ON REFLEXIVITY OF REPRESENTATIONS
OF LOCAL COMMUTATIVE ALGEBRAS

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An algebra $A$ of operators on a Hilbert space (or any complex vector space) $H$ is called reflexive in case no larger algebra of operators on $H$ induces the same lattice of invariant subspaces as $A$. In [2] E. A. Azoff and M. Ptak proved the following theorem that they say “should be regarded as the main result of [their] paper”.

**Theorem A.** Suppose $A$ is an operator algebra generated by a commuting family of nilpotents. Then in order for $A$ to be reflexive it is necessary that each rank two member of $A$ generate a one-dimensional ideal. If the underlying space is a finite-dimensional Hilbert space and the generators for $A$ commute with each other’s adjoints, then this condition is also sufficient.

If $A$ is an algebra of operators on a $C$-space $H$, then the action of $A$ induces a faithful module $A H$ whose submodules are the $A$-invariant subspaces of $H$, and if $A$ is generated by a commuting family of nilpotent operators, then $A$ is a split local commutative $C$-algebra in the sense that, as $C$-spaces, $A = C \oplus J$ with $J = J(A)$ the unique maximal ideal of $A$.

Let $K$ be an arbitrary field. A module $R M$ over a $K$-algebra $R$ is called reflexive (see [6] or [3], for example) if the only $K$-linear transformations of $M$ that preserve the submodule lattice of $R M$ are multiplications by elements of $R$. Thus, if $\lambda : R \to \text{End}(K M)$ is the ring homomorphism induced by $R$-scalar multiplication, then $\lambda(R) \cong R/\text{ann}(M)$ and $R M$ is reflexive if and only if $\lambda(R)$ is a reflexive algebra of $K$-operators on $M$. In the terminology of [3], the set of $K$-linear transformations of $M$ that preserve the submodule lattice of $R M$ is

$$\text{alglat}(M) = \{ \alpha \in \text{End}(K M) \mid \alpha m \in R m \text{ for all } m \in M \},$$

and $M$ is reflexive in case $\lambda(R) = \text{alglat}(M)$.

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Here we improve the first part of Theorem A by showing that: If $R$ is a split local commutative $K$-algebra, then in order for a faithful module $R M$ to be reflexive it is necessary that each $a \in R$ such that $a M$ is cyclic generates a minimal ideal; and we show that this stronger condition yields a larger class of reflexive modules (or algebras of operators): If in addition $R M$ is a finite-dimensional direct sum of cyclic modules, then this condition is also sufficient.

Assume that $R$ is a (not necessarily finite-dimensional) split local commutative $K$-algebra with unique maximal ideal $J = J(R)$, and consider a module $R M$. An element $a \neq 0$ in $R$ satisfies $J a M = 0$ if and only if $\lambda(a)$ generates a minimal (i.e., one-dimensional) ideal in $\lambda(R)$; and if $|a M : K| = 2$ (i.e., $\lambda(a)$ is a rank two member of $\lambda(R)$), then $a M$ is either cyclic or $J a M = 0$. Thus, as asserted in the preceding paragraph, the following theorem yields a necessary condition for reflexivity that is stronger than the one of Azoff and Ptak’s Theorem A.

**Theorem 1.** Let $R$ be a split local commutative $K$-algebra. If $R M$ is reflexive, then for $a \in R$, $J a M = 0$ whenever $a M$ is cyclic.

**Proof.** Assume that $0 \neq a M = R x = R a u$, for some $x, u \in M$. Then $\lambda(a) : M \to R a u$ is an $R$-epimorphism that splits over $K$, and if $L = \text{Ker}(\lambda(a)) = \text{ann}_M(a)$, then $\lambda(a)(R u + L) = R a u$. Thus $M = R u + L$ with $u \notin L$, so there is a subspace $U \leq M$ with

$$u \in U \subseteq R u \quad \text{and} \quad K M = U \oplus L.$$ 

Now we see that

$$\lambda(a)|_U : U \to R a u = K a u \oplus J a u$$

is a $K$-isomorphism. Suppose $J a u \neq 0$ and let

$$N = \{n \in U \mid a n \in J a u\} = (\lambda(a)|_U)^{-1}(J a u),$$

so that $K U = K u \oplus N$ and

$$K M = K u \oplus N \oplus L.$$ 

To see that $M$ is not reflexive, define $\alpha : M \to M$ via

$$\alpha(k u + n + l) = a n.$$ 

If $k = 0$, then $\alpha(k u + n + l) = a(k u + n + l)$. If $k \neq 0$, then $k + j$ is invertible, where $j \in J$ with

$$a n = j a u,$$

and letting

$$r = (k + j)^{-1} j a$$
we see that
\[
 r(ku + n + l) = (k + j)^{-1}(kjau + jan) \\
= (k + j)^{-1}(kan + jan) = an = \alpha(ku + n + l).
\]
Thus \( \alpha \in \text{alglat}(M) \). But, if \( ru = 0 \), then since \( N \subseteq U \subseteq Ru \), we would have \( rN = 0 \). But \( \alpha(N) = aN = Jau \neq 0 \), so \( \alpha \neq \lambda(r) \). ■

For a split local algebra \( R \) our Theorem 1 maintains that a necessary condition for reflexivity of \( aM \) is that \( aM \) is cyclic only if \( aM \) is simple (equivalently, one-dimensional). To show that it is sufficient in case \( M \) is a finite-dimensional direct sum of cyclic modules we use the following lemma, that appears as Proposition 4.2 in [2], where it is proved in a straightforward manner.

It will be convenient to employ the following notation from [5]: If \( \lambda : S \to \text{Hom}_K(U, X) \) is a \( K \)-vector space homomorphism, we write \( su = \lambda(s)(u) \) and denote such a system, a so-called \( S \)-representation, by \( s[U, X] \). Analogously to \text{alglat} we let
\[
\mathcal{A}(s[U, X]) = \{ \alpha \in \text{Hom}_K(U, X) \mid \alpha u \in Su \text{ for all } u \in U \}
\]
and we say that \( s[U, X] \) is reflexive if \( \lambda(S) = \mathcal{A}(s[U, X]) \). An element \( u \in U \) is called a separating vector in case \( su \neq 0 \) unless \( s = 0 \).

\textbf{Lemma 2.} Given \( s[U_i, X_i] \) with separating vectors \( u_i \) and \( \kappa T \leq S_1 \oplus \ldots \oplus S_l \), let
\[
T_i = \{ t_i \in T \mid t_iU_j = 0 \text{ for all } j \neq i \}.
\]
Then \( T[U_1 \oplus \ldots \oplus U_m, X_1 \oplus \ldots \oplus X_l] \) is reflexive if and only if \( T_i[U_i, X_i] \) is reflexive for all \( i = 1, \ldots, l \).

From this lemma we glean the following proposition that is one key to our sufficiency theorem.

\textbf{Proposition 3.} Let \( R \) be any (not necessarily local) commutative algebra, and suppose \( M_1, \ldots, M_l \) are cyclic \( R \)-modules with annihilators \( \text{ann}_R(M_i) = A_i \), respectively. Let \( M = M_1 \oplus \ldots \oplus M_l \) and \( I_i = \bigcap_{j \neq i} A_j \). Then \( aM \) is reflexive if and only if \( I_i[M_i, M_i] \) is reflexive for \( i = 1, \ldots, l \).

\textbf{Proof.} Assume, as we may, that \( M \) is faithful, and let \( S_i = R/A_i \) with representation \( s[M_i, M_i] \) which has a separating vector the \( R \)-generator \( m_i \in M_i \). Let
\[
T = \{ (r + A_1, \ldots, r + A_l) \in S_1 \oplus \ldots \oplus S_l \mid r \in R \}.
\]
Then the action of \( T \) on \( M \) is induced by canonical isomorphism \( R \cong T \), and \( I_i \cong (I_i + A_i)/A_i \cong T_i \) canonically, so Lemma 2 applies. ■

The next two lemmas can be found in [4, Proposition 3.3(b) and Lemma 2.3]; the first of them had appeared earlier in [1, Proposition 5.4].
Lemma 4. If \( \dim(K S) \leq 1 \), then \( S[U, X] \) is reflexive.

Lemma 5. If \( S = \sum_{j=1}^{n} S_j \) is a sum of subspaces such that the sum \( \sum_{j=1}^{n} S_j U \) is direct, then \( S[U, X] \) is reflexive if and only if each \( S_j[U, X] \) is reflexive.

These lemmas yield the second key to our sufficiency theorem.

Proposition 6. If \( I \) is a finitely generated semisimple ideal in a commutative split algebra \( S \), then \( I[S, S] \) is reflexive.

Proof. By hypothesis \( I = \bigoplus_{j=1}^{n} W_j \) with the \( W_j \) one-dimensional ideals, and clearly \( \sum_{j=1}^{n} W_j S = \bigoplus_{j=1}^{n} W_j \).

Finally, we are ready to complete our characterization of reflexive finite-dimensional direct sums of cyclic modules over split local algebras.

Theorem 7. Suppose \( R \) is a commutative finite-dimensional split local \( K \)-algebra, and that \( R_M = M_1 \oplus \ldots \oplus M_l \) is a faithful module with each \( M_i \) cyclic. If \( Ra \) is simple whenever \( aM \) is cyclic, then \( M \) is reflexive.

Proof. Let \( A_i, S_i = R/A_i \) and \( I_i \) be as in Proposition 3. Since \( M \) is faithful, \( I_i \cap A_i = 0 \), so we may assume \( I_i \subseteq S_i \), and \( S_i M_i \cong S_i S_i \) since \( S_i M_i \) is cyclic and faithful. Thus \( I_i[M_i, M_i] \) is reflexive if and only if \( I_i[S_i, S_i] \) is reflexive. To prove the latter, according to Proposition 6, we need only show that \( I_i \) is semisimple. Now if \( 0 \neq a \in I_i \) and \( M_i = Rm_i \), then \( aM_i = aM_i = Ram_i \) is cyclic, so by hypothesis, \( S_i a \cong Ra \) is simple (i.e., one-dimensional). Thus \( I_i \) is a semisimple. Finally, \( M \) is reflexive by Proposition 3.

Remarks. 1. If one wishes to eschew the faithful hypothesis in Theorem 7, the condition “\( Ra \) is simple” must be replaced by “\( aM \) is simple”.

2. A major portion of the proof of the sufficiency part [2, Theorem 5.7] of Azoff and Ptak’s Theorem A, namely [2, Propositions 5.4, 5.5 and 5.6], is devoted to proving that the finite-dimensional Hilbert space \( V \) over the (necessarily local split) algebra \( A(a) \) generated by a set of doubly commuting nilpotent linear transformations (matrices) \( \{a_1, \ldots, a_N\} \) in the hypothesis of their Theorem 5.7 is a direct sum of local-colocal (i.e., cyclic with a unique minimal submodule) \( A(a) \)-modules. It is in fact rather rare that a cyclic module over a local (\( C \)-)algebra is colocal. Thus our Theorem 7 is applicable to a significantly larger class of modules. The following example illustrates this fact.

Example. Let \( R = K[x, y]/I \), the ring of polynomials in \( x, y \) modulo the ideal \( I \) generated by \( \{x^3, x^2y, xy^2, y^3\} \), the monomials of degree 3. Thus, we may assume \( R \) has a multiplicative basis \( \{1, a, a^2, b, b^2, c\} \) with \( ab = c = ba \).

Also, \( R \) is a split local \( K \)-algebra with maximal ideal \( J = Ra + Rb \). Let

\[
M = R/Rb^2 \oplus R/Ra \oplus R/(Ra^2 + Rb)
\]
and

\[ N = \frac{R}{Rb^2} \oplus \frac{R}{Ra} \oplus \frac{R}{(Ra + Rb^2)}. \]

Then \(aN\) has basis \(\{a, a^2, c\}\), and is isomorphic to the cyclic module \(R/J^2\), so \(N\) is not reflexive by Theorem 1. On the other hand, \(M\) is reflexive by Theorem 7, because the only elements \(r \in R\) that have \(rN\) cyclic belong to \(J^2\).

Since \(RM\) is not a direct sum of cyclic-cocyclic modules, and \(\dim(rN) \neq 2\) unless \(r \in J^2\), Azoff and Ptak’s Theorem A does not apply here. To obtain their introductory example of direct sums of local-colocal modules that illustrates their theorem, simply factor the ideal generated by \(a^2\) out of \(R\).

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