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ON REFLEXIVITY OF REPRESENTATIONS OF LOCAL COMMUTATIVE ALGEBRAS

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An algebra A of operators on a Hilbert space (or any complex vector space) H is called *reflexive* in case no larger algebra of operators on H induces the same lattice of invariant subspaces as A. In [2] E. A. Azoff and M. Ptak proved the following theorem that they say "should be regarded as the main result of [their] paper".

THEOREM A. *Suppose A is an operator algebra generated by a commuting family of nilpotents. Then in order for A to be reflexive it is necessary that each rank two member of A generate a one-dimensional ideal. If the underlying space is a finite-dimensional Hilbert space and the generators for A commute with each other's adjoints, then this condition is also sufficient.*

If A is an algebra of operators on a $\mathbb{C}$-space H, then the action of A induces a faithful module ${}_AH$ whose submodules are the A-invariant subspaces of H, and if A is generated by a commuting family of nilpotent operators, then A is a split local commutative $\mathbb{C}$-algebra in the sense that, as $\mathbb{C}$-spaces, $A = \mathbb{C} \oplus J$ with $J = J(A)$ the unique maximal ideal of A.

Let K be an arbitrary field. A module ${}_RM$ over a K-algebra R is called *reflexive* (see [6] or [3], for example) if the only K-linear transformations of M that preserve the submodule lattice of ${}_RM$ are multiplications by elements of R. Thus, if $\lambda : R \to \mathrm{End}({}_KM)$ is the ring homomorphism induced by R-scalar multiplication, then $\lambda(R) \cong R/\mathrm{ann}(M)$ and ${}_RM$ is reflexive if and only if $\lambda(R)$ is a reflexive algebra of K-operators on M. In the terminology of [3], the set of K-linear transformations of M that preserve the submodule lattice of ${}_RM$ is

$$\mathrm{alglat}(M) = \{\alpha \in \mathrm{End}({}_KM) \mid \alpha m \in Rm \text{ for all } m \in M\},$$

and M is reflexive in case $\lambda(R) = \mathrm{alglat}(M)$.

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Here we improve the first part of Theorem A by showing that: *If R is a split local commutative K-algebra, then in order for a faithful module ${}_RM$ to be reflexive it is necessary that each $a \in R$ such that aM is cyclic generates a minimal ideal*; and we show that this stronger condition yields a larger class of reflexive modules (or algebras of operators): *If in addition ${}_RM$ is a finite-dimensional direct sum of cyclic modules, then this condition is also sufficient.*

Assume that R is a (not necessarily finite-dimensional) split local commutative K-algebra with unique maximal ideal $J = J(R)$, and consider a module ${}_RM$. An element $a \neq 0$ in R satisfies $JaM = 0$ if and only if $\lambda(a)$ generates a minimal (i.e., one-dimensional) ideal in $\lambda(R)$; and if $|aM : K| = 2$ (i.e., $\lambda(a)$ is a rank two member of $\lambda(R)$), then aM is either cyclic or $JaM = 0$. Thus, as asserted in the preceding paragraph, the following theorem yields a necessary condition for reflexivity that is stronger than the one of Azoff and Ptak's Theorem A.

THEOREM 1. *Let R be a split local commutative K-algebra. If ${}_RM$ is reflexive, then for $a \in R$, $JaM = 0$ whenever aM is cyclic.*

Proof. Assume that $0 \neq aM = Rx = Rau$, for some $x, u \in M$. Then $\lambda(a) : M \to Rau$ is an R-epimorphism that splits over K, and if $L = \mathrm{Ker}(\lambda(a)) = \mathrm{ann}_M(a)$, then $\lambda(a)(Ru + L) = Rau$. Thus $M = Ru + L$ with $u \notin L$, so there is a subspace $U \leq M$ with

$$u \in U \subseteq Ru \quad \text{and} \quad {}_KM = U \oplus L.$$

Now we see that

$$\lambda(a)|_U : U \to Rau = Kau \oplus Jau$$

is a K-isomorphism. Suppose $Jau \neq 0$ and let

$$N = \{n \in U \mid an \in Jau\} = (\lambda(a)|_U)^{-1}(Jau),$$

so that ${}_KU = Ku \oplus N$ and

$${}_KM = Ku \oplus N \oplus L.$$

To see that M is not reflexive, define $\alpha : M \to M$ via

$$\alpha(ku + n + l) = an.$$

If $k = 0$, then $\alpha(ku+n+l) = a(ku+n+l)$. If $k \neq 0$, then $k+j$ is invertible, where $j \in J$ with

$$an = jau,$$

and letting

$$r = (k + j)^{-1}ja$$

we see that

$$r(ku+n+l) = (k+j)^{-1}(kjau+jan)$$
$$= (k+j)^{-1}(kan+jan) = an = \alpha(ku+n+l).$$

Thus $\alpha \in \operatorname{alglat}(M)$. But, if $ru = 0$, then since $N \subseteq U \subseteq Ru$, we would have $rN = 0$. But $\alpha(N) = aN = Jau \neq 0$, so $\alpha \neq \lambda(r)$. ■

For a split local algebra R our Theorem 1 maintains that a necessary condition for reflexivity of ${}_RM$ is that aM is cyclic only if aM is simple (equivalently, one-dimensional). To show that it is sufficient in case M is a finite-dimensional direct sum of cyclic modules we use the following lemma, that appears as Proposition 4.2 in [2], where it is proved in a straightforward manner.

It will be convenient to employ the following notation from [5]: If $\lambda : S \to \operatorname{Hom}_K(U,X)$ is a K-vector space homomorphism, we write $su = \lambda(s)(u)$ and denote such a system, a so-called *S-representation*, by ${}_S[U,X]$. Analogously to alglat we let

$$\mathcal{A}({}_S[U,X]) = \{\alpha \in \operatorname{Hom}_K(U,X) \mid \alpha u \in Su \text{ for all } u \in U\}$$

and we say that ${}_S[U,X]$ is *reflexive* if $\lambda(S) = \mathcal{A}({}_S[U,X])$. An element $u \in U$ is called a *separating vector* in case $su \neq 0$ unless $s = 0$.

LEMMA 2. *Given ${}_{S_i}[U_i,X_i]$ with separating vectors u_i and ${}_KT \le S_1 \oplus \ldots \oplus S_l$, let*

$$T_i = \{t_i \in T \mid t_iU_j = 0 \text{ for all } j \neq i\}.$$

Then ${}_T[U_1 \oplus \ldots \oplus U_m,\, X_1 \oplus \ldots \oplus X_l]$ is reflexive if and only if ${}_{T_i}[U_i,X_i]$ is reflexive for all $i = 1,\ldots,l$.

From this lemma we glean the following proposition that is one key to our sufficiency theorem.

PROPOSITION 3. *Let R be any (not necessarily local) commutative algebra, and suppose $M_1,\ldots,M_l$ are cyclic R-modules with annihilators $\operatorname{ann}_R(M_i) = A_i$, respectively. Let $M = M_1 \oplus \ldots \oplus M_l$ and $I_i = \bigcap_{j\neq i} A_j$. Then ${}_RM$ is reflexive if and only if ${}_{I_i}[M_i,M_i]$ is reflexive for $i = 1,\ldots,l$.*

Proof. Assume, as we may, that M is faithful, and let $S_i = R/A_i$ with representation ${}_{S_i}[M_i,M_i]$ which has a separating vector the R-generator $m_i \in M_i$. Let

$$T = \{(r+A_1,\ldots,r+A_l) \in S_1 \oplus \ldots \oplus S_l \mid r \in R\}.$$

Then the action of T on M is induced by canonical isomorphism $R \cong T$, and $I_i \cong (I_i + A_i)/A_i \cong T_i$ canonically, so Lemma 2 applies. ■

The next two lemmas can be found in [4, Proposition 3.3(b) and Lemma 2.3]; the first of them had appeared earlier in [1, Proposition 5.4].

LEMMA 4. *If* $\dim({}_K S) \leq 1$, *then* ${}_S[U, X]$ *is reflexive.*

LEMMA 5. *If* $S = \sum_{j=1}^n S_j$ *is a sum of subspaces such that the sum* $\sum_{j=1}^n S_j U$ *is direct*, *then* ${}_S[U, X]$ *is reflexive if and only if each* ${}_{S_j}[U, X]$ *is reflexive.*

These lemmas yield the second key to our sufficiency theorem.

PROPOSITION 6. *If* I *is a finitely generated semisimple ideal in a commutative split algebra* S, *then* ${}_I[S, S]$ *is reflexive.*

Proof. By hypothesis $I = \bigoplus_{j=1}^n W_j$ with the W_j one-dimensional ideals, and clearly $\sum_{j=1}^n W_j S = \bigoplus_{j=1}^n W_j$. ■

Finally, we are ready to complete our characterization of reflexive finite-dimensional direct sums of cyclic modules over split local algebras.

THEOREM 7. *Suppose* R *is a commutative finite-dimensional split local* K*-algebra*, *and that* ${}_R M = M_1 \oplus \ldots \oplus M_l$ *is a faithful module with each* M_i *cyclic. If* Ra *is simple whenever* aM *is cyclic*, *then* M *is reflexive.*

Proof. Let A_i, $S_i = R/A_i$ and I_i be as in Proposition 3. Since M is faithful, $I_i \cap A_i = 0$, so we may assume $I_i \subseteq S_i$, and ${}_{S_i}M_i \cong {}_{S_i}S_i$ since ${}_{S_i}M_i$ is cyclic and faithful. Thus ${}_{I_i}[M_i, M_i]$ is reflexive if and only if ${}_{I_i}[S_i, S_i]$ is reflexive. To prove the latter, according to Proposition 6, we need only show that I_i is semisimple. Now if $0 \neq a \in I_i$ and $M_i = Rm_i$, then $aM = aM_i = Ram_i$ is cyclic, so by hypothesis, $S_i a \cong Ra$ is simple (i.e., one-dimensional). Thus I_i is a semisimple. Finally, M is reflexive by Proposition 3. ■

REMARKS. 1. If one wishes to eschew the faithful hypothesis in Theorem 7, the condition "Ra is simple" must be replaced by "aM is simple".

2. A major portion of the proof of the sufficiency part [2, Theorem 5.7] of Azoff and Ptak's Theorem A, namely [2, Propositions 5.4, 5.5 and 5.6], is devoted to proving that the finite-dimensional Hilbert space V over the (necessarily local split) algebra $\mathcal{A}(\mathbf{a})$ generated by a set of doubly commuting nilpotent linear transformations (matrices) $\{a_1, \ldots, a_N\}$ in the hypothesis of their Theorem 5.7 is a direct sum of local-colocal (i.e., cyclic with a unique minimal submodule) $\mathcal{A}(\mathbf{a})$-modules. It is in fact rather rare that a cyclic module over a local ($\mathbb{C}$-)algebra is colocal. Thus our Theorem 7 is applicable to a significantly larger class of modules. The following example illustrates this fact.

EXAMPLE. Let $R = K[x, y]/I$, the ring of polynomials in x, y modulo the ideal I generated by $\{x^3, x^2y, xy^2, y^3\}$, the monomials of degree 3. Thus, we may assume R has a multiplicative basis $\{1, a, a^2, b, b^2, c\}$ with $ab = c = ba$. Also, R is a split local K-algebra with maximal ideal $J = Ra + Rb$. Let

$$M = R/Rb^2 \oplus R/Ra \oplus R/(Ra^2 + Rb)$$

and

$$N = R/Rb^2 \oplus R/Ra \oplus R/(Ra + Rb^2).$$

Then aN has basis $\{a, a^2, c\}$, and is isomorphic to the cyclic module R/J^2, so N is not reflexive by Theorem 1. On the other hand, M is reflexive by Theorem 7, because the only elements $r \in R$ that have rN cyclic belong to J^2.

Since ${}_R M$ is not a direct sum of cyclic-cocyclic modules, and $\dim(rN) \neq 2$ unless $r \in J^2$, Azoff and Ptak's Theorem A does not apply here. To obtain their introductory example of direct sums of local-colocal modules that illustrates their theorem, simply factor the ideal generated by a^2 out of R.

REFERENCES

[1] E. A. Azoff, *On finite rank operators and preannihilators*, Mem. Amer. Math. Soc. 357 (1986).

[2] E. A. Azoff and M. Ptak, *On rank 2 linear transformations and reflexivity*, J. London Math. Soc. (2) 53 (1996), 383–396.

[3] K. R. Fuller, W. K. Nicholson and J. F. Watters, *Reflexive bimodules*, Canad. J. Math. 41 (1989), 592–611.

[4] —, —, —, *Algebras whose projective modules are reflexive*, J. Pure Appl. Algebra 98 (1995), 135–150.

[5] —, —, —, *Direct sums of reflexive modules*, Linear Algebra Appl. 239 (1996), 201–214.

[6] J. F. Habibi and W. H. Gustafson, *Reflexive serial algebras*, ibid. 99 (1988), 217–223.

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