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ENDOMORPHISM ALGEBRAS OF EXCEPTIONAL SEQUENCES OVER PATH ALGEBRAS OF TYPE $\widetilde{\mathbb{A}}_n$

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The notion of exceptional sequences originates from the study of vector bundles (see, for instance, [GR, B]) and was carried over to modules over hereditary artin algebras (see [CB, R2]). In this paper, we consider the following situation: let k be a commutative field, Q be a finite connected quiver without oriented cycles; then the path algebra $A = kQ$ is hereditary and we may study the exceptional sequences in the category $\operatorname{mod} A$ of finitely generated right A-modules. We recall that an indecomposable object E in $\operatorname{mod} A$ is called *exceptional* if $\operatorname{Ext}^1_A(E, E) = 0$. A sequence $\mathcal{E} = (E_1, \ldots, E_t)$ of exceptional objects in $\operatorname{mod} A$ is called an exceptional sequence if $\operatorname{Hom}_A(E_j, E_i) = 0$ and $\operatorname{Ext}^1_A(E_j, E_i) = 0$ for $j > i$. An exceptional sequence $\mathcal{E} = (E_1, \ldots, E_t)$ is called *complete* if t equals the number of isomorphism classes of simple A-modules, and *connected* if $\operatorname{End}(\bigoplus_{i=1}^t E_i)$ (which we denote briefly by $\operatorname{End} \mathcal{E}$) is a connected algebra. Ringel has asked whether, if $\mathcal{E}$ is a complete exceptional sequence in the module category over a representation-finite hereditary artin algebra, then $\operatorname{End} \mathcal{E}$ is also representation-finite. This question was answered affirmatively in case $A = kQ$, where Q is of type $\mathbb{A}_n$, first by H. Yao [Y] in case Q has a linear orientation, then by H. Meltzer [M] in case Q has an arbitrary orientation. It is reasonable to generalise Ringel's question as follows: let $\mathcal{E}$ be a complete exceptional sequence in the module category over a tame path algebra; is it then true that $\operatorname{End} \mathcal{E}$ is also tame? The objective of this paper is to answer this latter question affirmatively whenever $A = kQ$, where Q is of type $\widetilde{\mathbb{A}}_n$. More precisely, we prove the following theorem.

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THEOREM. *Let k be a commutative field, Q be a quiver with underlying graph $\widetilde{\mathbb{A}}_n$, and $A = kQ$ be its path algebra. Let $\mathcal{E}$ be a complete exceptional sequence in* $\operatorname{mod} A$. *Then* $\operatorname{End}\mathcal{E}$ *is either a direct product of one tilted algebra of type $\widetilde{\mathbb{A}}_m$ (with $m \le n$) and tilted algebras of type $\mathbb{A}_l$ (with $l \le n-m$), or a direct product of tilted algebras of type $\mathbb{A}_l$ (with $l \le n+1$). Each connected subsequence of $\mathcal{E}$ is a partial tilting module.*

We use essentially the description of the module category of a path algebra of type $\widetilde{\mathbb{A}}_n$, as in [DR, R1], and the structure of its indecomposable modules, as in [BR]. Notice that, if $(E_1, \ldots, E_t)$ is an exceptional sequence in $\operatorname{mod} A$, where $A = kQ$, then, in particular, each E_i is exceptional, hence $\operatorname{End} E_i = k$ (see, for instance, [K], (11.9)). If Q is an Euclidean quiver, this implies that E_i is postprojective, preinjective or regular lying in an exceptional tube of rank m (> 1), say, and, in this case, is of quasi-length at most $m - 1$.

We use without further reference properties of the Auslander–Reiten translations $\tau = \mathrm{DTr}$ and $\tau^{-1} = \mathrm{TrD}$, and the Auslander–Reiten quiver $\Gamma(\operatorname{mod} A)$ of A as in [ARS, R1]. In particular, we frequently use the Auslander–Reiten formulae

$$\operatorname{Ext}^1_A(M, N) \cong \mathrm{D}\operatorname{Hom}_A(N, \tau M) \cong \mathrm{D}\operatorname{Hom}_A(\tau^{-1}N, M).$$

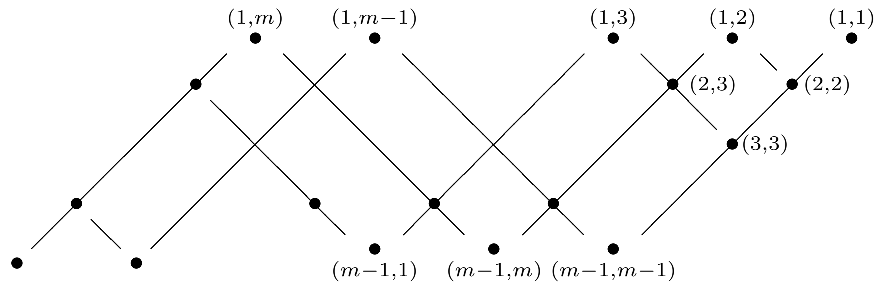
For the classification results of tilted and iterated tilted algebras of type $\mathbb{A}_n$ and $\widetilde{\mathbb{A}}_n$, we refer to [A1, AH, AS, R, H].

1. Regular exceptional modules. The aim of this section is to show that, if Γ is an exceptional tube of rank m, say, in the Auslander–Reiten quiver of the path algebra A of an Euclidean quiver, and $\mathcal{E}$ is a connected exceptional sequence all of whose terms lie in Γ, then $\operatorname{End}\mathcal{E}$ is a tilted algebra of type $\mathbb{A}_t$.

In this situation, the tube Γ is standard, thus we may identify the points in Γ with the corresponding indecomposable A-modules. Each point in Γ will be given by two coordinates: the first is the quasi-length of the corresponding indecomposable A-module (thus is a positive integer), and the second represents its regular socle (and is chosen from $\mathbb{Z}_m$). The modules E_i being exceptional, they have quasi-length at most $m - 1$. The figure on the next page shows the full translation subquiver Γ' of Γ consisting of all modules of quasi-length at most $m-1$. Associated to each point $M = (i, j)$ in Γ' are four sectional paths in Γ', these are:

(i) $(M\nearrow)$, the portion of coray from M to the mouth (that is, the sectional path from (i, j) to $(1, j - i + 1)$),

(ii) $(M\searrow)$, the portion of ray from M to infinity in Γ' (that is, the sectional path from (i, j) to $(m - 1, j)$),



(iii) ($\searrow M$), the portion of ray from the mouth to M (that is, the sectional path from $(1,j)$ to (i,j)), and

(iv) ($\nearrow M$), the portion of coray from infinity to M in Γ' (that is, the sectional path from $(m-1, m-1+j-i)$ to (i,j)).

It also follows from the standardness of Γ that, if $M=(i,j)$ is in Γ', then the support $\operatorname{Supp}\operatorname{Hom}_A(M,-)|_{\Gamma'}$ of the restriction to Γ' of the functor $\operatorname{Hom}_A(M,-)$ is a trapezoid with corners (i,j), $(1,j-i+1)$, $(m-1,j-i+1)$ and $(m-1,j)$, bounded by the sectional paths ($M\nearrow$), ($M\searrow$) and ($(1,j-i+1)\searrow$). Similarly, $\operatorname{Supp}\operatorname{Hom}_A(-,M)|_{\Gamma'}$ is a trapezoid with corners (i,j), $(m-1,m-1+j-i)$, $(1,j)$ and $(m-1,m-2+j)$, bounded by the sectional paths ($\searrow M$), ($\nearrow M$) and ($\nearrow(1,j)$).

Lemma 1.1. *Let $M\in\mathcal{E}$, and M,N,L lie in Γ.*

(a) *Let $N\in\mathcal{E}$. Then* $\operatorname{Hom}_A(M,N)\neq 0$ *if and only if* $N\in(M\nearrow)\cup(M\searrow)$.

(b) *Let $L\in\mathcal{E}$. Then* $\operatorname{Hom}_A(L,M)\neq 0$ *if and only if* $L\in(\searrow M)\cup(\nearrow M)$.

Proof. We only show (a), since the proof of (b) is similar.

For $M,N\in\mathcal{E}$, $\operatorname{Hom}_A(M,N)\neq 0$ implies that (M,N) is a subsequence of $\mathcal{E}$ so that $\operatorname{Hom}_A(\tau^{-1}M,N)=0$, that is, $N\in\operatorname{Supp}\operatorname{Hom}_A(M,-)|_{\Gamma'}$ but $N\notin\operatorname{Supp}\operatorname{Hom}_A(\tau^{-1}M,-)|_{\Gamma'}$. Therefore $N\in(M\nearrow)\cup(M\searrow)$. The converse is trivial. ∎

Lemma 1.2. *There exists no path $M\to N\to L$ in Γ with $M=(i,j)$, $N=(i-l,j-l)$, $l\geq 1$, $L=(k,j-l)$, $k>i-l$, and $M,N,L\in\mathcal{E}$.*

Proof. Assume the contrary. Since $N\in(M\nearrow)\cup(M\searrow)$ by Lemma 1.1, we have in fact $N\in(M\nearrow)$. Similarly, $L\in(N\searrow)$. But then we obtain $L\in\operatorname{Supp}\operatorname{Hom}_A(\tau^{-1}M,-)|_{\Gamma'}$ so that $\operatorname{Ext}_A^1(L,M)\neq 0$, a contradiction to the fact that (M,N,L) is a subsequence of $\mathcal{E}$. ∎

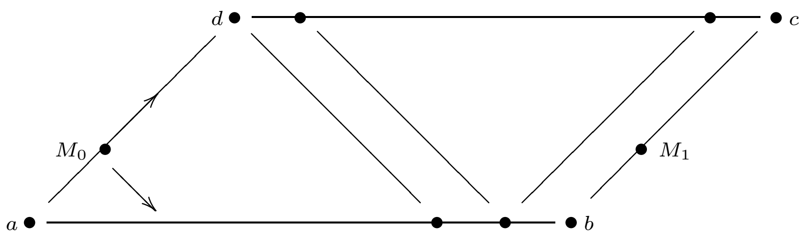
Lemma 1.3. *Assume there exists a path $M\xrightarrow{f}N\xrightarrow{g}L$ in Γ with $M=(i,j)$, $N=(k,j)$, $k>i$, $L=(k-l,j-l)$, $1\leq l<k$ and $M,N,L\in\mathcal{E}$. Then* $gf=0$.

Proof. By Lemma 1.1 and the hypothesis, we have $N \in (M\searrow)$. Also, since (M, N, L) is a subsequence of $\mathcal{E}$, we have $L \notin \operatorname{Supp}\operatorname{Hom}_A(\tau^{-1}M, -)|_{\Gamma'}$, hence $L \notin \operatorname{Supp}\operatorname{Hom}_A(M, -)|_{\Gamma'}$. That is, $\operatorname{Hom}_A(M, L) = 0$. ■

LEMMA 1.4. *Let $\mathcal{E}$ be a connected exceptional sequence lying in Γ. Then the quiver of* $\operatorname{End}\mathcal{E}$ *is a tree.*

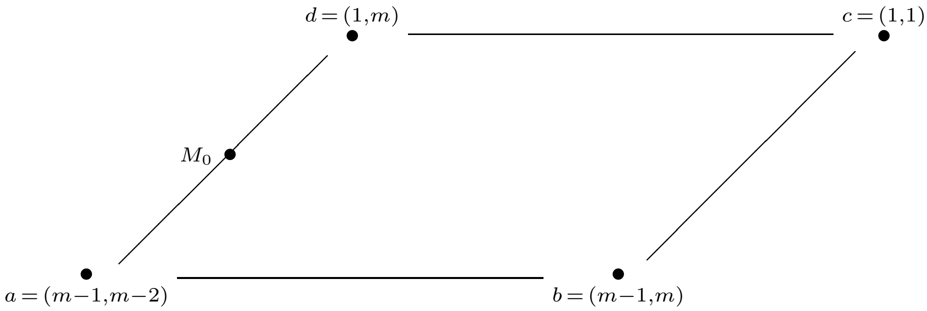
Proof. Assume the contrary; then the quiver of $\operatorname{End}\mathcal{E}$ contains a cycle, which, by [Y], Proposition 3.2, is not an oriented cycle. Let thus $\mathcal{F}$ be a subsequence of $\mathcal{E}$ such that the quiver of $\operatorname{End}\mathcal{F}$ is a cycle. We agree to say that $\mathcal{F}$ *passes through two neighbouring corays* $(\nearrow(1, j))$ and $(\nearrow(1, j-1))$ if there is an arrow α of the quiver of $\operatorname{End}\mathcal{F}$ representing a sectional path $\alpha_1 \dots \alpha_r$ where $\alpha_1, \dots, \alpha_r$ are arrows in Γ, and some $1 \le l \le r$ such that α_l is the arrow in Γ from $(i, j+i-1)$ to $(i+1, j+i-1)$. We also denote by $\widetilde{\Gamma}'$ the universal covering of the full translation subquiver Γ' of Γ of all modules of quasi-length at most $m-1$ (thus $\widetilde{\Gamma}' \cong \mathbb{Z}\mathbb{A}_{m-1}$). We consider two cases:

(a) Assume that $\mathcal{F}$ passes through all pairs of neighbouring corays $(\nearrow(1, j))$ and $(\nearrow(1, j-1))$, where j ranges over $\mathbb{Z}_m$. Let $M \in \mathcal{F}$; then there exist two points M_0, M_1 in $\widetilde{\Gamma}'$ lifting M, and a path of length $m+1$ from M_0 to M_1. The corays passing through M_0 and M_1 determine a parallelogram *abcd* in $\widetilde{\Gamma}'$ as shown:



There exists a walk $\widetilde{\mathcal{F}}$ inside *abcd* lifting the non-oriented path $\mathcal{F}$. Since the horizontal size of *abcd* is $m+1$, while its vertical size is $m-1$, the walk $\widetilde{\mathcal{F}}$ must necessarily contain a subpath as in Lemma 1.2. We thus obtain a contradiction.

(b) Assume that $\mathcal{F}$ does not pass through all pairs of neighbouring corays. Without loss of generality, we may suppose that $\mathcal{F}$ does not pass through the pair $(\nearrow(1, 1))$, $(\nearrow(1, m))$ and that there exists a point M of $\mathcal{F}$ on the coray $(\nearrow(1, m))$. We may further assume that M is the point of $\mathcal{F}$ on $(\nearrow(1, m))$ having the largest first coordinate (that is, quasi-length). We construct as in (a) a point M_0 of $\widetilde{\Gamma}'$ lifting M, we consider the coray from $a = (m-1, m-2)$ to $d = (1, m)$ passing through M_0, then construct a parallelogram $abcd$, where $b = (m-1, m)$ and $c = (1, 1)$. The hypothesis (b) says that there exists a lifting $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ which is entirely contained inside $abcd$.

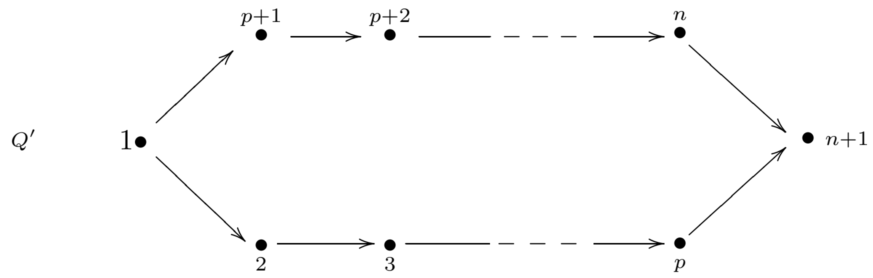


We claim that $M_0 \neq (1, m)$, $M_0 \neq (m-1, 1)$ and that M_0 is a source in $\widetilde{\mathcal{F}}$. Indeed, if $M_0 = (1, m)$, then there is a single ray $(M_0\searrow)$ starting at M_0, no other paths in $abcd$ starting or ending at M_0, so that we cannot form a cycle. If $M_0 = (m-1, 1)$, then there is a single coray $(M_0\nearrow)$ starting at M_0, no other paths starting or ending at M_0, so that we cannot form a cycle. Finally, let $M_0 \neq (1, m), (m-1, m-2)$. Then, by the choice of M, the only walks through M_0 which may lie in $\widetilde{\mathcal{F}}$ start with arrows from $(M_0\nearrow) \cup (M_0\searrow)$, that is, M_0 is a source in $\widetilde{\mathcal{F}}$. But then $\widetilde{\mathcal{F}}$ must contain a subpath as in Lemma 1.2, a contradiction. ■

THEOREM 1.5. *Let Γ be an exceptional tube in the Auslander–Reiten quiver of the path algebra of a Euclidean quiver*, *and $\mathcal{E} = (E_1, \ldots, E_t)$ be a connected exceptional sequence whose terms lie in Γ. Then* $\operatorname{End}\mathcal{E}$ *is a tilted algebra of type $\mathbb{A}_t$.*

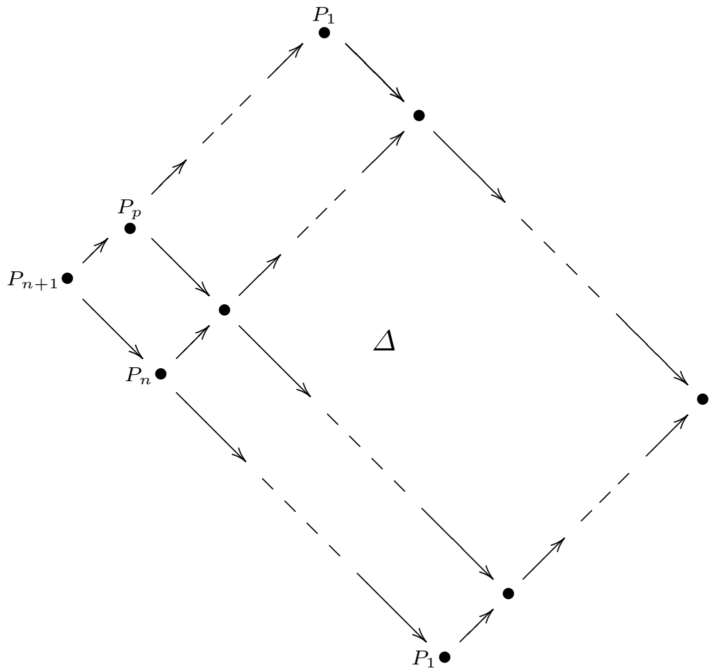
P r o o f. By [A1, H], we must show that the bound quiver of $\operatorname{End}\mathcal{E}$ is a gentle tree without double zeros. By Lemma 1.4, this quiver is a tree. It follows from Lemma 1.1 that the number of arrows entering or leaving a given point is at most two. By Lemmata 1.2 and 1.3, the bound quiver of $\operatorname{End}\mathcal{E}$ is gentle. Finally, Lemma 1.2 also implies that it has no double zeros. ■

2. Postprojective components. Let $A = kQ$ be the path algebra of a quiver Q of type $\widetilde{\mathbb{A}}_n$, with an arbitrary orientation. Assume that Q has p arrows in the counterclockwise sense, and q in the clockwise sense (thus $p + q = n + 1$). We may clearly assume that $p \geq q$. Let Q' be the quiver of type $\widetilde{\mathbb{A}}_n$ having just one source 1, and one sink $n+1$, and having p arrows in the counterclockwise sense, and q in the clockwise sense, and let $B = kQ'$.

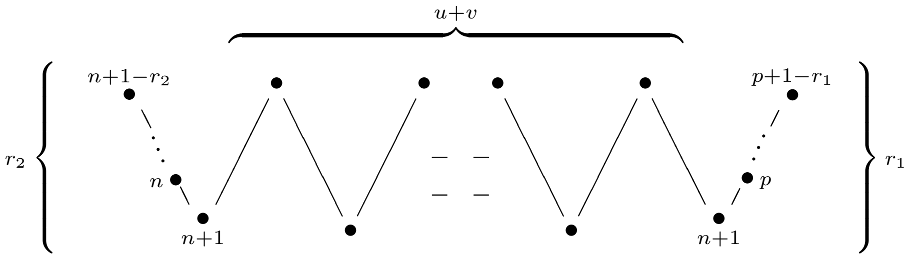


For a point i, we denote by P_i (or I_i) the corresponding indecomposable projective (or injective, respectively) module. There exists a tilting B-module T_B, which is the slice module of a complete slice in the postprojective component $\mathcal{P}$ of $\Gamma(\operatorname{mod} B)$, having as summand P_1, such that $A = \operatorname{End} T_B$. The tilting module T_B determines a torsion pair in each of $\operatorname{mod} B$ and $\operatorname{mod} A$ such that the full subcategory of $\operatorname{mod} A$ consisting of the postprojective A-modules is equivalent to the full subcategory of $\operatorname{mod} B$ consisting of the torsion postprojective B-modules [A2].

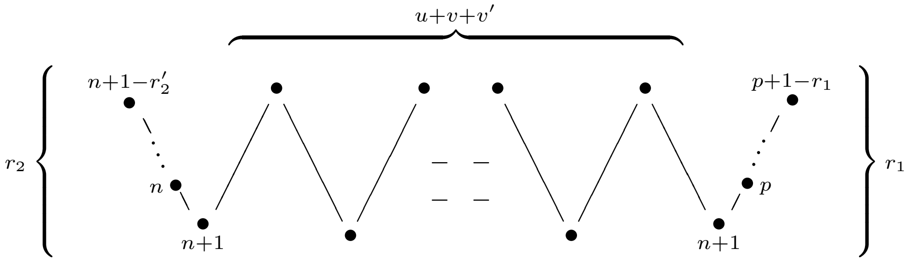
The postprojective component $\mathcal{P}$ contains two types of sectional paths, those parallel to the path from P_{n+1} to P_1 via P_p, which we call (q)-*paths*, and those parallel to the path from P_{n+1} to P_1 via P_n, which we call (p)-*paths*. We denote by Δ the full translation subquiver of $\mathcal{P}$ bounded by the two paths from P_{n+1} to P_1, and the (q)-path, and the (p)-path starting at P_1.



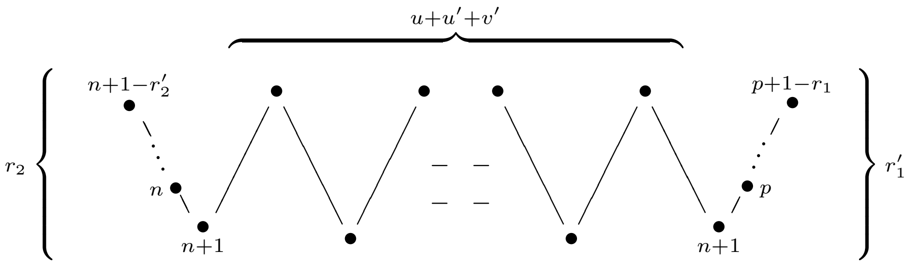
The indecomposable modules in $\mathcal{P}$ are described by lines in $\widetilde{Q}'$, the universal covering of Q' (see [BR]). Thus, for any $r \geq 0$, $\tau^{-r}P_{n+1}$ is given by the line



where the integers u, v, r_1, r_2 are defined by $r = pu + r_1 = qv + r_2$, $u, v \geq 0$, $0 \leq r_1 < p$, $0 \leq r_2 < q$. For $p < k \leq n+1$ and $r \geq 0$, $\tau^{-r}P_k$ is given by the line



where u, v, r_1, r_2 are as above, and v', r_2' are defined by $r_2 + (n+1-k) = v'q + r_2'$, $v' \geq 0$, $0 \leq r_2' < q$. Finally, for $1 \leq l \leq p$ and $r \geq 0$, $\tau^{-r}P_l$ is given by the line



where u, v, r_1, r_2 are as above and u', r_1' are defined by $r_1 + (p+1-l) = u'p + r_1'$, $u' \geq 0$, $0 \leq r_1' < p$.

We call $n+1-r_2$ (or $n+1-r_2'$) the *left endpoint* and $p+1-r_1$ (or $p+1-r_1'$) the *right endpoint* of the module.

LEMMA 2.1. *In $\mathcal{P}$, the modules lying on a (p)-path have the same right endpoint, and those lying on a (q)-path have the same left endpoint. Moreover, each path in the postprojective (or preinjective component) of $\Gamma(\operatorname{mod} A)$ is a monomorphism (or epimorphism, respectively).*

Proof. The first statement follows from the above description of the modules in $\mathcal{P}$, the second from this description and the tilting functor $\mathrm{Hom}_B(T,-)\colon \mathrm{mod}\,B \to \mathrm{mod}\,A$. ∎

Let now $\mathcal{E} = (E_1, \ldots, E_t)$ be an exceptional sequence in the postprojective component $\mathcal{P}$ of $\Gamma(\mathrm{mod}\,B)$. Applying the functor $\tau = \mathrm{D}\,\mathrm{Ext}^1_B(-, B)$, we may assume that one of the modules of $\mathcal{E}$ is projective. But now, if M, N are two modules in $\mathcal{P}$, we have $\mathrm{Hom}_B(M,N) \neq 0$ if and only if there exists a path from M to N in $\mathcal{P}$. Thus, (M,N) is a subsequence of $\mathcal{E}$ if and only if there exists a path from M to N in $\mathcal{P}$, but no path from M to τN. Since, for any indecomposable projective B-module P, and indecomposable module X which is not in Δ, there exists a path from P to τX, we deduce that $\mathcal{E}$ lies entirely in Δ.

LEMMA 2.2. *Let $\mathcal{E} = (E_1, \ldots, E_t)$ be an exceptional sequence in $\mathcal{P}$. Then there exists a complete slice $\mathcal{S}$ of $\mathcal{P}$ such that all terms of $\mathcal{E}$ lie on $\mathcal{S}$.*

Proof. Assume that E_i, E_j are two terms in $\mathcal{E}$. We claim that E_i, E_j belong to different τ-orbits in $\mathcal{P}$. Indeed, if this is not the case, then there exist an indecomposable projective module P_B and integers $r < s$ such that $E_i = \tau^{-r}P$, $E_j = \tau^{-s}P$. But then $\mathrm{Hom}_B(E_i, E_j) \neq 0$ implies that (E_i, E_j) is a subsequence of $\mathcal{E}$, and this contradicts $\mathrm{Ext}^1_B(E_j, E_i) \cong \mathrm{D}\,\mathrm{Hom}_B(\tau^{-1}E_i, E_j) \neq 0$.

Let again E_i, E_j be two terms of $\mathcal{E}$. We may assume without loss of generality that (E_i, E_j) is a subsequence of $\mathcal{E}$ and such that the τ-orbits of E_i and E_j are neighbours among the orbits of the terms of $\mathcal{E}$ in the orbit graph of $\mathcal{P}$. Now $\mathrm{Hom}_B(\tau^{-1}E_i, E_j) = 0$ implies that E_j is not a successor of $\tau^{-1}E_i$ in Δ and $\mathrm{Hom}_B(E_j, E_i) = 0$ implies that E_j is not a predecessor of E_i. This shows that, if there exists a path from E_i to E_j, then this path is sectional. Consequently, E_i and E_j lie on a complete slice $\mathcal{S}$ of $\mathcal{P}$, and hence so do all terms in $\mathcal{E}$. ∎

COROLLARY 2.3. *Let $\mathcal{E} = (E_1, \ldots, E_t)$ be an exceptional sequence in the postprojective component of $\Gamma(\mathrm{mod}\,A)$. Then $\mathrm{End}\,\mathcal{E}$ is a direct product of path algebras of type $\mathbb{A}_m$ (with $m \leq t$), or is a connected path algebra of type $\widetilde{\mathbb{A}}_{t-1}$.* ∎

LEMMA 2.4. *If (M,N) is an exceptional sequence in $\mathrm{mod}\,A$, with M postprojective and N preinjective, then $\mathrm{Hom}_A(M,N) = 0$.*

Proof. Applying the functor τ^{-1}, we may assume that N is injective. By Lemma 2.1, there exists a monomorphism $f : M \to \tau^{-1}M$. Assume that $g : M \to N$ is non-zero. The injectivity of N implies the existence of $g' : \tau^{-1}M \to N$ such that $g = g'f$. Thus $g' \neq 0$. Hence $\mathrm{Ext}^1_A(N,M) \cong \mathrm{D}\,\mathrm{Hom}_A(\tau^{-1}M, N) \neq 0$, a contradiction to the fact that (M,N) is an exceptional sequence. ∎

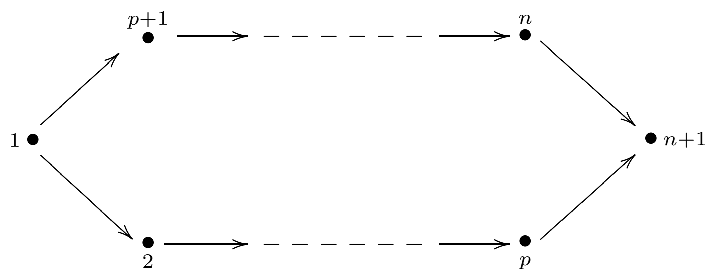
LEMMA 2.5. *Let $\mathcal{E} = (E_1, \ldots, E_t, F_1, \ldots, F_s)$ be an exceptional sequence in* $\operatorname{mod} A$, *with the E_i postprojective and the F_j preinjective, and $t, s \geq 1$. Then* $\operatorname{End}(\bigoplus_{i=1}^t E_i)$ *is not the path algebra of a quiver of type $\widetilde{\mathbb{A}}_{t-1}$.*

Proof. We assume that $\operatorname{End}(\bigoplus_{i=1}^t E_i)$ is the path algebra of a quiver of type $\widetilde{\mathbb{A}}_{t-1}$ and show that $\mathcal{E}$ cannot contain any preinjective term.

Let the quiver Q of A have sources $i_1, \ldots, i_r$ and sinks $j_1, \ldots, j_r$ such that we have paths from i_k to j_{k-1} and j_k, for each $1 < k \leq r$, and paths from i_1 to j_r and j_1. Then, for each j lying on the reduced walk from j_{k-1} to j_k containing i_k, we have $\operatorname{Hom}_A(P_{i_k}, I_j) \neq 0$. Let $m > 0$ be an arbitrary integer. By Lemma 2.1, there exists an epimorphism $\tau^m I_j \to I_j$, hence an epimorphism $\operatorname{Hom}_A(P_{i_k}, \tau^m I_j) \to \operatorname{Hom}_A(P_{i_k}, I_j)$ so that $\operatorname{Hom}_A(P_{i_k}, \tau^m I_j) \neq 0$. Furthermore, for any monomorphism $f : P_{i_k} \to X$ with X postprojective and morphism $g : P_{i_k} \to \tau^m I_j$, there exists a morphism $g' : X \to \tau^m I_j$ such that $g'f = g$, because we may apply the functor τ^{-m} to these modules. Thus $\operatorname{Hom}_A(X, \tau^m I_j) \neq 0$.

It follows from the proof of Lemma 2.2 that $\operatorname{End}(\bigoplus_{i=1}^t E_i)$ is hereditary of type $\widetilde{\mathbb{A}}_{t-1}$ if and only if the terms E_i lie on a complete slice $\mathcal{S}$, of which all the sources and sinks are themselves terms of the sequence. If all of $P_{i_1}, \ldots, P_{i_r}$ are terms of $\mathcal{E}$, we are done. If P_{i_k} is not a term of $\mathcal{E}$, there exists a sink X of $\mathcal{S}$ that is a term of $\mathcal{E}$, and such that P_{i_k} is a submodule of X. Therefore $\mathcal{E}$ cannot contain any preinjective term. ■

3. The arrows from postprojective to regular. In this section, we assume that A is a hereditary algebra of type $\widetilde{\mathbb{A}}_n$, and that $\mathcal{E}$ is an exceptional sequence in $\operatorname{mod} A$ such that some terms of $\mathcal{E}$ are postprojective, and some are regular. It follows from the considerations at the beginning of Section 2 that we may assume A to be given by the following quiver:



Then $\Gamma(\operatorname{mod} A)$ has two exceptional tubes Γ_p and Γ_q, of respective ranks p and q. We denote, as in Section 1, by Γ'_p and Γ'_q the full translation subquiver of Γ_p and Γ_q, respectively, consisting of the exceptional modules. We need one more notation: let M be a mouth module in an exceptional tube; the *mitre* $\widehat{M}$ of M is the full translation subquiver consisting of those exceptional modules N in the tube such that there exist sectional paths $X \to \ldots \to N$ for some X in $(\nearrow M)$ and $N \to \ldots \to Y$ for some Y in $(M\searrow)$.

Lemma 3.1. *Let* (M, N) *be an exceptional sequence with* M *postprojective and* N *regular. Assume the left endpoint of* M *is* k (*with* $p+1 \le k \le n+1$) *and its right endpoint is* l (*with* $2 \le l \le p$ *or* $l = n+1$). *Then* $\mathrm{Hom}_A(M, N) \neq 0$ *if and only if one of the following conditions is satisfied*:

(a) $N \in (\nearrow(1, i))$ *in* Γ_q, *where* $i = 1$ *whenever* $k = n+1$ *and* $i = k-p+1$ *whenever* $p+1 \le k \le n$, *or*

(b) $N \in (\nearrow(1, i))$ *in* Γ_p, *where* $i = 1$ *whenever* $l = n+1$ *and* $i = l$ *whenever* $2 \le l \le p$.

Proof. By the description [DR] of the indecomposable regular A-modules, $\mathrm{Hom}_A(M, N) \neq 0$ implies $N \in (\widehat{1, i})$ and, since (M, N) is an exceptional sequence, we have $\mathrm{Hom}_A(\tau^{-1}M, N) = 0$ so that $N \notin (1, \widehat{i-1})$. ∎

We shall need the dual of Lemma 3.1, which we state here for future reference.

Lemma 3.2. *Let* (M, N) *be an exceptional sequence with* M *regular and* N *preinjective. Assume the right endpoint of* N *is* k (*with* $k = 1$ *or* $p + 1 \le k \le n + 1$) *and the left endpoint of* N *is* l (*with* $1 \le l \le p$). *Then* $\mathrm{Hom}_A(M, N) \neq 0$ *if and only if one of the following conditions is satisfied*:

(a) $M \in ((1, k)\searrow)$ *in* Γ_q, *or*

(b) $M \in ((1, l)\searrow)$ *in* Γ_p. ∎

Lemma 3.3. *Let* (M, N) *be an exceptional sequence with* M *postprojective and* N *regular. If* $\mathrm{Hom}_A(M, N) \neq 0$, *there exists no* $L \in (N\searrow)$ *such that* (M, N, L) *is an exceptional sequence.*

Proof. Indeed, if this is the case, then $L \in (1, \widehat{i-1})$ so that we have $\mathrm{Hom}_A(\tau^{-1}M, L) \neq 0$, a contradiction. ∎

We shall again need the dual.

Lemma 3.4. *Let* (M, N) *be an exceptional sequence with* M *regular and* N *preinjective. If* $\mathrm{Hom}_A(M, N) \neq 0$, *there exists no* $L \in (\nearrow M)$ *such that* (L, M, N) *is an exceptional sequence.* ∎

Lemma 3.5. *Let* $\mathcal{E} = (E_1, \ldots, E_r, F_1, \ldots, F_s)$ *be an exceptional sequence with the* E_i *postprojective*, *the* F_j *regular*, *and* $(F_1, \ldots, F_s)$ *connected. Then there exist a unique* E_i *and a unique* F_j *such that* $\mathrm{Hom}_A(E_i, F_j) \neq 0$, *and the non-zero morphisms from* E_i *to* F_j *factor through no other module in* $\mathcal{E}$.

Proof. Since $(F_1, \ldots, F_s)$ is connected, we may assume without loss of generality that the F_j lie in Γ_p. By Lemma 3.1, we must consider the right endpoint of any postprojective term of $\mathcal{E}$ which maps non-trivially to them.

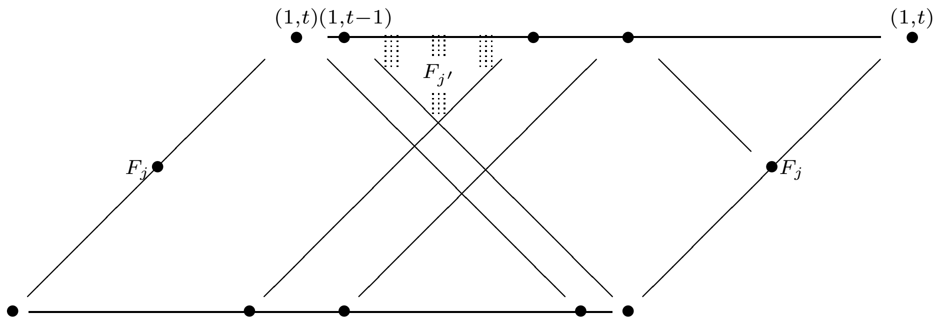
Assume that $E_{i_1} \to \ldots \to E_{i_u}$ in $\mathcal{E}$, where all these modules have the same right endpoint l; then, by Lemma 2.1, these modules are linearly ordered by inclusion. If these modules map non-trivially to some regular term in $\mathcal{E}$, then these regular terms $F_{j_1} \to \ldots \to F_{j_v}$ belong to $(\nearrow(1,t))$, where $t = l$ if $2 \le l \le p$, or $t = 1$ if $l = n+1$, and hence are linearly ordered by the quotient relation. Since

$$\begin{aligned}\operatorname{End}\mathcal{E} = \Big[\bigoplus \operatorname{Hom}_A(E_{i_f}, E_{i_g})\Big] \oplus \Big[\bigoplus \operatorname{Hom}_A(E_{i_g}, F_{j_h})\Big] \\ \oplus \Big[\bigoplus \operatorname{Hom}_A(F_{j_h}, F_{j_k})\Big]\end{aligned}$$

we choose $E_i = E_{i_u}$ and $F_j = F_{j_1}$. By construction, $\operatorname{Hom}_A(E_i, F_j) \neq 0$ and the non-zero morphisms from E_i to F_j factor through no other module in $\mathcal{E}$.

It remains to prove the uniqueness of the pair (E_i, F_j). Since, clearly, any pair satisfying the conditions of the statement is constructed in the above way, assume that there exist $E_{i'}$ with right endpoint $l' \neq l$, and $F_{j'}$, on the line $(\nearrow(1,t'))$ where $t' = l'$ if $2 \le l' \le p$ and $t' = 1$ if $l' = n+1$.

Since $\operatorname{Hom}_A(\tau^{-1}E_i, F_{j'}) = 0$, we have $F_{j'} \notin (1, \widehat{t-1})$. Also, notice that $F_{j'} \notin (\nearrow(1,t))$ by construction of F_j. Since $\operatorname{Hom}_A(\tau^{-1}E_{i'}, F_j) = 0$, we have similarly $F_j \in (1, \widehat{t'-1})$. Therefore, $F_{j'}$ belongs to the shaded area in the figure below.



By Lemma 3.3, there is no $L \in \mathcal{E}$ such that $L \in (\searrow F_j)$ or $L \in (\searrow F_{j'})$. By Lemma 1.2, there is no path $F_j \to L \to L'$ (or $F_{j'} \to L \to L'$) with $L \in (F_j \nearrow)$ (or $L \in (F_{j'} \nearrow)$, respectively) and $L' \in (L \searrow)$. Therefore, F_j and $F_{j'}$ are disconnected in Γ_p, a contradiction. ∎

LEMMA 3.6. *With the assumptions and notation of Lemma* 3.3, *we have*:

(a) *If* $\operatorname{End}(\bigoplus_{l=1}^{r} E_l)$ *is representation-infinite*, *then* E_i *is a sink of* $\mathcal{S}$.

(b) *If we have two morphisms* $f : E_l \to E_i$, $g : E_{l'} \to E_i$, *where* f *is induced by a* (q)*-path*, *and* g *is induced by a* (p)*-path*, *and if* $h : E_i \to F_j$ *is a non-zero morphism*, *then* $hf = 0$ *whenever* $F_j \in \Gamma_p$ *and* $hg = 0$ *whenever* $F_j \in \Gamma_q$.

Proof. (a) This follows from the choice of E_i in Lemma 3.5, and the structure of the complete slice $\mathcal{S}$ (see Lemma 2.2).

(b) This follows from the description of the indecomposable A-modules. ■

4. Proof of the main result. Assume now that A is a tame hereditary algebra of type $\widetilde{\mathbb{A}}_n$ (with any orientation), and that $\mathcal{E} = (E_1, \ldots, E_{n+1})$ is a complete exceptional sequence in $\operatorname{mod} A$. It follows easily from the considerations of Sections 2 and 3 that it suffices to consider the case where there exist t, s such that $(E_1, \ldots, E_t)$ are postprojective, $(E_{t+1}, \ldots, E_s)$ are regular and $(E_{s+1}, \ldots, E_{n+1})$ are preinjective.

We first recall the classification results from [AS, R, H] that will be needed. A triangular algebra is called *gentle* if it is isomorphic to a bound quiver algebra kQ/I, where (Q, I) satisfies:

(a) The number of arrows in Q with a given source or target is at most two.

(b) For any $\alpha \in Q_1$, there is at most one $\beta \in Q_1$ and one $\gamma \in Q_1$ such that $\alpha\beta$, $\gamma\alpha \notin I$.

(c) For any $\alpha \in Q_1$, there is at most one $\xi \in Q_1$ and one $\zeta \in Q_1$ such that $\alpha\xi$, $\zeta\alpha \in I$.

(d) I is generated by a set of paths of length two.

Then we have:

THEOREM 4.1 [AS]. *An algebra is iterated tilted of type $\widetilde{\mathbb{A}}_n$ if and only if it is gentle and its quiver contains a unique (non-oriented) cycle on which the number of clockwise oriented relations equals the number of counterclockwise oriented relations.* ■

THEOREM 4.2 [R, H]. *An iterated tilted algebra of type $\widetilde{\mathbb{A}}_n$ is tilted if and only if it contains no full subcategory of one of the following forms or their duals*:

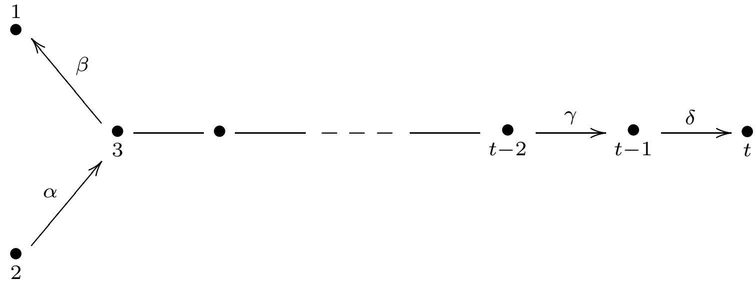
(a) $$\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet} \;\text{———}\; - - - \;\text{———}\; \underset{t-2}{\bullet} \xrightarrow{\gamma} \underset{t-1}{\bullet} \xrightarrow{\delta} \underset{t}{\bullet}$$

with $t \geq 4$, $\alpha\beta = 0$, $\gamma\delta = 0$.

(b) $$\underset{1}{\bullet} \xleftarrow{\alpha} \underset{2}{\bullet} \xleftarrow{\beta} \underset{3}{\bullet} \;\text{———}\; - - - \;\text{———}\; \underset{t-2}{\bullet} \xrightarrow{\gamma} \underset{t-1}{\bullet} \xrightarrow{\delta} \underset{t}{\bullet}$$

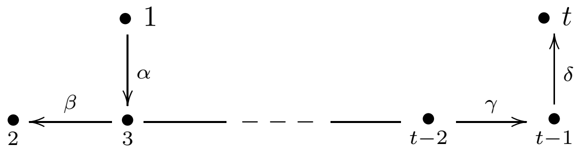
with $t \geq 4$, $\beta\alpha = 0$, $\gamma\delta = 0$, 1 *and* 2 *lie on the cycle while* $t-1$ *and* t *do not.*

(c)

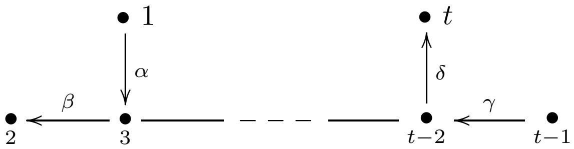


with $t \geq 6$, $\alpha\beta = 0$, $\gamma\delta = 0$, 1, 2 *and* 3 *lie on the cycle while* $t-2$, $t-1$ *and* t *do not.*

(d)



or



with $t \geq 5$, $\alpha\beta = 0$ $\gamma\delta = 0$, *all points* i *with* $2 \leq i \leq t-1$ *lie on the cycle while* 1 *and* t *do not.*

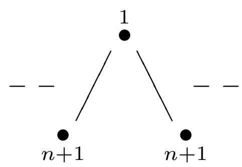
In each case, there are no other relations than the specified ones, and the arrows between 3 *and* $t-2$ *are oriented arbitrarily.* ■

LEMMA 4.3. *If* (E, F) *is an exceptional sequence in* $\operatorname{mod} A$ *with* E *postprojective and* F *preinjective, then*:

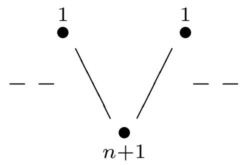
(a) E *belongs to the rectangle in the postprojective component* $\mathcal{P}$ *consisting of the* (p)*-paths starting at* P_{n+1} *and* P_3, *and the* (q)*-paths starting at* P_{n+1} *and* P_{p+2}.

(b) F *belongs to the rectangle in the preinjective component* $\mathcal{Q}$ *consisting of the* (p)*-paths ending at* I_{p-1} *and* I_1, *and the* (q)*-paths ending at* I_{n-1} *and* I_1.

Proof. This follows from the fact that, if E is a module of the form



and F is any preinjective module, then $\mathrm{Hom}_A(E,F) \neq 0$. Dually, if E is any postprojective module, while F is a module of the form

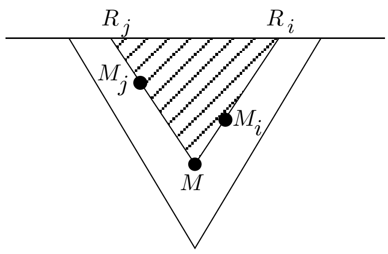


then $\mathrm{Hom}_A(E,F) \neq 0$. ■

LEMMA 4.4. *Let $(E, M_1, \ldots, M_s, F)$ be a connected shortest walk in the exceptional sequence $\mathcal{E}$, with E postprojective, F preinjective and all the M_l regular lying in the same exceptional tube. Then $s = 1$.*

Proof. We may assume that all the M_l belong to Γ_p. The connectedness of the given walk implies that $\mathrm{Hom}_A(E, M_i) \neq 0$ and $\mathrm{Hom}_A(M_j, F) \neq 0$ for some $1 \le i, j \le s$. Let l be the right endpoint of E. Since (E, M_i) is an exceptional sequence with $\mathrm{Hom}_A(E, M_i) \neq 0$, we see, by Lemma 3.1, that $M_i \in (\nearrow(1,l))$ whenever $2 \le l \le p$, and $M_i \in (\nearrow(1,1))$ whenever $l = n+1$. Dually, if k is the left endpoint of F, then, by Lemma 3.2, we have $M_j \in ((1,k)\searrow)$. Denote by R_i the point $(1,l)$ if $2 \le l \le p$, or $(1,1)$ if $l = n+1$, and by R_j the point $(1,k)$. By Lemma 3.5, we may assume that E is a sink (among the terms of $\mathcal{E}$) in a (p)-path, M_i is a source in $(\nearrow R_i)$, M_j is a sink in $(\searrow R_j)$, and F is a source in a (p)-path.

Since $\mathrm{Hom}_A(\tau^{-1}E, M_j) \cong \mathrm{D}\,\mathrm{Ext}^1_A(M_j, E) = 0$, it follows that $M_j \in (1, \widehat{l-1})$ when $2 \le l \le p$, and $M_j \notin (\widehat{1,p})$ when $l = n+1$. Dually, since $\mathrm{Hom}_A(M_i, \tau F) \cong \mathrm{D}\,\mathrm{Ext}^1_A(F, M_i) = 0$, we have $M_i \notin (1, \widehat{k+1})$ when $k \neq p$, and $M_i \notin (\widehat{1,p})$ when $k = p$. Letting M be the module of least quasi-length in the intersection of $(\nearrow R_i)$ and $(\searrow R_j)$, we find that M_i, M_j lie on the sides of the triangle $R_i M R_j$. Similarly, if $1 \le l \le s$, then M_l belongs neither to $(1, \widehat{l-1})$ when $2 \le l \le p$, to $(\widehat{1,p})$ when $l = p+1$, nor to $(1, \widehat{k+1})$ when $k \neq p$, to $(\widehat{1,p})$ when $k = p$. The connectedness of the given walk then implies that M_l belongs to the triangle $R_i M R_j$.



We claim that $M_i = M_j$. Assume that $M_i \neq M_j$ and that $\mathrm{Hom}_A(M_j, M_i) \neq 0$. Then $M_i \in \mathrm{Supp}\,\mathrm{Hom}_A(M_j, -)$ but $M_i \notin \mathrm{Supp}\,\mathrm{Hom}_A(\tau^{-1}M_j, -)$.

Hence $M_i = M$, and this contradicts the assumption that M_j is a sink in $(\searrow R_j)$. On the other hand, if $M_i \neq M_j$ and $\mathrm{Hom}_A(M_j, M_i) = 0$, then, by Lemma 1.1, there exists M_l inside the triangle $R_i M R_j$ such that $\mathrm{Hom}_A(M_i, M_l) \neq 0$ or $\mathrm{Hom}_A(M_l, M_i) \neq 0$, that is, $M_l \in (M_i \nearrow)$ or $M_l \in (\searrow M_i)$, since M_i is a source in $(\nearrow R_i)$. By the connectedness of the given sequence, there exists M_h such that $\mathrm{Hom}_A(M_l, M_h) \neq 0$ or $\mathrm{Hom}_A(M_h, M_l) \neq 0$, that is, $M_h \in (M_l \nearrow) \cup (M_l \searrow)$ or $M_h \in (\searrow M_l) \cup (\nearrow M_l)$. By induction and Lemma 1.2, we obtain a walk of the form

$$\begin{tikzcd}[column sep=small]
 & \bullet & & & \bullet \arrow[dr] & \\
M_h\bullet \arrow[ur] \arrow[dr] & & \text{or} & & & \bullet M_h \\
 & \bullet M_l & & & M_l\bullet \arrow[ur] \arrow[dr] & \\
M_i\bullet \arrow[ur] & & & & & \bullet M_i
\end{tikzcd}$$

Thus we cannot reach M_j, a contradiction. This shows that $M_i = M_j$. Hence $s = 1$. ∎

Lemma 4.5. *Let (E, M, F) be a connected subsequence of $\mathcal{E}$, with E postprojective, M regular and F preinjective. Then the simple module S_{n+1} is a direct summand of the socle of M.*

P r o o f. We observe that S_{n+1} is a direct summand of $\operatorname{soc} M$ if and only if

$$M \neq \begin{matrix} i \\ i+1 \\ \vdots \\ i+k \end{matrix}$$

(with $2 \leq i \leq p$, $i + k < p + 1$ if $M \in \Gamma_p$ or $p + 1 \leq i \leq n$, $i + k < n + 1$ if $M \in \Gamma_q$), or, equivalently, if and only if $M \in \widehat{R}$ (where

$$R = \begin{matrix} 1 \\ p+1 \\ \vdots \\ n \\ n+1 \end{matrix}$$

if $M \in \Gamma_p$, or

$$R = \begin{array}{c} 1 \\ 2 \\ \vdots \\ p \\ n+1 \end{array}$$

if $M \in \Gamma_q$). If S_{n+1} is not a direct summand of $\operatorname{soc} M$, and $M \in \Gamma_p$, then the right endpoint of E is i, and the left endpoint of F is $i+k$. Therefore the left endpoint of τF is $i+k+1$. Hence $\operatorname{Ext}^1_A(F,E) = \mathrm{D}\operatorname{Hom}_A(E,\tau F) \neq 0$, a contradiction. The proof is similar if $M \in \Gamma_q$. ∎

LEMMA 4.6. *Let (E_1, M_1, F_1) and (E_2, M_2, F_2) be two connected subsequences of $\mathcal{E}$, with E_1, E_2 postprojective, M_1, M_2 regular and F_1, F_2 preinjective. If $M_1 \neq M_2$, then M_1 and M_2 lie in two different tubes.*

Proof. Assume that this is not the case, and that both M_1 and M_2 lie in Γ_p (say). Suppose the right endpoint of E_1, and therefore of M_1, is l_1 where $3 \leq l_1 \leq p$, or $l_1 = n+1$, by Lemma 4.3, and similarly that the right endpoint of E_2, and therefore of M_2, is l_2, where $3 \leq l_2 \leq p$, or $l_2 = n+1$.

(a) Assume $l_1 = l_2 = l$, say; then $M_1, M_2 \in (\nearrow(1,l))$ when $l \leq p$, or $(\nearrow(1,1))$ whenever $l = n+1$. Without loss of generality, we may assume that $\operatorname{Hom}_A(M_1, M_2) \neq 0$. Now, $\operatorname{Hom}_A(M_2, F_2) \neq 0$, therefore $\operatorname{Hom}_A(F_2, M_1) = 0$ (or, equivalently, (M_1, F_2) is a subsequence of $\mathcal{E}$). Letting k_2 denote the left endpoint of M_2 and F_2, where $k_2 = 1, 2, \ldots, p-1$, we get $M_1 \notin (\widehat{1, k_2})$ and this contradicts the fact that $M_1 \in (\nearrow M_2)$.

(b) If $l_1 < l_2$, then $M_1 \in (\widehat{1, l_2})$ when $l_2 \leq p$, or $M_1 \in (\widehat{1,1})$ when $l_2 = n+1$. Hence $\operatorname{Hom}_A(E_2, M_1) \neq 0$. On the other hand, $M_1 \in (1, \widehat{l_2 - 1})$ when $l_2 \neq n+1$, and $M_1 \in (\widehat{1,p})$ when $l_2 = n+1$, thus $\operatorname{Hom}_A(\tau^{-1}E_2, M_1) \neq 0$, that is, $\operatorname{Ext}^1_A(M_1, E_2) \neq 0$. This is impossible, since E_2, M_1 belong to the same exceptional sequence $\mathcal{E}$. ∎

LEMMA 4.7. *Let (E, M) be a connected subsequence of $\mathcal{E}$, with E postprojective and a sink on a (p)-path (among the terms of $\mathcal{E}$), and $M \in \Gamma_p$. Then*

(a) $\operatorname{Hom}_A(E', E) \neq 0$, *with E' postprojective and in $\mathcal{E}$, implies that the path from E' to E is a (p)-path.*

(b) $\operatorname{Hom}_A(E, E'') \neq 0$, *with E'' postprojective and in $\mathcal{E}$, implies that the path from E to E'' is a (q)-path.*

Furthermore, there cannot exist at the same time in $\mathcal{E}$ terms such as E' and E'' above.

Proof. To show (a), assume that the path from E' to E is a (q)-path. The right endpoint of E' is larger than the right endpoint of E, and $M \in \widehat{R}$, where R is regular having the same right endpoint as that of $\tau^{-1}E'$, a contradiction. (b) is proven similarly. The last statement follows from the fact that, if E' and E'' both occur, then the points E', E, E'' cannot lie on a complete slice, a contradiction to Lemma 2.2. ∎

We shall also need the dual statement.

LEMMA 4.8. *Let (M,F) be a connected subsequence of $\mathcal{E}$, with F preinjective and a source on a (q)-path (among the terms of $\mathcal{E}$), and $M \in \Gamma_q$. Then*

(a) $\mathrm{Hom}_A(F,F') \neq 0$, *with F' preinjective and in $\mathcal{E}$, implies that the path from F to F' is a (q)-path.*

(b) $\mathrm{Hom}_A(F'',F) \neq 0$, *with F'' preinjective and in $\mathcal{E}$, implies that the path from F'' to F is a (p)-path.*

Furthermore, there cannot exist at the same time in $\mathcal{E}$ terms such as F' and F'' above. ∎

LEMMA 4.9. *If (E,M,F) is a connected subsequence of $\mathcal{E}$, with E postprojective, M regular and F preinjective, then $\mathrm{Hom}_A(E,F)=0$. Further, if M_1, M_2 are regular and $\mathrm{Hom}_A(M_1,M)\neq 0$, $\mathrm{Hom}_A(M,M_2)\neq 0$, then $M_1 \in (\searrow M)$, $M_2 \in (M\nearrow)$ and $\mathrm{Hom}_A(M_1,M_2)=0$.*

Proof. The first statement is clear by Lemma 2.4. The second statement follows from Lemmata 1.1, 3.3, 3.4 and 1.3. ∎

PROPOSITION 4.10. *Let $A = kQ$ be a path algebra of type $\widetilde{\mathbb{A}}_n$, and $\mathcal{E}$ be an exceptional sequence in $\mathrm{mod}\,A$. Assume that $\mathcal{E}$ contains a cycle $\mathcal{C}$ consisting of postprojective, regular and preinjective terms. Then the connected component of $\mathrm{End}\,\mathcal{E}$ containing the cycle corresponding to $\mathcal{C}$ is a representation-finite tilted algebra of type $\widetilde{\mathbb{A}}_l$, with $l \le n$.*

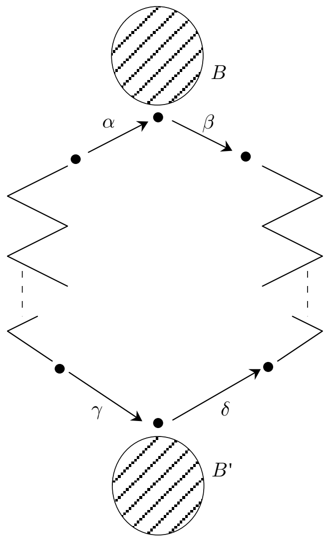
Proof. It follows from Lemmata 4.6, 4.4, 4.7, 4.8 and Theorem 1.5 that, if E belongs to $\mathcal{C}$, and E' belongs to $\mathcal{E}\backslash\mathcal{C}$, and both are postprojective, then

$$\mathrm{Hom}_A(E,E')=0 \quad \text{and} \quad \mathrm{Hom}_A(E',E)=0,$$

and, dually, if F belongs to $\mathcal{C}$, and F' belongs to $\mathcal{E}\backslash\mathcal{C}$, and both are preinjective, then

$$\mathrm{Hom}_A(F,F')=0 \quad \text{and} \quad \mathrm{Hom}_A(F',F)=0;$$

consequently, the quiver of the connected component of $\mathrm{End}\,\mathcal{E}$ containing the points corresponding to the cycle $\mathcal{C}$ is as follows:



where $\alpha\beta = 0$, $\gamma\delta = 0$, all unoriented edges on the cycle may be oriented arbitrarily, and B, B' are tilted algebras of type $\mathbb{A}_m$. The statement then follows from Theorem 4.2. ■

We may thus assume that, if a cycle occurs in the bound quiver of $\operatorname{End}\mathcal{E}$, then all the points of this cycle are postprojective (or, dually, preinjective). The main theorem follows from the next two lemmata.

LEMMA 4.11. *With the above notation*, $\operatorname{End}\mathcal{E}$ *is either a direct product of one representation-infinite iterated tilted algebra of type* $\widetilde{\mathbb{A}}_n$ (*with* $m \leq n$) *and iterated tilted algebras of type* $\mathbb{A}_l$ (*with* $l \leq n - m$), *or else a direct product of iterated tilted algebras of type* $\mathbb{A}_l$ (*with* $l \leq n + 1$).

Proof. By Theorem 1.5, Corollary 2.3 and Lemmata 2.5, 3.5, the ordinary quiver of $\operatorname{End}\mathcal{E}$ contains at most one cycle and, if it does, then this cycle is not bound by any relation. We thus only need to show that $\operatorname{End}\mathcal{E}$ is a gentle algebra. Assume that $\mathcal{F} = (F_1, \ldots, F_t)$ is a connected subsequence of $\mathcal{E}$. If $\mathcal{F}$ lies entirely in the regular part, then, by Theorem 1.5, $\operatorname{End}\mathcal{F}$ is gentle. If $\mathcal{F}$ lies in the postprojective (or the preinjective) component then, by Corollary 2.3, $\operatorname{End}\mathcal{F}$ is also gentle. Assume that we have non-zero morphisms

$$\begin{tikzcd}
E_{i_1} \arrow[dr, "f"] & & \\
& E_{i_3} \arrow[r, "h"] & E_{i_4} \\
E_{i_2} \arrow[ur, "g"] & &
\end{tikzcd}$$

where $E_{i_1}, E_{i_2}, E_{i_3}$ are postprojective, and E_{i_4} is regular (and E_{i_1} and E_{i_2} are not necessarily distinct). Then, by Lemma 3.6, we have either $hf = 0$ or $hg = 0$. Finally, assume that we have non-zero morphisms

$$E_{i_1} \xrightarrow{f} E_{i_2} \xrightarrow{g} E_{i_3}$$

which do not factor through other modules in $\mathcal{E}$, with E_{i_1} postprojective, and E_{i_2}, E_{i_3} regular. Then by Lemma 3.1, there exists no non-zero morphism $h : E_{i_2} \to E_{i_4}$ with $E_{i_4} \in \mathcal{E}$ regular and distinct from E_{i_3} and such that E_{i_2} does not factor through other modules in $\mathcal{E}$. Furthermore, if there exists a non-zero morphism $h : E_{i_4} \to E_{i_2}$ with $E_{i_4} \in \mathcal{E}$ regular and such that h does not factor through other modules in $\mathcal{E}$, then, by Lemmata 3.1, 3.5, we have $gh = 0$. Invoking the duality between postprojective and preinjective modules completes the proof. ∎

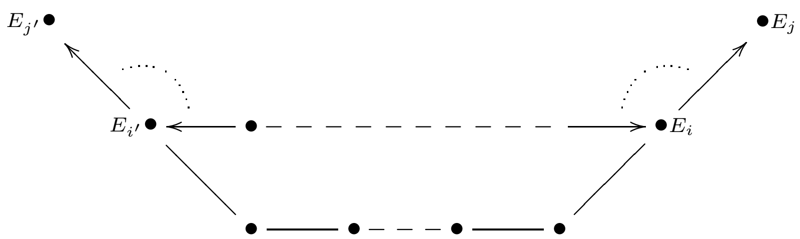
LEMMA 4.12. *With the notation above*, *each of the connected components of* $\operatorname{End}\mathcal{E}$ *is in fact a tilted algebra.*

Proof. By Theorem 1.5, Corollary 2.3 and Lemma 3.5, if a cycle occurs in the bound quiver of $\operatorname{End}\mathcal{E}$, then the corresponding terms of $\mathcal{E}$ are all postprojective (and then $\mathcal{E}$ has no preinjective terms, by Lemma 2.5) or all preinjective (and then, dually, $\mathcal{E}$ has no postprojective terms). Assume thus that a cycle occurs and that the corresponding terms of $\mathcal{E}$ all lie in $\mathcal{P}$. Then $\mathcal{E} = (E_1, \ldots, E_r, E_{r+1}, \ldots, E_{n+1})$ with $E_1, \ldots, E_r \in \mathcal{P}$, $E_{r+1}, \ldots, E_{n+1} \in \Gamma_p \vee \Gamma_q$ (here, $2 \le r \le n$) and $\operatorname{End}(\bigoplus_{i=1}^{r} E_i)$ is a path algebra of type $\widetilde{\mathbb{A}}_{r-1}$. In order to show our claim, we need to prove that the bound quiver of $\operatorname{End}\mathcal{E}$ contains no full bound subquiver of one of the forms (a)–(d) listed in Theorem 4.2.

We first notice that the arrows between $\mathcal{P}$ and $\Gamma_p \vee \Gamma_q$ are all from $\mathcal{P}$ to $\Gamma_p \vee \Gamma_q$, therefore case (c) cannot occur. Assume that (a) occurs, that is, there exists a walk of the form

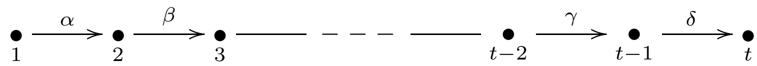
$$\underset{1}{\bullet} \xrightarrow{\alpha} \underset{2}{\bullet} \xrightarrow{\beta} \underset{3}{\bullet} \text{———} - - - \text{———} \underset{t-2}{\bullet} \xrightarrow{\gamma} \underset{t-1}{\bullet} \xrightarrow{\delta} \underset{t}{\bullet}$$

with $\alpha\beta = 0$, $\gamma\delta = 0$ and $t \ge 4$, in the bound quiver of $\operatorname{End}\mathcal{E}$. Then, by Theorem 1.5 and Corollary 2.3, not all the terms of $\mathcal{E}$ corresponding to the points of this walk lie in the same component. Since $\operatorname{End}(\bigoplus_{i=1}^{r} E_i)$ is hereditary, this means that the terms corresponding to $1, 2, \ldots, t-2$ are all regular. But this is impossible by Lemmata 1.2, 3.3. Thus (a) does not occur. Finally, for (b) and (d), we notice that the only possibility of occurrence of two zero-relations in the same walk pointing in different directions is of the form

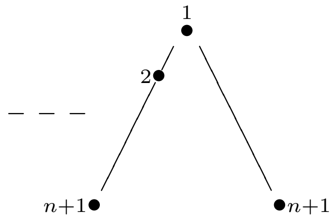


where the dotted lines indicate zero-relations. Therefore (b) and (d) do not occur. This completes the proof in case the bound quiver of $\operatorname{End}\mathcal{E}$ contains a cycle.

If the bound quiver of $\operatorname{End}\mathcal{E}$ contains no cycle we need to show that it contains no walk of the form

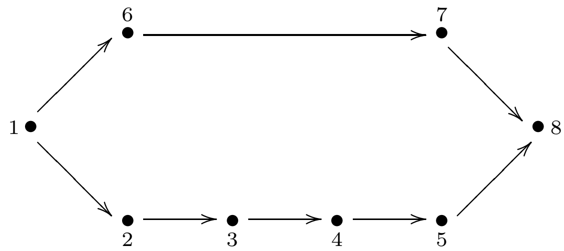


with $\alpha\beta = 0$, $\gamma\delta = 0$ and $t \geq 4$. If there is no path from $\mathcal{P}$ to the preinjective component $\mathcal{I}$ in $\mathcal{E}$, we are done by the argument above. If there exists such a path, then we have a subsequence (E, F, G) of $\mathcal{E}$ with $E \in \mathcal{P}$, $F \in \Gamma_p \vee \Gamma_q$, $G \in \mathcal{I}$ and $\operatorname{Hom}_A(E, F) \neq 0$, $\operatorname{Hom}_A(F, G) \neq 0$ by Lemmata 3.3, 3.4. Assume that $F \in \Gamma_p$ (the other case is similar) with right endpoint l, and $2 \leq l \leq p$, then E has right endpoint l, by Lemma 3.1, and G has left endpoint l, by Lemma 3.2. Then, if $l \neq 2$, $\tau^{-1}E$ has right endpoint $l-1$ or $\tau^{-1}E$ is given by a line of the form



(see [BR]). Since G has l as left endpoint, we have $\operatorname{Hom}_A(\tau^{-1}E, G) \neq 0$, a contradiction. ■

Remark 4.13. (a) With the above notation $\bigoplus_{i=1}^{n+1} E_i$ is generally not a tilting module; for instance, if A is given by the quiver



then the sequence (S_2, S_3) consisting of the simple modules corresponding to the points $2, 3$ is clearly exceptional, but $\mathrm{Ext}^1_A(S_2, S_3) \neq 0$ shows that $S_2 \oplus S_3$ is not a partial tilting module.

(b) The methods of Section 1 can be used with only slight modifications to prove the following theorem:

THEOREM. *Let k be a commutative field, Q be a quiver with underlying graph $\mathbb{A}_n$, and $A = kQ$ be its path algebra. Let $\mathcal{E}$ be an exceptional sequence in* $\operatorname{mod} A$. *Then* $\operatorname{End}\mathcal{E}$ *is a direct product of tilted algebras of type $\mathbb{A}_l$ (with $l \leq n$). Each connected subsequence of $\mathcal{E}$ is a partial tilting module.* ■

This strengthens the result of [Y]. We omit the proof, since we learned later that it was proved independently by Meltzer [M], using the derived category. In the same paper, Meltzer gives an example showing that a similar statement does not hold for other Dynkin diagrams.

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