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SMALL NON-σ-POROUS SETS IN TOPOLOGICALLY COMPLETE METRIC SPACES

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1. Introduction. The notion of a σ-porous set was defined by E. P. Dolzhenko [1] in 1967 and has been used many times since then (cf. [3]). Most these applications use σ-porous subsets of $\mathbb{R}$ or $\mathbb{R}^2$, but there also exist applications of σ-porosity in infinite-dimensional Banach spaces and even in metric spaces without a linear structure (cf. [7]).

Each σ-porous subset of $\mathbb{R}^n$ is of the first category and of Lebesgue measure zero. These facts easily imply that for $X = \mathbb{R}^n$ the following proposition holds.

PROPOSITION A. *There exists a closed nowhere dense set $F \subset X$ which is not σ-porous.*

In fact, it is sufficient to take an arbitrary closed nowhere dense set F of positive Lebesgue measure.

Proposition A is also true for an arbitrary Banach space X. In fact, it is sufficient to take a nonzero continuous functional $f \in X^*$ and a closed nowhere dense non-σ-porous set $D \subset \mathbb{R}$, and then to put $F := f^{-1}(D)$. This observation was presented in [3] with an argument which is correct in separable Banach spaces only. A more complicated argument, which works also in nonseparable spaces, is contained in [6]. As far as I know, Proposition A has not been proved in a more general setting.

In the present article we use a construction which shows that Proposition A holds for each topologically complete metric space X without isolated points.

A basic relatively deep result ([2], for references to other proofs see [3]) concerning σ-porous sets says that there exists a non-σ-porous set $P \subset \mathbb{R}^n$ which is of the first category and is null for the Lebesgue measure μ.

Our construction improves this result, since it shows that such a P can be chosen to be even of Hausdorff dimension zero (we obtain a closed set

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P for $n = 1$ and a G_δ set P for $n > 1$). It also generalizes this result to a separable topologically complete metric space X without isolated points, where μ is now an arbitrary fixed Borel locally finite measure on X.

These results were announced without proof in [5].

Since our construction also works for a more general notion of σ-$\langle g\rangle$-porosity, which gives for some g a much weaker notion of "σ-porosity" (e.g. for $g(x) = x^q$, $0 < q < 1$), we present here the results in this more general setting.

Note that the notion of a σ-$\langle x^q\rangle$-porous set, $0 < q < 1$, was applied in some special problems of the theory of boundary behaviour of complex functions and does not depend on q (cf. [3], pp. 323–325). As far as I know, any existing notion of "σ-porosity" is stronger than the notion of σ-$\langle g\rangle$-porosity for a suitable g. In this sense the results of the present article are the best possible.

In some applications, other notions of "σ-porous sets" which are stronger than Dolzhenko's basic notion of (ordinary) σ-porous sets are used. Some such "σ-porous" sets (for example, σ-very porous sets [3] or "σ-porous" sets used in [7]) have the property that they can be covered by countably many closed (ordinary) porous sets. Thus the Baire category theorem easily implies that no nonempty closed subset F of a topologically complete space which is (ordinary) porous at no point of a dense subset D of F is "σ-porous". To construct a nowhere dense set F with the above property in any topologically complete space is an easy task. Thus the analogues of the results of the present article for such stronger notions of "σ-porosity" are rather easy.

From many equivalent definitions of (ordinary) porosity, we choose the following one.

Let X be a metric space. An open ball with center c and radius r will be denoted by $B(c, r)$.

A set $P \subset X$ is said to be *porous at* $x \in X$ if there exists $K > 1$ and a sequence of open balls $B(c_n, r_n)$ such that $c_n \to x$, $B(c_n, r_n) \cap P = \emptyset$ and $x \in B(c_n, Kr_n)$ for each n. Clearly, if we replace the condition $c_n \to x$ by $r_n \to 0$ (or by $c_n \to x$ and $r_n \to 0$), then we obtain the same notion.

A set $P \subset X$ is said to be *porous* if it is porous at all its points. A set is said to be *σ-porous* if it is a countable union of porous sets.

Further, let $g : [0, \infty) \to [0, \infty)$ be a function such that

(1) g is nondecreasing, $g(0) = 0$, g is right continuous at 0 and $g(x) > x$ for $x > 0$.

A set $P \subset X$ is said to be *$\langle g\rangle$-porous at* $x \in X$ (cf. [2], [3]) if there exists a sequence of open balls $B(c_n, r_n)$ such that $c_n \to x$, $B(c_n, r_n) \cap P = \emptyset$ and $x \in B(c_n, g(r_n))$ for each n. Clearly, if we replace the condition $c_n \to x$ by $r_n \to 0$ (or by $c_n \to x$ and $r_n \to 0$), then we obtain the same notion.

A set $P \subset X$ is said to be *$\langle g \rangle$-porous* if it is $\langle g \rangle$-porous at all its points. A set is said to be *σ-$\langle g \rangle$-porous* if it is a countable union of $\langle g \rangle$-porous sets.

Note that in the case $g(x) = 3x$ (in fact in each case $g(x) = qx,\ q > 1$) the notion of σ-$\langle g \rangle$-porosity coincides with the notion of (ordinary) σ-porosity (Lemma E of [6]; it follows easily from results of [2]).

In the sequel, the symbol (X, ϱ) will always denote a fixed topologically complete metric space which has no isolated point. We shall suppose that σ is a fixed complete metric on X which is equivalent to ϱ.

Further, g will always be a fixed function $g : [0,\infty) \to [0,\infty)$ for which (1) holds.

We shall prove that in each topologically complete space X without isolated points there exists a closed nowhere dense set F which is not σ-$\langle g \rangle$-porous. Moreover, if X is separable, we can demand that this set is of measure zero for an arbitrary prescribed locally finite Borel measure μ. If $X = \mathbb{R}$, we can also demand that F is of Hausdorff dimension zero. As pointed out above, we obtain in particular theorems for ordinary σ-porosity. It is not difficult to see (cf. Note 1(b) of [4]) that our result generalizes Konyagin's result ([3], Theorem 5.2) which asserts that there exists a closed Lebesgue null set $F \subset \mathbb{R}$ which is not σ-$[g]$-porous.

Our construction gives, in a general topologically complete separable space X without isolated points, a nowhere dense G_δ set of Hausdorff dimension zero which is not σ-$\langle g \rangle$-porous, but unfortunately not a closed set with these properties (also the case $X = \mathbb{R}^n$ presents troubles). Thus the theorem (which deals with the ordinary porosity) announced in [5] without proof is not proved here in full generality.

However, also the assertion of [5] which is not proved here is true, since any Suslin non-σ-porous subset of a topologically complete metric space Y contains a closed non-σ-porous subset. This deep result was recently proved by M. Zelený in the special case of a compact space X and by J. Pelant in the general case. These results were obtained independently (after my seminar lecture on the results of the present article), and their proofs (which are much more complicated than the proof of the present article) use different methods.

I know Pelant's result from his seminar lecture where the main ideas of the proof were presented; as far as I know, the result is still unpublished. Zelený's result is proved in an unpublished manuscript and will be contained (among other results) in a long paper which is almost prepared for publication. Both proofs deal with the ordinary porosity only and thus the σ-$\langle g \rangle$-porosity analogue has not been proved yet.

The case when X contains isolated points is briefly discussed in Note 8 below.

The basic technical tool of the present article is the generalized Foran lemma for G_δ sets which is proved in [6] and is a simple generalization of the generalized Foran lemma for closed sets of [3]. It involves the notion of abstract porosity, but we formulate it here only in the special case of $\langle g\rangle$-porosity.

We say that a nonempty system $\mathcal{A}$ of nonempty G_δ subsets of X is a *generalized Foran system* (for $\langle g\rangle$-porosity) if the folowing condition holds:

If $A \in \mathcal{A}$, H is open and $A \cap H \neq \emptyset$, then there exists $A^* \in \mathcal{A}$ such that $A^* \subset A \cap H$ and $A \cap H$ is $\langle g\rangle$-porous at no point of A^*.

The *generalized Foran lemma* (for $\langle g\rangle$-porosity) is the following result.

LEMMA 1. *Let X be a topologically complete metric space and let $\mathcal{A}$ be a generalized Foran system of G_δ subsets of X. Then $\mathcal{A}$ contains no σ-$\langle g\rangle$-porous set.*

2. The basic lemma

DEFINITION. Let $G \subset X$, $\emptyset \neq G \neq X$, be an open set. We say that a system $\mathcal{B}$ of open subsets of X is a *G-system* if the following assertions hold:

(2) The system $\{\overline{B} : B \in \mathcal{B}\}$ does not cover G and is discrete in G (i.e. for each $x \in G$ there exists a neigbourhood of x which intersects at most one member of $\{\overline{B} : B \in \mathcal{B}\}$).

(3) If $y \in G$, $r > 0$ and $B(y, g(r))\setminus G \neq \emptyset$, then $B(y, r)$ contains a member of $\mathcal{B}$.

NOTE 1. (i) Since $G \neq X$, (3) and (1) imply that each G-system is nonempty.

(ii) The condition (2) easily implies that $\{\overline{B} : B \in \mathcal{B}\}$ is a disjoint system,

$$G \setminus \bigcup\{\overline{B} : B \in \mathcal{B}\} = G \setminus \overline{\bigcup\{B : B \in \mathcal{B}\}}$$

is a nonempty open set and

$$\partial\Big(\bigcup\mathcal{B}\Big) \subset \partial G \cup \bigcup\{\partial B : B \in \mathcal{B}\}.$$

(iii) If $\mathcal{B}$ is a G-system and for each $B \in \mathcal{B}$ a nonempty open set $C(B) \subset B$ is given, then $\{C(B) : B \in \mathcal{B}\}$ is a G-system as well.

The basic element of our construction is the following lemma.

LEMMA 2. *Let $G \subset X$, $\emptyset \neq G \neq X$, be an open set. Then there exists a G-system.*

Proof. In the first step find by the Zorn lemma a maximal set $M \subset G$ with the following property:

(4) If $x, y \in M$ and $x \neq y$, then
$$g(2\varrho(x,y)) \geq \min(\varrho(x, X \setminus G), \varrho(y, X \setminus G)).$$

We show that M has the following property:

(5) If $y \in G$, $r > 0$ and $B(y, g(r)) \setminus G \neq \emptyset$, then $B(y, r/2) \cap M \neq \emptyset$.

In fact, suppose that $y \in G$, $r > 0$, $B(y, g(r)) \setminus G \neq \emptyset$ and $B(y, r/2) \cap M = \emptyset$. Then, for each $x \in M$, we have $\varrho(x, y) \geq r/2$ and consequently
$$g(2\varrho(x,y)) \geq g(r) \geq \varrho(y, X \setminus G) \geq \min(\varrho(y, X \setminus G), \varrho(x, X \setminus G)).$$
Therefore $M \cup \{y\}$ also has the property (4), which is a contradiction.

Further observe that
$$M' \cap G = \emptyset. \tag{6}$$

In fact, suppose that there exists some $z \in M' \cap G$. Then we can choose sequences (x_n), (y_n) in M such that $x_n \neq y_n$, $x_n \to z$ and $y_n \to z$. By (4) we have
$$\begin{aligned} 0 = \lim_{n\to\infty} g(2\varrho(x_n, y_n)) &\geq \lim_{n\to\infty} \min(\varrho(x_n, X \setminus G), \varrho(y_n, X \setminus G)) \\ &= \varrho(z, X \setminus G) > 0, \end{aligned}$$
which is a contradiction.

Since X has no isolated points, (6) implies that we can choose a point $a \in G \setminus M$. Now observe that that if we put
$$\sigma_x = \min\left(\tfrac{1}{2}\varrho(x, a), \tfrac{1}{2}\varrho(x, X \setminus G), \tfrac{1}{3}\varrho(x, M \setminus \{x\})\right) \tag{7}$$
for each $x \in M$ (where $\varrho(x, \emptyset) = \infty$), then

(8) $\{\overline{B(x, \sigma_x)} : x \in M\}$ is a system of pairwise disjoint subsets of G which is locally finite in G and whose union does not contain a.

Now we show that for each $x \in M$ we can choose a $\delta_x > 0$ such that the following statement holds:

(9) If $y \in G$, $r > 0$, $B(y, g(r)) \setminus G \neq \emptyset$ and $x \in B(y, r/2)$, then $r/2 > \delta_x$.

In fact, suppose that, on the contrary, such a $\delta_x > 0$ does not exist. Then there exist sequences $y_n \to x$ and $r_n \to 0$ such that $g(r_n) > \varrho(y_n, X \setminus G)$. Consequently,
$$0 = \lim_{n\to\infty} g(r_n) \geq \lim_{n\to\infty} \varrho(y_n, X \setminus G) = \varrho(x, X \setminus G) > 0,$$
which is a contradiction. The condition (9) clearly implies that

(10) $\overline{B(x, \delta_x)} \subset B(y, r)$ if $y \in G$, $r > 0$, $B(y, g(r)) \setminus G \neq \emptyset$ and $x \in B(y, r/2) \cap M$.

If we now put $r_x = \min(\sigma_x, \delta_x)$, we easily see that the conditions (8), (5) and (10) imply that $\{B(x, r_x) : x \in M\}$ is a G-system.

We shall also need the following simple observation on G-systems.

LEMMA 3. *Let $\mathcal{B}$ be a G-system and $z \in \partial G$. Then each neighbourhood of z contains a member of $\mathcal{B}$.*

Proof. Choose an arbitrary $r > 0$ and $x \in B(z, r/2) \cap G$. Since $z \in B(x, g(r/2))$, by (3) there exists $B \in \mathcal{B}$ such that $B \subset B(x, r/2) \subset B(z, r)$.

3. Constructions

CONSTRUCTION 1. Let $G \subset X$, $\emptyset \neq G \neq X$ be an open set and let $m \in \mathbb{N}$. Then we choose a system $\mathcal{D}(G, m)$ such that

(i) $\mathcal{D}(G, m)$ is a G-system,
(ii) If $B \in \mathcal{D}(G, m)$, then $\operatorname{diam}_\sigma B < 1/m$.

Further put

(iii) $R(G, m) = G \setminus \bigcup\{\overline{B} : B \in \mathcal{D}(G, m)\}$.

NOTE 2. (i) We can choose $\mathcal{D}(G, m)$ by Lemma 2 and Note 1(iii).
(ii) $R(G, m)$ is a nonempty open subset of G by Note 1(ii).
(iii) Using Note 1(ii), we easily obtain

$$\partial\Big(\bigcup \mathcal{D}(G, m)\Big) \subset \partial G \cup \bigcup\{\partial B : B \in \mathcal{D}(G, m)\} \quad \text{and}$$
$$\partial R(G, m) \subset \partial G \cup \bigcup\{\partial B : B \in \mathcal{D}(G, m)\}.$$

CONSTRUCTION 2. Let $G \subset X$, $\emptyset \neq G \neq X$, be an open set and $m \in \mathbb{N}$. Then we define a sequence of nonempty systems of nonempty open sets

$$\mathcal{S}_1(G, m),\ \mathcal{S}_2(G, m),\ \ldots$$

and a sequence of nonempty open sets

$$G \supset R_1(G, m) \supset R_2(G, m) \supset \ldots$$

inductively in the following way:

(i) $\mathcal{S}_1(G, m) = \mathcal{D}(G, m)$ and $R_1(G, m) = R(G, m)$.
(ii) If $\mathcal{S}_k(G, m)$ and $R_k(G, m)$ are defined, then we put

$$\mathcal{S}_{k+1}(G, m) = \mathcal{D}(R_k(G, m), m) \quad \text{and} \quad R_{k+1}(G, m) = R(R_k(G, m), m).$$

Further put $\Phi_k(G, m) = \bigcup \mathcal{S}_k(G, m)$.

NOTE 3. For each natural k we have

$$\partial R_k(G, m) \cup \partial \Phi_k(G, m) \subset \partial G \cup \bigcup_{s=1}^{k} \bigcup\{\partial B : B \in \mathcal{S}_s(G, m)\}.$$

In fact, for $k = 1$ this immediately follows from Note 2(iii). For $k > 1$ we can use induction, since Note 2(iii) gives

$$\partial R_k(G,m) \cup \partial \Phi_k(G,m) \subset \partial R_{k-1}(G,m) \cup \bigcup \{\partial B : B \in \mathcal{S}_k(G,m)\}.$$

CONSTRUCTION 3. (i) Choose an open set $\emptyset \neq U \neq X$.

(ii) Put $\mathcal{K}_1^k = \mathcal{S}_k(U,1)$ and $\mathcal{K}_1 = \bigcup_{k=1}^{\infty} \mathcal{K}_1^k$. If $B \in \mathcal{K}_1^k$, then we say that B has *level* 1 and *order* k. Thus $\mathcal{K}_1$ is the set of all sets of level 1.

(iii) If $\mathcal{K}_n$ (i.e. the set of all sets of level n) is defined, then we put

$$\mathcal{K}_{n+1}^k = \bigcup \{\mathcal{S}_k(B, n+1) : B \in \mathcal{K}_n\} \quad \text{and} \quad \mathcal{K}_{n+1} = \bigcup_{k=1}^{\infty} \mathcal{K}_{n+1}^k.$$

We say that sets from $\mathcal{K}_{n+1}^k$ have *level* $n+1$ and *order* k.

NOTE 4. It is easy to verify the following properties:

(i) $\mathcal{K}_n^k \cap \mathcal{K}_n^{k^*} = \emptyset$ for $k \neq k^*$.

(ii) If B and C are different sets of the same level, then $\overline{B} \cap \overline{C} = \emptyset$.

(iii) If $n_1 < n_2$ are natural numbers and B is a set of level n_2, then there exists precisely one set C of level n_1 such that $\overline{B} \subset C$; we have $B \neq C$. If $D \neq C$ also has level n_1, then (ii) gives $\overline{D} \cap \overline{B} = \emptyset$.

(iv) The properties (iii) and (ii) show that no set can have two different levels or two different orders.

(v) If B is a set of level n and k is a natural number, then there exists a set C of level $n+1$ and order k for which $\overline{C} \subset B$.

(vi) If B is a set of level n, then $\operatorname{diam}_\sigma B < 1/n$.

DEFINITION. For each natural n and $x \in X$ there exists (Note 4(ii)) at most one set B of level n which contains x. If such a B exists, it will be denoted by $B(x,n)$ and its order by $k(x,n)$.

Analogously, by (ii) and (iii), if C is a set of level m, for each $1 \leq n \leq m$ there exists precisely one set $B(C,n)$ of level n which contains C. Its order will be denoted by $k(C,n)$.

Let $p = (p_n)_{n=1}^{\infty}$ be a sequence of natural numbers. We say that an $x \in X$ is *p-admissible* if $k(x,n)$ is defined for all n and $k(x,n) \leq p_n$. Similarly we say that a set C of level m is *p-admissible* if $k(C,n) \leq p_n$ for each $1 \leq n \leq m$.

NOTE 5. It is easy to see that $x \in X$ is p-admissible iff $B(x,n)$ is p-admissible for all n.

LEMMA 4. *Let p be a sequence of natural numbers and C be a set of level m. Then C is p-admissible iff there exists $x \in C$ which is p-admissible.*

Proof. If C contains a p-admissible point x, then $C = B(x,m)$ is p-admissible by Note 5. Now suppose that C is p-admissible. By Note 4(v) there exist sets

$$C \supset \overline{B}_{m+1} \supset B_{m+1} \supset \overline{B}_{m+2} \supset \dots$$

such that B_k is a set of level k and order 1 for $k > m$. Using Note 4(vi) and the fact that σ is complete we conclude that there exists $x \in \bigcap_{k=m+1}^{\infty} B_k$. It is easy to see that x is p-admissible.

Definition. If p is a sequence of natural numbers and B is a p-admissible set then we denote by $A(B,p)$ the set of all $x \in B$ which are p-admissible.

Note 6. If we denote by G_n the union of all subsets of B which have level n and order at most p_n, then Note 4(iii) easily implies $A(B,p) = \bigcap_{n=1}^{\infty} G_n$. Consequently, $A(B,p)$ is a G_δ set.

Lemma 5. *Each set of the form $A(B,p)$ is a nonempty nowhere dense G_δ set.*

Proof. Lemma 4 and Note 5 immediately give that $A(B,p)$ is a nonempty G_δ set. Now suppose contrary to our claim that $A(B,p)$ is dense in an open set $H \neq \emptyset$. Let n^* be the level of B and $x \in H \cap A(B,p)$. By Note 4(vi) we can choose $n > n^*$ such that $B(x,n) \subset H$. Further choose (cf. Note 4(v)) a set $C \subset B(x,n)$ of level $n+1$ and order $p_{n+1}+1$. Then by Lemma 4, $C \cap A(B,p) = \emptyset$ and $C \subset H$, which is a contradiction.

Lemma 6. *Let p be a sequence of natural numbers and let B be a p-admissible set. Then*

$$\overline{A(B,p)} \setminus A(B,p) \subset \bigcup \{\partial C : C \subset B \text{ is } p\text{-admissible}\}.$$

Proof. Let $c \in \overline{A(B,p)} \setminus A(B,p)$ and $c \notin \partial B$. Denote by n^* the level of B. Since $c \in B \setminus A(B,p)$, Note 5 easily implies that there is the least $n > n^*$ with the property that c lies in no p-admissible set of level n. The set $B^* := B(c, n-1)$ is clearly a p-admissible subset of B. We have

$$B^* \cap A(B,p) \subset \bigcup_{k=1}^{p_n} \Phi_k(B^*, n),$$

since the right-hand side is the union of all p-admissible subsets of B^* which have level n. Therefore there exists $k \leq p_n$ such that $c \in \overline{\Phi}_k(B^*, n)$. By the definition of n we have $c \in \partial \Phi_k(B^*, n)$. Thus Note 3 implies that c belongs to the boundary of a p-admissible set $C \subset B$, which completes the proof.

4. Results

Definition. Let $\mathcal{A}$ be the system of all sets of the form $A(B,p)$ for which $\lim_{n\to\infty} p_n = \infty$.

Proposition. *$\mathcal{A}$ is a generalized Foran system for $\langle g \rangle$-porosity.*

P r o o f. Lemma 5 implies that $\mathcal{A}$ is a nonempty system of nonempty G_δ sets.

Now suppose that $A \in \mathcal{A}$ and $H \subset X$ is an open set such that $A \cap H \neq \emptyset$.

Let $A = A(B,p)$, where B is a set of level s. Choose an arbitrary $x \in A \cap H$ and find a natural $u > s$ so large that $\overline{B(x,u)} \subset H \cap B$ and

$$p_n \geq 2 \quad \text{for } n > u. \tag{11}$$

Put $B^* = B(x,u)$,

$$p^* = (p_1, \ldots, p_u, p_{u+1} - 1, p_{u+2} - 1, \ldots) \quad \text{and} \quad A^* = A(B^*, p^*).$$

Clearly $A^* \in \mathcal{A}$ and $A^* \subset A \cap H$. Now choose an arbitrary $z \in A^*$. We prove that A is not $\langle g \rangle$-porous at z. Suppose otherwise. Then there exist $y \in B^*$ and $r > 0$ such that

$$z \in B(y, g(r)) \quad \text{and} \quad B(y,r) \cap A = \emptyset.$$

We have $y \in B^* = B(z,u)$. Since $y \neq z$, by Note 4(vi) we can find the least natural n for which $y \notin B(z,n)$; of course $n > u$ and therefore $B(z, n-1) \subset B^*$. Now we distinguish two cases:

(α) There exists a set $E \subset B(z, n-1)$ of level n and order $k \leq p_n^* = p_n - 1$ such that $y \in \overline{E}$.

Let $\mathcal{P}$ be the set of all subsets of E of level $n+1$ and order 1; in other words,

$$\mathcal{P} = \mathcal{S}_1(E, n+1) = \mathcal{D}(E, n+1).$$

If $y \in \partial E$, Lemma 3 shows that $B(y,r)$ contains an element of $\mathcal{P}$. If $y \in E$, then obviously $E \neq B(z,n)$. Consequently (Note 4(ii)), $\overline{E} \cap \overline{B(z,n)} = \emptyset$ and therefore $z \notin E$. Since $\mathcal{P}$ is an E-system, again $B(y,r)$ contains an element $P \in \mathcal{P}$. In both cases Lemma 4 implies that $\emptyset \neq P \cap A^* \subset P \cap A$. Consequently, $B(y,r) \cap A \neq \emptyset$, which is a contradiction.

(β) No E as in (α) exists, in other words,

$$y \in R_{p_n^*}(B(z, n-1), n) =: R.$$

In this case consider the system $\mathcal{M}$ of all sets of level n and order $p_n = p_n^* + 1$ which are contained in $B(z, n-1)$, in other words, $\mathcal{M} = \mathcal{D}(R, n)$. Thus $\mathcal{M}$ is an R-system and since $B(z,n)$ is of order at most p_n^*, it follows that $z \notin R$. Consequently, $B(y,r)$ contains a set M from $\mathcal{M}$. Since M is clearly p-admissible, Lemma 4 implies that $M \cap A \neq \emptyset$. Consequently, $B(y,r) \cap A \neq \emptyset$, which is a contradiction again.

As an immediate consequence of the Proposition we obtain the following result.

THEOREM 1. *Let X be a topologically complete metric space without isolated points and let g be as in the Introduction. Then there exists a closed nowhere dense set $F \subset X$ which is not σ-$\langle g \rangle$-porous.*

P r o o f. By Lemma 5 and the Proposition, it is sufficient to take an arbitrary $A \in \mathcal{A}$ and put $F = \overline{A}$.

COROLLARY. *Let X be a topologically complete metric space without isolated points. Then there exists a closed nowhere dense set $F \subset X$ which is not σ-porous.*

If we want to obtain a set $A \in \mathcal{A}$ such that A (or even $\overline{A}$) is in a sense "small" (e.g. it is μ-null for a prescribed measure μ), we can repeat Constructions 1–3 with the difference that we do not fix one operation $\mathcal{D}(G,m)$ in Construction 1, but we observe that we can use in Constructions 2–3 any operation $\mathcal{D}(G,m)$ satisfying the conditions (i) and (ii) of Construction 1. Note 1(iii) can then be used to show that for a suitable choice of $\mathcal{D}(G,m)$ the resulting set $A \in \mathcal{A}$ (or even $\overline{A}$) is small in a certain sense. In this way we obtain the following result.

THEOREM 2. *Let X be a separable topologically complete metric space without isolated points, g be as in the Introduction and let μ be a locally finite Borel measure on X. Then there exists a closed nowhere dense set $F \subset X$ such that $\mu F = 0$ and F is not σ-$\langle g\rangle$-porous.*

P r o o f. First note that μ is clearly σ-finite. Further observe that for any open set $\emptyset \neq B \subset X$ and $\varepsilon > 0$ there exists an open ball $C \subset B$ such that $\mu C < \varepsilon$ and $\mu(\partial C) = 0$. In fact, since B is uncountable, we can choose $a \in B$ with $\mu\{a\} = 0$, and since μ is locally finite, we can find $\delta > 0$ such that $B(a,\delta) \subset B$ and $\mu B(a,\delta) < \varepsilon$. For any $0 < r < \delta$ we have $\partial B(a,r) \subset \{x \in X : \varrho(a,x) = r\}$, which implies that $\mu(\partial B(a,r)) = 0$ for some $0 < r < \delta$. Now we can put $C := B(a,r)$.

Using this observation, Note 1(iii) and separability of X which implies that any operation $\mathcal{D}(G,m)$ yields a countable system of sets of a fixed level, it is easy to see that for a suitable operation $\mathcal{D}(G,m)$ we obtain

(12) $$\mu\Big(\bigcup \mathcal{K}_m\Big) < 1/m \quad \text{for each natural } m \text{ and}$$

(13) any set of any level has a μ-null boundary.

In fact, when we define sets of level k, we can use an operation $\mathcal{D}(G,m)$ such that $\mu(\bigcup \mathcal{K}_m^k) < 1/(m2^{k+1})$ and all members of $\mathcal{K}_m^k$ are open balls having μ-null boundaries.

Now we choose an arbitrary $A \in \mathcal{A}$ and put $F = \overline{A}$. As above we see that F is nowhere dense. Note 6 and (12) imply that $\mu A = 0$ for each $A \in \mathcal{A}$ and Lemma 6 and (13) show that $\mu(\overline{A}) = 0$ as well.

Quite similarly we can obtain the following theorem.

Theorem 3. *Let X and g be as in Theorem 2. Then there exists a nowhere dense G_δ set $A \subset X$ of Hausdorff dimension zero which is not σ-$\langle g\rangle$-porous. If $X = \mathbb{R}$, then A can be chosen closed.*

Proof. We proceed quite analogously to the proof of Theorem 2; instead of the condition (10) it is sufficient to satisfy the condition

$$\sum_{B\in\mathcal{K}_m} (\operatorname{diam} B)^{1/m} < 1/m. \tag{14}$$

Then clearly each $A \in \mathcal{A}$ has Hausdorff dimension zero. If $X = \mathbb{R}$, then also $\overline{A}$ has Hausdorff dimension zero, since we can suppose that all members of $\mathcal{K}_m$ are intervals and therefore by Lemma 6 the set $\overline{A} \setminus A$ is countable.

Note 7. (i) As stated in the Introduction, from an unpublished result of Pelant it follows that, for ordinary porosity, we can obtain a closed A in Theorem 3 even in the general case.

(ii) Of course, by the above method we can also easily obtain some more general results; e.g. A from Theorem 3 can also be chosen null w.r.t. a prescribed locally finite Borel measure μ, or even $H_\phi(A) = 0$, where H_ϕ is the Hausdorff measure given by a function ϕ. In the last case it is sufficient to use an operation $\mathcal{D}(G, m)$ for which

$$\sum_{B\in\mathcal{K}_m} \phi(\operatorname{diam} B) < 1/m. \tag{15}$$

Note 8. Let X be a metric topologically complete space which now has (exceptionally) isolated points (i.e. $X \setminus X' \neq \emptyset$). Then

(i) If $\operatorname{int} X' \neq \emptyset$, then the conclusion of Theorem 1 clearly holds. In fact, we can apply Theorem 1 to $X_1 := \operatorname{int} X'$.

(ii) If $X' = \emptyset$, then the conclusion of Theorem 1 clearly fails.

(iii) It is easy to find a closed subspace $X \subset \mathbb{R}$ such that $\operatorname{int} X' = \emptyset$ and the conclusion of Theorem 1 holds.

(iv) It is not difficult to construct a closed subspace $X \subset \mathbb{R}^2$ such that $X' = \mathbb{R}$ and the conclusion of Theorem 1 fails.

REFERENCES

[1] E. P. Dolzhenko, *Boundary properties of arbitrary functions*, Izv. Akad. Nauk SSSR Ser. Mat. 31 (1967), 3–14 (in Russian).

[2] L. Zajíček, *Sets of σ-porosity and sets of σ-porosity (q)*, Časopis Pěst. Mat. 101 (1976), 350–359.

[3] —, *Porosity and σ-porosity*, Real Anal. Exchange 13 (1987–88), 314–350.

[4] —, *Porosity, derived numbers and knot points of typical continuous functions*, Czechoslovak Math. J. 39 (1989), 45–52.

[5] L. Zajíček, *A note on σ-porous sets*, Real Anal. Exchange 17 (1991–92), 18.
[6] —, *Products of non-σ-porous sets and Foran systems*, Atti Sem. Mat. Fis. Univ. Modena 44 (1996), 497–505.
[7] T. Zamfirescu, *Porosity in convexity*, Real Anal. Exchange 15 (1989–90), 424–436.

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