the long exact sequence

$$\begin{aligned} 0 &\to H^0_{\Phi}(\operatorname{Ker} d^0) \to H^0_{\Phi}(M) \to H^0_{\Phi}(\operatorname{Im} d^0) \\ &\to H^1_{\Phi}(\operatorname{Ker} d^0) \to H^1_{\Phi}(M) \to H^1_{\Phi}(\operatorname{Im} d^0) \to \ldots \end{aligned}$$

in which $H^0_{\Phi}(\operatorname{Im} d^0) = 0$ (by (1)) and $H^j_{\Phi}(\operatorname{Ker} d^0) = 0$ for all $j \in \mathbb{N}$. Therefore $\operatorname{Ker} d^0 \cong H^0_{\Phi}(\operatorname{Ker} d^0) \cong H^0_{\Phi}(M)$ and $H^j_{\Phi}(\operatorname{Im} d^0) \cong H^j_{\Phi}(M)$ for all $j \in \mathbb{N}$. To complete the proof, it is therefore enough to show that

$$H^i_{\Phi}(\operatorname{Im} d^0) \cong \operatorname{Ker} d^i / \operatorname{Im} d^{i-1} \quad \text{for all } i \in I \setminus \{0\}. \tag{4}$$

We can now deduce from (1), (3) and the exact sequence

$$0 \to \operatorname{Im} d^{i-1} \to \operatorname{Ker} d^{i} \to \operatorname{Ker} d^i / \operatorname{Im} d^{i-1} \to 0$$

that, for each $i \in I \setminus \{0\}$,

$$H^1_{\Phi}(\operatorname{Im} d^{i-1}) \cong \operatorname{Ker} d^i / \operatorname{Im} d^{i-1} \tag{5}$$

and

$$H^j_{\Phi}(\operatorname{Im} d^{i-1}) \cong H^j_{\Phi}(\operatorname{Ker} d^{i}) \quad \text{for all } j \geq 2. \tag{6}$$

Now (4) follows from (6), (2) and (5).

2.4. Lemma. *Let Φ be a system of ideals of A and G be an A-module. Then G is Φ-torsion if and only if* $\operatorname{Supp}(G) \subseteq \bigcup_{\mathfrak{a} \in \Phi} V(\mathfrak{a})$.

Proof. The "if" part is obvious. Let g be an arbitrary element of G and $\operatorname{Ass}(Ag) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_s\}$. Then there exist $\mathfrak{a}_1, \ldots, \mathfrak{a}_s \in \Phi$ such that $\mathfrak{p}_i \in V(\mathfrak{a}_i)$ for all $i = 1, \ldots, s$. Since Φ is a system of ideals of A there exists $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}_1 \ldots \mathfrak{a}_s$. Hence $\mathfrak{c} \subseteq \bigcap_{i=1}^s \mathfrak{p}_i = \sqrt{\operatorname{Ann}_A(Ag)}$ and so $\mathfrak{c}^l g = 0$ for some $l \in \mathbb{N}$. Therefore G is Φ-torsion and the proof is complete.

Now, we describe the general local cohomology modules $H^i_{\Phi}(M)$ in terms of generalized fractions.

2.5. *Reminder*: *Complexes of modules of generalized fractions.* The concept of a chain of triangular subsets on A is explained in [8, p. 420]. Such a chain $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ determines a complex of modules of generalized fractions

$$0 \to M \xrightarrow{e^0} U_1^{-1}M \xrightarrow{e^1} \ldots \to U_n^{-n}M \xrightarrow{e^n} U_{n+1}^{-n-1}M \to \ldots$$

in which $e^0(m) = m/(1)$ for all $m \in M$ and $e^n(m/(u_1,\ldots,u_n)) = m/(u_1,\ldots,u_n,1)$ for all $n \in \mathbb{N}$, $m \in M$ and $(u_1,\ldots,u_n) \in U_n$. We denote this complex by $C(\mathcal{U}, M)$.

2.6. THEOREM. *Let Φ be a system of ideals of A. For each $n \in \mathbb{N}$, let $U_n := \{(x_1,\ldots,x_n) \in A^n :$ there exists j with $0 \le j \le n$ such that $x_1,\ldots,x_j$ is an $\mathfrak{a}$-filter regular M-sequence for some $\mathfrak{a} \in \Phi$ and $x_{j+1} = \ldots = x_n = 1\}$. Then $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a chain of triangular subsets on A, and*

$$H^i(C(\mathcal{U}, M)) \cong H^i_\Phi(M) \quad \textit{for all } i \ge 0.$$

Proof. Note that, for each $n \in \mathbb{N}$, U_n is an expansion of $U_n(\Phi, M)$ in the sense of [11, p. 38], and that $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$ is a chain of triangular subsets on A. Thus we may form the complex $C(\mathcal{U}, M)$. In order to prove the second part, in view of 2.3, it suffices to prove that

(7) $$H^j_\Phi(U_i^{-i}M) = 0 \quad \text{for all } i \in \mathbb{N} \text{ and } j \ge 0$$

and

(8) $$H^i(C(\mathcal{U}, M)) \quad \text{is } \Phi\text{-torsion for all } i \ge 0.$$

Let $i \in \mathbb{N}$. By [2, 2.1] and [8, 1.4],

$$H^j_\Phi(U_i^{-i}M) \cong \varinjlim_{\mathfrak{a}\in\Phi} \varinjlim_{x\in U_i} H^j_\mathfrak{a}(U_{xi}^{-i}(M)) \quad \text{for all } j \ge 0,$$

where $U_{xi} = \{(x_1^{\alpha_1},\ldots,x_i^{\alpha_i}) : \alpha_1,\ldots,\alpha_i \in \mathbb{N}\}$ for some $\mathfrak{a}$-filter regular M-sequence $x = x_1,\ldots,x_i$ and $\mathfrak{a} \in \Phi$. So, for the proof of (7), it is enough to show that, for all $i \in \mathbb{N}$, $x \in U_i$ and $\mathfrak{b} \in \Phi$, there exist $z \in U_i$ and $\mathfrak{c} \in \Phi$ such that $x \le z$, $\mathfrak{b} \le \mathfrak{c}$ (i.e. $\mathfrak{c} \subseteq \mathfrak{b}$) and $H^j_\mathfrak{c}(U_{zi}^{-i}M) = 0$ for every $j \ge 0$. Since $\mathfrak{b} \in \Phi$, there exists $(y_1,\ldots,y_i) \in U_i$ such that $y_1,\ldots,y_i$ form a $\mathfrak{b}$-filter regular M-sequence. Thus, in view of 1.4, there exist $(z_1,\ldots,z_i) \in U_i$ and $\mathfrak{c} \in \Phi$ such that $x \le z$, $\mathfrak{b} \le \mathfrak{c}$ and $z_1,\ldots,z_i$ form a $\mathfrak{c}$-filter regular M-sequence. Now by [12, 2.2], $H^j_\mathfrak{c}(U_{zi}^{-i}M) = 0$ for all $j \ge 0$ and (7) follows.

For the proof of (8), by 2.4, we need to show that

$$\operatorname{Supp}(H^i(C(\mathcal{U}, M))) \subseteq \bigcup_{\mathfrak{a}\in\Phi} V(\mathfrak{a}) \quad \text{for all } i \ge 0.$$

Assume the contrary. Then there exists $\mathfrak{p} \in \operatorname{Supp}(H^i(C(\mathcal{U}, M)))\setminus\bigcup_{\mathfrak{a}\in\Phi} V(\mathfrak{a})$ for some $i \ge 0$. Let $\Psi : A \to A_\mathfrak{p}$ be the natural homomorphism and, for all $l \in \mathbb{N}$, set

$$U_{l\mathfrak{p}} := \{(\Psi(y_1),\ldots,\Psi(y_l)) : (y_1,\ldots,y_l) \in U_l\},$$

which is a triangular subset of $(A_\mathfrak{p})^l$. Hence, by 1.2, each element of $U_{l\mathfrak{p}}$ is a poor $M_\mathfrak{p}$-sequence for all $l \in \mathbb{N}$. Therefore, in view of [6, 2.1] and [8, 3.1], the complex $C(\mathcal{U}, M) \otimes_A A_\mathfrak{p}$ is exact. Thus $\mathfrak{p} \notin \mathrm{Supp}(H^i(C(\mathcal{U}, M)))$, which is the required contradiction.

We now come to the second main theorem of this paper.

2.7. Theorem. *Let Φ be a system of ideals of A and let n be a positive integer. Then*

$$H^i_\Phi(M) \cong \varinjlim_{x \in U_n(\Phi, M)} H^i(x, M) \quad \text{for all } i = 0, 1, \dots, n-1,$$

where $U_n(\Phi, M)$ is the triangular subset of A^n specified in 1.4.

Proof. Let Θ be the set of all ideals which are generated by an element of $U_n(\Phi, M)$. Then Θ is a system of ideals of A. By [5, Statement 2],

$$H^i_\Theta(M) \cong \varinjlim_{x \in U_n(\Phi, M)} H^i(x, M) \quad \text{for all } i = 0, 1, \dots, n-1. \tag{9}$$

Next, it follows from 2.6 (with the same notations) that

$$H^i_\Phi(M) \cong H^i(C(\mathcal{U}, M)) \quad \text{for all } i \geq 0. \tag{10}$$

Hence, in particular, $H^i(C(\mathcal{U}, M))$ is Φ-torsion for all $i \geq 0$. Therefore, since $\bigcup_{\mathfrak{a} \in \Phi} V(\mathfrak{a}) \subseteq \bigcup_{\mathfrak{b} \in \Theta} V(\mathfrak{b})$, it follows that $H^i(C(\mathcal{U}, M))$ is Θ-torsion for all $i \geq 0$. Also, one may use the same arguments as in the proof of (7) to see that

$$H^j_\Theta(U_i^{-i} M) = 0 \quad \text{for all } j \geq 0 \text{ and } i = 1, \dots, n.$$

Now, we can apply 2.3 to get

$$H^i_\Theta(M) \cong H^i(C(\mathcal{U}, M)) \quad \text{for all } i = 0, 1, \dots, n-1. \tag{11}$$

The assertion follows from (9)–(11).

Acknowledgments. The authors are deeply grateful to the referee for his careful reading of the manuscript and helpful suggestions. Also, the authors would like to thank Prof. H. Zakeri for helpful discussions and the Institute for Studies in Theoretical Physics and Mathematics for financial support.

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Received 20 March 1997;
revised 7 January 1998