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THE LOCAL DUALITY FOR HOMOMORPHISMS AND AN APPLICATION TO PURE SEMISIMPLE PI-RINGS

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The local duality $\mathsf{L} : M_R \mapsto {}_R\mathsf{L}M$ defined below is a useful tool both in module theory and in representation theory. For example, it is applied in [4, I, Theorem 3.9] to construct Auslander–Reiten sequences for finitely presented modules. It is shown in [17] that the local duality induces a dichotomy for the finite length modules over an artinian ring R which satisfies a polynomial identity. The consequences of this dichotomy for the representation theory of R are studied in [18] and [19]. If k is a commutative artinian ring and R a k-artin algebra, the local duality coincides on the finite length modules with the (functorial) duality $\mathsf{D} = \mathrm{Hom}(-, \mathrm{E}({}_k\overline{k}))$, where $\overline{k}$ is the factor ring $k/\operatorname{Rad} k$.

The local duality $\mathsf{L} : M_R \mapsto {}_R\mathsf{L}M$ is not functorial in general. The aim of this article is to show that L has the following related properties.

• The local duality commutes with finite direct sums, up to isomorphism, provided each summand has perfect endomorphism ring (Theorem 1.6). Its relation to further dualities given by proper subrings of the endomorphism ring is investigated in Propositions 1.2 and 1.5.

• The local duality can be defined for homomorphisms $f : M_R \to N_R$ between R-modules and it behaves well on a class of homomorphisms which we call "endofinite" (Theorem 3.2). However, this class may not be closed under addition or composition (Examples 1 and 2).

• For artinian right pure semisimple PI-rings, the local duality induces a bijection between the isoclasses of indecomposable finite length left and right modules. We use this bijection to obtain a new proof for the fact proved by Herzog [10] that such rings are of finite representation type.

Notation. Throughout this article by a ring we mean an associative ring with an identity element. A ring R is called *semilocal* provided its factor $\overline{R} = R/\operatorname{Rad} R$ modulo the (Jacobson) radical $\operatorname{Rad} R$ is semisimple. We denote by Mod-R and R-Mod the categories of all right and left R-modules,

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respectively. For the full subcategory of Mod-R consisting of the *finite length modules* we will write mod-R. Homomorphisms of modules will be written on the side of the elements which is opposite the scalars. For M, N in Mod-R the group of R-homomorphisms $\mathrm{Hom}(M_R, N_R)$ will also be denoted by (M_R, N_R). Obviously, (M_R, N_R) has a natural structure of an $\mathrm{End}\, N_R$-$\mathrm{End}\, M_R$-bimodule. Furthermore we write $M \Subset N_R$ for "M is isomorphic to a direct summand of the R-module N". For a right R-module M with semilocal endomorphism ring $S = \mathrm{End}\, M_R$, the *local dual* is defined as the left R-module

$$\mathsf{L}M = {}_R\mathrm{Hom}({}_SM, {}_SI)$$

of S-homomorphisms from ${}_SM$ to the S-injective envelope $I = {}_S\mathrm{E}({}_S\overline{S})$ of the factor $\overline{S} = S/\mathrm{Rad}\, S$. For the notion of purity (pure submodules, (Σ-) pure injective modules, finite matrix subgroups) we refer the reader to [11, Ch. 6–8].

1. Dualizing modules using subrings of endomorphism rings. Suppose that M is a right R-module and has a local right perfect endomorphism ring $S = \mathrm{End}\, M_R$. Let T be a subring of S and let ${}_TJ$ be an injective cogenerator of T-Mod. We denote by

$$\mathsf{L}_{T,J}M = {}_R({}_TM, {}_TJ)$$

the dual of M constructed using T and J. Proposition 1.2 is concerned with the relation between $\mathsf{L}M$ and $\mathsf{L}_{T,J}M$: There exists a set X such that the sum $(\mathsf{L}M)^{(X)}$ is a pure submodule of $\mathsf{L}_{T,J}M$ and $\mathsf{L}_{T,J}M$ is isomorphic to a summand of the product $(\mathsf{L}M)^X$. Sometimes we can obtain an isomorphism $\mathsf{L}M \cong \mathsf{L}_{T,J}M$: If R is a semilocal ring whose radical factor $\overline{R}$ is an artin algebra, the isomorphism class of the dual module ${}_R\mathsf{L}_{T,J}M$ does not depend on the subring T provided M has finite length both as a right R-module and as a left T-module and ${}_TJ = \mathrm{E}({}_T\overline{T})$ (Proposition 1.5). This extends a previous result of the author that the composition structure of the dual module does not depend on the subring T (see [17, Theorem 9]). Moreover, for a finite direct sum M of modules M_i with perfect endomorphism ring we obtain ${}_R\mathsf{L}M \cong \bigoplus \mathsf{L}M_i$ (Theorem 1.6).

For the proof of Proposition 1.2 we will need the following lemma.

LEMMA 1.1. *Let T be a subring of a right perfect ring S, let ${}_TJ$ be an injective T-module and put ${}_SI' = ({}_TS, {}_TJ)$. There exists a set X and injective envelopes I_x of simple S-modules for $x \in X$ such that $\coprod_{x\in X} I_x \subseteq {}_SI'$ is a pure and large submodule and ${}_SI' \Subset \prod_{x\in X} I_x$.*

Proof. We decompose the socle $\mathrm{Soc}\, {}_SI' = \coprod_{x\in x} E_x$ of ${}_SI'$ into simple modules ${}_SE_x$ and write I_x for their injective envelopes. Consider the following diagram of left S-modules, where ι_1, ι_2 and ι_3 are the canonical inclusions:

$$\begin{array}{ccccc} \operatorname{Soc}{}_S I' & \xrightarrow{\iota_2} & \coprod I_x & \xrightarrow{\iota_3} & \prod I_x \\ & \searrow^{\iota_1} & \downarrow{\scriptstyle \exists f} & \nearrow_{\exists g} & \\ & & {}_S I' & & \end{array}$$

Since the functor $({}_T S, -) : T\text{-Mod} \to S\text{-Mod}$ preserves injective modules (and cogenerators), ${}_S I'$ is injective and there is f such that $\iota_2 f = \iota_1$. This f is a monomorphism, since $\operatorname{Im} \iota_2$ is a large submodule [1, Prop. 6.17]. Hence we have g such that $fg = \iota_3$. By Bass' theorem, $\operatorname{Soc}{}_S I' \subseteq {}_S I'$ is a large submodule of ${}_S I'$, so g is a monomorphism and splits. Finally, ι_3 is a pure monomorphism, hence so is f. ∎

Proposition 1.2. *Let T be a subring of a local right perfect ring S, let ${}_T J$ be an injective cogenerator and ${}_S M_R$ a bimodule.*

(1) $\mathsf{L}M \mathrel{\in\!\!\mid} {}_R \mathsf{L}_{T,J} M$.

(2) *There is a set X and a pure embedding $(\mathsf{L}M)^{(X)} \subseteq {}_R \mathsf{L}_{T,J} M$ such that $\mathsf{L}_{T,J} M \mathrel{\in\!\!\mid} {}_R (\mathsf{L}M)^X$.*

(3) *If ${}_S M$ is finitely generated and $\mathrm{E}({}_S \overline{S})$ is Σ-pure injective, then $\mathsf{L}_{T,J} M \cong {}_R (\mathsf{L}M)^{(X)}$ for some set X.*

Proof. Put ${}_S I = \mathrm{E}({}_S \overline{S})$ and ${}_S I' = ({}_T S, {}_T J)$ and note that $\mathsf{L}M = {}_R({}_S M, {}_S I)$, whereas $\mathsf{L}_{T,J} M = {}_R({}_T M, {}_T J) \cong ({}_S M, {}_S I')$.

(1) We have seen in the proof of Lemma 1.1 that ${}_S I'$ is an injective cogenerator, so $I \mathrel{\in\!\!\mid} {}_S I'$. Hence $\mathsf{L}M \mathrel{\in\!\!\mid} {}_R \mathsf{L}_{T,J} M$.

(2) By Lemma 1.1, we have a set X such that the sum $I^{(X)}$ is isomorphic to a large pure submodule of ${}_S I'$ and I' is isomorphic to a summand of the product ${}_S I^X$. Hence we have

$$(*) \quad (\mathsf{L}M)^{(X)} = ({}_S M, {}_S I)^{(X)} \subseteq ({}_S M, {}_S I^{(X)}) \subseteq ({}_S M, {}_S I') \cong {}_R \mathsf{L}_{T,J} M$$

and $\mathsf{L}_{T,J} M \mathrel{\in\!\!\mid} {}_R({}_S M, {}_S I^X) = (\mathsf{L}M)^X$. Since $(\mathsf{L}M)^{(X)} \subseteq {}_R (\mathsf{L}M)^X$ is a pure submodule, also the embedding $(\mathsf{L}M)^{(X)} \subseteq {}_R \mathsf{L}_{T,J} M$ is pure.

(3) If ${}_S I$ is Σ-pure injective, the pure embedding $I^{(X)} \subseteq {}_S I'$ splits and we have $I^{(X)} \cong {}_S I'$ since $I^{(X)}$ is large in ${}_S I'$. Furthermore, if ${}_S M$ is finitely generated, we have equality in $({}_S M, {}_S I)^{(X)} \subseteq ({}_S M, {}_S I^{(X)})$ and it follows from $(*)$ in (2) that $\mathsf{L}_{T,J} M \cong {}_R (\mathsf{L}M)^{(X)}$. ∎

In several situations we can obtain an isomorphism ${}_R \mathsf{L}M \cong \mathsf{L}_{T,J} M$ using the following

Lemma 1.3. *Let ${}_S M_R$ be a bimodule, S a right perfect ring, $T \subseteq S$ a subring and ${}_T J$ an injective module. If $({}_T \overline{S}, {}_T J) \cong {}_S \overline{S}$, then $({}_S M, {}_S \mathrm{E}({}_S \overline{S})) \cong {}_R({}_T M, {}_T J)$.*

Proof. Since S is a semilocal ring, we have $\mathrm{Soc}\,_S({}_TS, {}_TJ) = ({}_T\overline{S}, {}_TJ) \cong {}_S\overline{S}$ and it can be easily seen (as in the proof of Lemma 1.1) that the injective modules ${}_S({}_TS, {}_TJ)$ and ${}_S\mathrm{E}({}_S\overline{S})$ are isomorphic. The claim follows from an application of the Hom-$\otimes$-adjoint isomorphism to ${}_R({}_SM, {}_S({}_TS, {}_TJ))$. ■

The following immediate consequence is well known [5, proof of Prop. 2.7]. It shows that the dualities L and D coincide for modules over artin algebras.

Corollary 1.4. *Suppose M_R is a finitely generated module over a k-artin algebra R. Then ${}_R\mathsf{L}M \cong \mathsf{D}M$ where $\mathsf{D} = (-, {}_k\mathrm{E}({}_k\overline{k}))$: mod-$R \to$ R-mod is the classical duality.* ■

The following proposition shows that the isoclass of the L-dual module does not depend on the subring.

Proposition 1.5. *Let R be a semilocal ring such that $\overline{R}$ is an artin algebra. Suppose that M_R is a finite length module and $T \subseteq \mathrm{End}\, M_R$ a subring such that ${}_TM$ has finite length. Then ${}_R\mathsf{L}M \cong ({}_TM, {}_TJ)$, where ${}_TJ = \mathrm{E}({}_T\overline{T})$.*

Proof. In the first step we show that we may assume that R is an artinian PI-ring, i.e. that R is artinian and $\overline{R}$ is an artin algebra. Since ${}_TM$ is finitely generated, say by $m_1, \ldots, m_t$, there is a monomorphism $R/A \to M_R^t$, $\overline{r} \mapsto (m_1r, \ldots, m_tr)$, where $A = \mathrm{ann}\, M_R$ is the annihilator ideal. So R/A is right artinian. Since the dual module $\mathsf{L}M$ also has finite length as a left R-module and as a right $\mathrm{End}\,_TJ$-module [17, Theorem 11], it follows from the same argument that R/A is also left artinian. Now ${}_R\mathsf{L}(M_R) = {}_R\mathsf{L}(M_{R/A})$ and also ${}_R({}_TM_{R/A}, {}_TJ) = {}_R({}_TM_R, {}_TJ)$ are equal, so the claim of the first step has been shown.

In the second step we assume that M_R is a finite length module over an artinian PI-ring R. Since $T \subseteq \mathrm{End}\, M_R$ is a subring such that ${}_TM$ has finite length, we deduce from [17, Cor. 13] that T is an artinian PI-ring. By Rosenberg and Zelinsky's theorem [15, Theorem 3] the module ${}_TJ$ is finitely generated, hence it induces a Morita duality. We claim that ${}_R({}_TM, {}_TJ) \cong \mathsf{L}M$. Let $S = \mathrm{End}\, M$. The bimodule $\overline{S}$ is, both as a left T-module and as a right S-module, a finite length module over a semiprimary PI-ring. In this case the multiplicity of $\overline{Se}$ as a composition factor of the Morita dual module ${}_S({}_T\overline{S}, {}_TJ)$ coincides with the multiplicity of $\overline{eS}$ as a composition factor of $\overline{S}_S$ for each primitive idempotent $e \in S$ [17, Theorem 11]. Thus $({}_T\overline{S}, {}_TJ) \cong {}_S\overline{S}$ and the claim follows from Lemma 1.3. ■

Now we are able to show that the local duality commutes with finite direct sums, up to isomorphism.

THEOREM 1.6. *Let $M_R = \coprod_{i=1}^{n} M_i$ be a sum of modules, each with right perfect endomorphism ring. Then ${}_R\mathsf{L}M \cong \coprod_{i=1}^{n} \mathsf{L}M_i$.*

Proof. We may assume that M_R has the decomposition $M = \coprod_{i=1}^{t} M_i^{n_i}$ where the modules M_i have local perfect endomorphism ring S_i and are pairwise nonisomorphic; otherwise decompose the modules M and $M_1, \ldots, M_n$ in the theorem.

Consider $S = \mathrm{End}(M_1^{n_1} \oplus \ldots \oplus M_t^{n_t})$ as $n \times n$-matrix ring and take for T the diagonal subring $S_1^{n_1} \times \ldots \times S_t^{n_t}$, where $n = n_1 + \ldots + n_t$. Put ${}_S I = \mathrm{E}({}_S\overline{S})$ and ${}_T J = \mathrm{E}({}_T\overline{T})$. Now, $\overline{S} = \overline{S}_1^{n_1 \times n_1} \times \ldots \times \overline{S}_t^{n_t \times n_t}$ is a $\overline{T}$-module satisfying ${}_S\overline{S} \cong ({}_T\overline{S}, {}_T J)$.

The ring S is right perfect by [1, Prop. 28.11], so it follows from Lemma 1.3 that ${}_R({}_S M, {}_S I) \cong ({}_T M, {}_T J)$. Observe that the ith factor of T acts trivially on the jth summand of $M_1^{n_1} \oplus \ldots \oplus M_t^{n_t}$ for $1 \le i, j \le n$ and $j \ne i$, so ${}_R({}_T M, {}_T J) \cong \coprod_{i=1}^{n} ({}_{S_i} M_i, {}_{S_i}\mathrm{E}({}_{S_i}\overline{S}_i)) = \coprod_{i=1}^{n} \mathsf{L}M_i$ and the claim has been shown. ∎

2. The endomorphism ring of a homomorphism. In this section we introduce the endomorphism ring of a homomorphism $f : M_R \to N_R$ as the endomorphism ring of the triple (M, N, f) when considered as a module over the triangular matrix ring $T_2(R) = \left(\begin{smallmatrix} R & R \\ 0 & R \end{smallmatrix}\right)$, and list several properties.

Let R be a ring. Recall that a right module over $T_2(R)$ is a triple (M, N, f) where M, N are R-modules and $f : M_R \to N_R$ is a homomorphism. The ring $T_2(R)$ acts on (M, N, f) as

$$(*) \qquad (m, n) \cdot \begin{pmatrix} r_1 & r_3 \\ 0 & r_2 \end{pmatrix} = (mr_1, f(m)r_3 + nr_2).$$

Homomorphisms between $T_2(R)$-modules (M, N, f) and (M', N', f') are those pairs of R-homomorphisms $h = (\mu, \nu)$ where $\mu : M \to M'$ and $\nu : N \to N'$ satisfy $f'\mu = \nu f$. We also write $\mu = \pi_1(h)$ and $\nu = \pi_2(h)$. Thus the category Mod-$T_2(R)$ is equivalent to the category of homomorphisms in Mod-R (see e.g. [7, III, Prop. 2.2] and [6]).

DEFINITION. For a homomorphism $f : M_R \to N_R$ define the *endomorphism ring* of f as

$$\mathrm{End}\, f = \mathrm{End}(M, N, f)_{T_2(R)}.$$

We say that f is *endofinite* if (M, N, f) has finite length when viewed as an $\mathrm{End}\, f$-module. The *endolength* of f is the length of the left $\mathrm{End}\, f$-module (M, N, f). We will consider it as an element of $\mathbb{N} \cup \{\infty\}$. Note that this length coincides with the length of the left $\mathrm{End}\, f$-module ${}_{\pi_1}M \oplus {}_{\pi_2}N$.

Proposition 2.1. *Let R be a ring and $f : M_R \to N_R$ a homomorphism.*

(1) *The map $f : {}_{\pi_1}M \to {}_{\pi_2}N$ is an* $\operatorname{End} f$*-R-bimodule homomorphism. Suppose $\varrho_1 : S \to \operatorname{End} M$ and $\varrho_2 : S \to \operatorname{End} N$ are ring homomorphisms such that $f : {}_{\varrho_1}M \to {}_{\varrho_2}N$ is an S-R-bimodule homomorphism. Then there exists a uniquely determined ring homomorphism $\sigma : S \to \operatorname{End} f$ such that $\varrho_1 = \pi_1 \circ \sigma$ and $\varrho_2 = \pi_2 \circ \sigma$.*

(2) *If M_R and N_R are finite length modules*, *then* $\operatorname{End} f$ *is a semiprimary ring.*

(3) *Every homomorphism in* Mod-R *and in* R-Mod *is endofinite if and only if $T_2(R)$ is an artinian ring of finite representation type.*

Proof. (1) The proof is straightforward.

(2) The endomorphism ring of a finite length module is semiprimary (see e.g. [1, 29.3]).

(3) Recall that a ring T is artinian of finite representation type if and only if every left T-module and every right T-module is endofinite (cf. [26, Theorem 6] and [13, 11.38]). ■

3. The local dual of a homomorphism

Definition. Let $f : M \to N$ be a homomorphism in Mod-R and (M, N, f) the corresponding $T_2(R)$-module. Suppose that the endomorphism ring $S = \operatorname{End} f$ is semilocal and ${}_SI = \mathrm{E}({}_S\overline{S})$. We define the *local dual* of f as $\mathsf{L}f = \mathsf{L}(M, N, f) = ({}_S(M, N, f), {}_SI)$. We will consider $\mathsf{L}f$ also as a homomorphism of left R-modules

$$\mathsf{L}f = (f, {}_SI) : ({}_{\pi_2}N, {}_SI) \to ({}_{\pi_1}M, {}_SI).$$

If R is a semilocal ring with $\overline{R}$ an artin algebra, we characterize those homomorphisms f in mod-R for which $\mathsf{L}f$ is well behaved. We show in Theorem 3.2 that the following properties are equivalent: (1) f is endofinite, (2) $\mathsf{L}f$ is endofinite, (3) $\mathsf{L}f$ is a homomorphism between the finite length modules ${}_R\mathsf{L}N$, and ${}_R\mathsf{L}M$, and (4) f occurs as the L-dual of a homomorphism in R-mod. However, the class of these homomorphisms may not be closed under addition (Example 1) and composition (Example 2).

The local dual of a homomorphism has the following basic properties.

Proposition 3.1. *Let $f : M_R \to N_R$ be a homomorphism with semilocal endomorphism ring S. Put $I = {}_S\mathrm{E}({}_S\overline{S})$.*

(1) *If M and N are finitely presented R-modules*, *then* $\operatorname{End} \mathsf{L}f \cong \operatorname{End} {}_SI$. *In particular*, $\mathsf{L}f$ *has semiperfect endomorphism ring.*

(2) *The homomorphism f is endofinite if and only if $\mathsf{L}f$ is endofinite. Moreover*, *f and $\mathsf{L}f$ have the same endolength in $\mathbb{N} \cup \{\infty\}$.*

(3) *Assume that* R *is a semilocal ring with* $\overline{R} = R/\operatorname{Rad} R$ *an artin algebra. If* f *is endofinite, and* M_R *and* N_R *have finite length, then there are left* R*-module isomorphisms*

$$({}_{\pi_1}M, {}_SI) \cong {}_R\mathsf{L}M \quad \text{and} \quad ({}_{\pi_2}N, {}_SI) \cong {}_R\mathsf{L}N.$$

P r o o f. (1) If M_R and N_R are finitely presented, then so is f when considered as a $T_2(R)$-module. Hence the assertion follows from [4, I, Cor. 11.3].

(2) This is a consequence of [26, Prop. 3] applied to the $T_2(R)$-module f.

(3) If f is an endofinite homomorphism, then ${}_{\pi_1}M$ and ${}_{\pi_2}N$ have finite length as $\operatorname{End} f$-modules, so we may apply Proposition 1.5. ∎

DEFINITION. Suppose $f : M_R \to N_R$ is a homomorphism between finitely presented modules and $S = \operatorname{End} f$. If ${}_SI = \mathrm{E}({}_S\overline{S})$ induces a Morita duality S-mod $\to$ mod-S', where $S' = \operatorname{End}{}_SI$, with respect to which (M, N, f) is reflexive, then we call f *reflexive with respect to* L. In this case we see from Proposition 3.1(1) that $S' = \operatorname{End} \mathsf{L}f$ and the following diagram of S-R-bimodules commutes, where η is the evaluation map:

$$\begin{array}{ccc} M & \xrightarrow{\quad f \quad} & N \\ \downarrow{\scriptstyle \eta_M} & & \downarrow{\scriptstyle \eta_N} \\ (({}_{\pi_1}M, {}_SI)_{S'}, I_{S'}) & \xrightarrow[\mathsf{LL}f]{} & (({}_{\pi_2}N, {}_SI)_{S'}, I_{S'}) \end{array}$$

THEOREM 3.2 (A dichotomy for homomorphisms). *Let* R *be a semilocal ring whose radical factor* $\overline{R}$ *is an artin algebra and let* $f : M_R \to N_R$ *be a homomorphism between finite length modules. Write* $S = \operatorname{End} f$ *and* $I = {}_S\mathrm{E}({}_S\overline{S})$.

(1) *Suppose* f *is endofinite. Then* $\mathsf{L}f$ *is an endofinite homomorphism between the modules* ${}_R\mathsf{L}N \cong ({}_{\pi_2}N, {}_SI)$ *and* ${}_R\mathsf{L}M \cong ({}_{\pi_1}M, {}_SI)$ *of finite length. Moreover*, f *is* L*-reflexive.*

(2) *Suppose* f *is not endofinite. Then* $\mathsf{L}f$ *is not endofinite*, *not both modules* ${}_R({}_{\pi_1}M, {}_SI)$ *and* ${}_R({}_{\pi_2}N, {}_SI)$ *have finite length*, *and* f *is not isomorphic to the* L*-dual or the* L*-bidual of a homomorphism between finite length modules.*

P r o o f. If R is a semilocal ring with radical factor an artin algebra, then so is the triangular matrix ring $T = T_2(R)$ [7, III, Prop. 2.5]. Since the L-dual of a right T-module (M, N, f) is the left T-module $(({}_{\pi_1}M, {}_SI), ({}_{\pi_2}N, {}_SI), \mathsf{L}f)$, the result follows from the corresponding statement for modules [17, Theorem 9] and from Proposition 3.1(3). ∎

EXAMPLE 1. *The sum of two endofinite homomorphisms* $f, f' : M_R \to N_R$ *may not be endofinite.* Let K be a field, ϕ an automorphism of K such that $\dim_{\mathrm{Fix}\,\phi} K = \infty$. By ${}_\phi K$ we denote the K-K-bimodule K with multiplication $a \cdot b \cdot c = \phi(a)bc$ for $a, c \in K$ and $b \in {}_\phi K$. We consider the hereditary artinian PI-ring $R = \left(\begin{smallmatrix} K & K \oplus_\phi K \\ 0 & K \end{smallmatrix}\right)$. Note that the centre of R is the field $k = \mathrm{Fix}\,\phi$, so R is not an artin algebra; moreover, the duality $\mathsf{D} = (-, {}_k k)$ sends every nonzero R-module to an R-module of infinite length. We consider homomorphisms between the projective indecomposable modules $P_{1R} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) R$ and $P_{2R} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) R$. Since R is left artinian, P_1 and P_2 have finite length over their endomorphism ring, which is canonically isomorphic to K. Of course, the dual modules $\mathsf{L}P_1$ and $\mathsf{L}P_2$ are the indecomposable injective left R-modules, which are endofinite finite length modules.

Let $f, f' : P_2 \to P_1$ be the homomorphisms given by $(0,1) \mapsto (0,(1,0))$ and $(0,1) \mapsto (0,(0,1))$, respectively. Both homomorphisms have endomorphism ring K, but the endostructure of the K-R-bimodule homomorphisms $f : P_2 \to P_1$ and $f : {}_\phi P_2 \to P_1$ is not "compatible". Thus, their duals are R-K-bimodule homomorphisms $\mathsf{L}f : \mathsf{L}P_1 \to \mathsf{L}P_2$ and $\mathsf{L}f' : \mathsf{L}P_1 \to (\mathsf{L}P_2)_\phi$; however, their sum $f + f'$ is "only" a k-R-bimodule homomorphism and the R-k-bimodule homomorphism $\mathsf{L}(f + f')$ is a homomorphism between modules of infinite length. In particular, irreducible morphisms between endofinite finite length modules may or may not be endofinite.

Example 2. *The composition $g \circ f$ of two endofinite homomorphisms f and g between indecomposable modules may not be endofinite.* Let T be an infinite set, ϕ_1 and ϕ_2 bijections of T such that ϕ_1 and ϕ_2 have finite order but $\phi_2 \circ \phi_1$ acts transitively on T. (Take e.g. $T = \mathbb{Z}$, $\phi_1(z) = -z$ and $\phi_2(z) = -z + 1$.) Let $K = k(X_i : i \in T)$ be the field of rational functions in variables indexed by T. We denote the k-linear action on K given by $X_i \mapsto X_{\phi_j(i)}$ also by ϕ_j for $j = 1, 2$. Since ϕ_j has finite order, the dimension of K over $\mathrm{Fix}\,\phi_j$ is finite for $j = 1, 2$ [9, Theorem 3.5.5]. Since $\phi_2 \circ \phi_1$ acts transitively on the infinite set T, we have $\mathrm{Fix}(\phi_2 \circ \phi_1) = k$. Put $B_j = K \oplus_{\phi_j} K$ for $j = 1, 2$ and

$$R = \begin{pmatrix} K & B_1 & B_1 \otimes B_2 \\ 0 & K & B_2 \\ 0 & 0 & K \end{pmatrix}.$$

Then R is a hereditary artinian PI-ring. Furthermore, $f : P_{3R} \to P_{2R}$ and $g : P_{2R} \to P_{1R}$ given by $(0,0,1) \mapsto (0,0,(1,1))$ and $(0,1,0) \mapsto (0,(1,1),0)$, respectively, are both endofinite with $\mathrm{End}\, f = \mathrm{Fix}\,\phi_2$ and $\mathrm{End}\, g = \mathrm{Fix}\,\phi_1$, but $g \circ f$ is not endofinite since $\mathrm{End}(g \circ f) = \mathrm{Fix}(\phi_2 \circ \phi_1)$.

4. On the pure semisimplicity conjecture. According to a theorem of Auslander [2], Ringel–Tachikawa [14] and Simson [20], an artinian ring R of finite representation type is right pure semisimple. Recall that a ring R

is said to be *right pure semisimple* if every right R-module is pure injective, or equivalently, if every right R-module is a direct sum of modules in ind-R, the class of finitely presented right R-modules with local endomorphism ring [11, Theorem 8.4]. It is an open question, called the *pure semisimplicity conjecture* (pss-conjecture), whether the converse of this result also holds. The aim of this section is to give a new short module-theoretic proof of the pss-conjecture for left artinian polynomial identity rings.

THEOREM 4.1 (Herzog). *A left artinian PI-ring R is right pure semisimple if and only if R is of finite representation type.*

The pss-conjecture for artin algebras has been shown by Auslander [3]; the proof of this theorem for local PI-rings, for hereditary PI-rings and for PI-rings such that the square of the Jacobson radical is zero is due to Simson [21], [22]. For arbitrary PI-rings, the pss-theorem has been established by Herzog [10]; the result could be extended by Krause [12] to *right dualizing rings*, i.e. to rings for which the local dual of every finitely presented endofinite right R-module is finitely presented. The reader is referred to [26], [24] and [25] for a discussion of the pure semisimplicity conjecture. In [23], [24] and [25] also potential counterexamples in the class of hereditary rings are discussed in relation with Artin problems for division ring extensions.

Note that the assumption in Theorem 4.1 that R is left artinian can be avoided by passing to a Morita dual ring R', which is left artinian and right pure semisimple [21, Prop. 2.4(a)]. Since our proof also collects information about the category of R-modules, we would like to avoid this change of rings.

The validity of the pss-conjecture for artin algebras is an immediate consequence of the existence of almost split sequences [7], and of the following proposition, due to Auslander [3, Cor. 2.3].

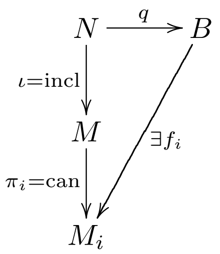
PROPOSITION 4.2. *Let R be a right pure semisimple ring such that there exists a left almost split morphism $N \to B$ in the category* Mod-R *for every module N in* ind-R. *Then there are only finitely many modules in* ind-R, *up to isomorphism.*

We include a module-theoretic version of Auslander's proof.

Proof. Suppose that $(M_i)_{i\in I}$ is a family of pairwise nonisomorphic modules in ind-R and $M = \prod_{i\in I} M_i$ is their product. We show in three steps that the canonical pure monomorphism $\sigma : \coprod_{i\in I} M_i \to \prod_{i\in I} M_i$ is an isomorphism. Then I must be finite, and we are done.

STEP 1. Each M_i occurs as a summand of M: By assumption, the sum $\coprod_{i\in I} M_i$ is a pure injective module, hence σ is a split monomorphism.

STEP 2. Since R is right pure semisimple, M is a direct sum of modules in ind-R. We show that any direct summand N of M with N in ind-R is isomorphic to one of the modules M_i: Assume that N is not isomorphic to any module M_i and let $q : N \to B$ be a left almost split map for N. Consider the diagram



By assumption on N, no $\pi_i \iota$ is a split monomorphism, so for every $i \in I$ there is a map $f_i : B \to M_i$ with $f_i q = \pi_i \iota$. Hence the product map $f = (f_i)_{i \in I}$ makes the upper part of the diagram commutative, i.e. $fq = \iota$. Since ι is a split monomorphism, so is q—a contradiction.

Step 3. Every M_{i_0} occurs at most once in a direct sum decomposition of M: Apply the argument in Step 2 to $M' = \prod_{i \neq i_0} M_i$ instead of $M = M' \oplus M_{i_0}$. ■

So for the proof of Theorem 4.1 we have to show that for every M in ind-R there exists a left almost split map $M \to N$ in the category Mod-R. This is the case if M is an endofinite module over an artinian PI-ring R. The following lemma will be used to "transform" one chain condition on finite matrix subgroups (see [11, Prop. 6.3]) into endofiniteness.

Lemma 4.3. *Let M_R be a finitely presented module such that the endomorphism ring $S = \operatorname{End} M_R$ is right perfect. If M satisfies acc for finite matrix subgroups*, *then M is endofinite.*

Proof. Since finitely generated endo-submodules of M are finite matrix subgroups (of type $Sm_1 + \ldots + Sm_n = \{f(m_1, \ldots, m_n) : f \in \operatorname{Hom}(M_R^n, M_R)\}$), the module M has acc for finitely generated endo-submodules. Hence every endo-submodule of M is finitely generated. Since S is right perfect, every left S-module has dcc for cyclic submodules, hence by Björk's theorem [8, Theorem 2] also dcc for finitely generated submodules. Thus M is endofinite. ■

Now we can give a new proof of Herzog's Theorem 4.1.

Proof of Theorem 4.1. The ring R is right pure semisimple, so every right R-module is Σ-pure injective and hence satisfies dcc for finite matrix subgroups [26, Theorem 8.1]. Thus every left R-module has acc for finite matrix subgroups [26, Theorem 6]. Since R is left artinian, every module $M \in R$-ind has finite length. By [1, Cor. 29.3] the endomorphism ring of ${}_R M$ is semiprimary, hence right perfect, so it follows from Lemma 4.3 that

M is endofinite. In particular, R is twosided artinian. Moreover, since R is a PI-ring, the transpose preserves finite endolength [18, Theorem 8], so also every module in ind-R is endofinite. But every endofinite module M in ind-R is the L-dual of a module in R-ind [17, Theorem 1] and thus there exists a left almost split map $M \to N$ in the category Mod-R [4, I, Theorem 3.9]. By Proposition 4.2, R is of finite representation type. ∎

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