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ONE-PARAMETER GLOBAL BIFURCATION IN A MULTIPARAMETER PROBLEM

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We study the nonlinear eigenvalue problem $F(x,\lambda) = L(\lambda)x + R(x,\lambda) = 0$ where $F : X \times \mathbb{R}^k \to Y$ with X and Y Banach spaces and $k > 1$ a positive integer. If $L(\lambda)$ is linear in λ, then it is shown that λ_0 is a one-parameter global bifurcation point of the eigenvalue problem provided: a standard transversality condition is satisfied, the dimension of the null space of $L(\lambda_0)$ is an odd number and the component of the set of characteristic values of $L(\lambda)$ containing λ_0 is a bounded one-codimensional continuum.

0. Introduction. Consider the nonlinear eigenvalue problem

$$F(x,\lambda) = L(\lambda)x + R(x,\lambda) = 0, \tag{0.1}$$

where $F : X \times \mathbb{R}^k \to Y$, k is a positive integer and X, Y and $X \times \mathbb{R}^k$ are Banach spaces. $L(\lambda) : X \to Y$ is the Fréchet derivative of F at the origin and $F(0,\lambda) = 0$ for all $\lambda \in \mathbb{R}^k$.

A global bifurcation point of (0.1) is a point $\lambda_0 \in \mathbb{R}^k$ such that a branch of nontrivial solutions to (0.1) (i.e., $F(x,\lambda) = 0$ and $x \neq 0$) emanates from $(0,\lambda_0) \in X \times \mathbb{R}^k$ and satisfies certain nonlocal properties which will be made precise in the sequel. We seek sufficient conditions for a given $\lambda_0 \in \mathbb{R}^k$ to be a global bifurcation point of (0.1).

The results given in this paper represent a generalization of a real parameter ($k = 1$) global bifurcation theorem proved by the author in [26] to multiparameter ($k > 1$) global bifurcation problems which possess a one-parameter nature. We use topological degree methods by applying a recent theorem of the author [27] which will be stated below.

The first significant multiparameter bifurcation results were due to Ize [13] in 1976 and Alexander and Yorke [3] in 1978. These two sets of authors used cohomology arguments and discovered, independently, the important role played by the Whitehead's J-homomorphism. In 1980, Alexander and Fitzpatrick [2] extended the scope of multiparameter bifurcation to include

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the general class of A-proper mappings. A critical assumption was that $L(\lambda_0)$ should be a Fredholm operator of index zero, which seems to be essential to the analysis. In recent years, Ize [14] has used cohomology theory to refine and extend earlier results to obtain necessary and sufficient conditions for multiparameter global bifurcation.

The objective of this study is to prove a global bifurcation result for a multiparameter problem which admits a one-parameter analysis. We sacrifice generality at the expense of treating the specific problem where $L(\lambda)$ is linear in λ. This enables us to invoke topological degree theory in a manner that naturally extends the previous methods for $\lambda \in \mathbb{R}$.

The key to our techniques is the Theorem stated in Section 1, which appeared recently [27]. We assume that our mappings are A-proper and $L(\lambda_0)$ is a Fredholm operator of index zero. Then, provided that a standard transversality condition is satisfied, λ_0 will be a global bifurcation point of (0.1) if the dimension of the null space of $L(\lambda_0)$ is an odd number and the maximal-connected subset of characteristic values of $L(\lambda)$ containing λ_0 is a bounded one-dimensional continuum in $\mathbb{R}^k$. Although we invoke degree-theoretic arguments in the analyses of the proofs, the statement of the main bifurcation theorem is written in terms of easily verifiable hypotheses which are free of degree considerations. This contrasts sharply with [27] in which we sought greater generality of the operator $L(\lambda)$ at the expense of technically difficult sufficient conditions for global bifurcation.

Our main theorem treats a similar type of problem to that studied by Alexander and Fitzpatrick [2] who also took $L(\lambda)$ to be linear in λ. Their results are applicable to situations where λ_0 is isolated in the set of characteristic values of $L(\lambda)$; however, we demand that the bifurcation point λ_0 should *not* be an isolated characteristic value and we attempt only to study one-parameter bifurcation along a straight-line curve $\lambda_0 \pm r\widehat{\lambda}$ through λ_0 which is transverse to the component of the set of characteristic values through λ_0.

It should be stressed that the affine dependence of the family of linear operators $L(\lambda)$ on λ together with the A-properness of $L(\lambda)$ and $F(\cdot,\lambda)$ allow us to transform the problem into an equivalent form in which we can take advantage of the rich theory of compact perturbations of the identity. To achieve our conclusions, it is not necessary to impose restrictions on the dimension k of the parameter space, nor to require the technical conditions on the Galerkin approximations assumed in [2].

Although this method of studying multiparameter global bifurcation by seeking one-parameter bifurcation surfaces has been studied by other authors, including: Alexander and Antman [1], Cantrell [4], [5], Fitzpatrick, Massabó and Pejsachowicz [11], Ize, Massabó, Pejsachowicz and Vignoli [15] and Esquinas and López-Gómez [7], [8], our method applied to this specific problem seems to be much simpler by the natural application of the generalized topological degree. In addition, the definition of global bifurcation in this paper is more restrictive and our global bifurcating surface emanating from the *nonisolated* characteristic value λ_0 may possibly return to a different component of the set of characteristic values.

It should be emphasized that a large number of multiparameter bifurcation problems do not possess a one-parameter nature. See, for example, Hale [12], Esquinas and López-Gómez [9] and López-Gómez [16] where the problems arose by working with either overdetermined boundary-value problems, or systems of equations, or higher-order equations.

The simple example given in the final section indicates that our hypotheses are easily satisfied and are a natural extension of the ideas studied in [26] to multiparameter problems possessing a one-parameter nature in which $L(\lambda)$ is linear in λ.

General concepts. Let X_n and Y_n be sequences of oriented finite-dimensional subspaces of X and Y, respectively, and let $Q_n : Y \to Y_n$ be continuous linear projections of Y onto Y_n, for each $n \in \mathbb{N}$. We say that the approximation scheme $\varGamma = \{X_n, Y_n, Q_n\}$ is *admissible* for maps from X into Y if: $\dim X_n = \dim Y_n$ (where "dim" denotes the dimension of the space), for each n; $\operatorname{dist}(x, X_n) \to 0$ (where $\operatorname{dist}(x, X_n) = \inf\{\|x - z\| : z \in X_n\}$) as $n \to \infty$, for each $x \in X$; and $Q_n y \to y$ as $n \to \infty$, for each $y \in Y$.

Suppose that $U \subset X$ is an open set and define $U_n = U \cap X_n$, $\overline{U}_n = \overline{U} \cap X_n$ and $\partial U_n = \partial U \cap X_n$. A mapping $f : \overline{U} \to Y$ is said to be *A-proper* with respect to the admissible scheme $\varGamma$ if $f_n = Q_n f|_{U_n} : U_n \to Y_n$ is continuous for each n, and, whenever $\{x_{n_j}\}$ is any bounded sequence with $x_{n_j} \in U_{n_j}$ for each $j \in \mathbb{N}$ and $f_{n_j}(x_{n_j}) \to y$ as $j \to \infty$ for some $y \in Y$, there exists a subsequence (which, without loss of generality, we denote by $\{x_{n_j}\}$) and $x \in \overline{U}$ such that $x_{n_j} \to x$ as $j \to \infty$ and $f(x) = y$.

Examples of A-proper maps include: $I - K$ with I the identity operator and K a linear compact operator; $I - B$ with B taken to be k-ball contractive ($k < 1$); monotone and accretive mappings; and many others. See the survey article [21].

The generalized degree of an A-proper mapping f at the point $0 \in X$ relative to an open set $U \subset X$, denoted by $\operatorname{Deg}(f, U, 0)$, is defined whenever $0 \notin f(\partial U)$. If $\mathbb{Z}$ represents the set of integers, let $\mathbb{Z}' = \mathbb{Z} \cup \{-\infty, \infty\}$ be the extended integers. Then define

$$\operatorname{Deg}(f, U, 0) = \{\gamma \in \mathbb{Z}' : \text{there is a sequence } \{n_j\} \text{ with } \mathrm{d}(f_{n_j}, U_{n_j}, 0) \to \gamma \text{ as } j \to \infty\},$$

where $\mathrm{d}(f_{n_j}, U_{n_j}, 0)$ denotes the classical Brouwer degree for mappings between oriented normed spaces of equal finite dimension.

The generalized degree is, in general, multivalued. However, if $f = I - K$ with K a linear compact operator, then $\mathrm{Deg}(f, U, 0) = \deg_{\mathrm{LS}}(f, U, 0)$ where $\deg_{\mathrm{LS}}$ denotes the classical Leray–Schauder degree for linear compact perturbations of the identity.

Many of the useful properties of classical topological degree hold for generalized degree. The following homotopy property, due to Toland, will play a crucial role in the proof of our main theorem.

Homotopy Property (Toland [25]). *Suppose that* $H : X \times [a, b] \to Y$ *satisfies the condition that* $H(\cdot, s) : X \to Y$ *is A-proper with respect to* Γ *for each* $s \in [a, b]$ *and* $H(x, \cdot) : [a, b] \to Y$ *is continuous, uniformly for* x *in closed bounded subsets of* X. *Let* $W \subset X \times [a, b]$ *be a bounded open set and define* $W_s = \{x \in X : (x, s) \in W\}$. *Then* $\mathrm{Deg}(H(\cdot, s), W_s, 0)$ *is independent of* s *in* $[a, b]$ *provided that* $0 \notin H(\partial W_s)$ *for* $a \le s \le b$.

1. Main results. Consider the nonlinear eigenvalue problem represented by (0.1). Assume that the following hypotheses are satisfied:

(H1) $F(\cdot, \lambda) : X \to Y$ is an A-proper mapping with respect to the admissible scheme $\Gamma = \{X_n, Y_n, Q_n\}$ for λ in a closed connected set $G \subset \mathbb{R}^k$ where G is assumed to have nonempty interior, denoted by $\operatorname{int} G$.

(H2) For each $\lambda \in \mathbb{R}^k$, $L(\lambda) = A - T(\lambda)$ with $T(\lambda) = \sum_{j=1}^k \lambda_j B_j$ where $A, B_j : X \to Y$, for $j = 1, \ldots, k$, are bounded linear operators and for all $\lambda \in G$, $L(\lambda)$ is an A-proper operator with respect to Γ.

(H3) $R : X \times \mathbb{R}^k \to Y$ is a continuous mapping such that $R(x, \lambda) = o(\|x\|)$ as $\|x\| \to 0$, uniformly for λ in bounded subsets of $\mathbb{R}^k$.

(H4) The mapping $\lambda \mapsto R(x, \lambda)$ is continuous from $\mathbb{R}^k$ into Y, uniformly for x in bounded subsets of X.

Clearly, $L(\lambda)$ is the Fréchet derivative of $F(\cdot, \lambda)$ at the point $0 \in X$. Also the mapping $\lambda \mapsto L(\lambda)$ is continuous from $\mathbb{R}^k$ into the space of bounded linear operators.

We call $\{(0, \lambda) \in X \times \mathbb{R}^k\}$ the set of *trivial solutions* of (0.1). Let S denote the set of nontrivial solutions in the sense that $(x, \lambda) \in S$ if and only if $F(x, \lambda) = 0$ and $x \neq 0$.

Define the set of *characteristic values* of $L(\lambda)$ to be

$$C(L) = \{\lambda \in \mathbb{R}^k \cap G : N(L(\lambda)) \neq \{0\}\},$$

and let

$$S' = S \cup \{(0, \lambda) \in X \times \mathbb{R}^k : \lambda \in C(L)\}.$$

It is shown in [18] that if $\lambda \in G$ but $\lambda \notin C(L)$, then $L(\lambda)$ is a homeomorphism.

In order to proceed, we need more hypotheses.

(H5) For $\lambda_0 \in C(L) \cap \operatorname{int} G$, let $\mathcal{C}$ denote the maximal connected subset (component) of $C(L)$, in $\mathbb{R}^k$, containing λ_0. Assume that $\mathcal{C}$ is bounded and there exist constants $\varrho > 0$ and $\widehat{\lambda} \in \mathbb{R}^k$, with $|\widehat{\lambda}| = 1$, such that: $\{\lambda_0 + r\widehat{\lambda} : r \in (-\infty, 0) \cup (0, \infty)\} \cap \mathcal{C} = \emptyset$; $\overline{B}(\lambda_0, \varrho) \setminus \mathcal{C}$ is a disconnected set in $\mathbb{R}^k$; the sets

$$\{\lambda_0 + r\widehat{\lambda} : 0 < r \leq \varrho\} \quad \text{and} \quad \{\lambda_0 + r\widehat{\lambda} : -\varrho \leq r < 0\}$$

belong to distinct components of $\overline{B}(\lambda_0, \varrho) \setminus \mathcal{C}$ in $\mathbb{R}^k$; and for all r in the set $0 < |r| \leq \varrho$, the following criteria are satisfied:

(a) $\lambda_0 + r\widehat{\lambda} \in G$;

(b) whenever $0 \neq x \in N(L(\lambda_0))$, then $L(\lambda_0 + r\widehat{\lambda})x \notin R(L(\lambda_0))$.

(H6) $L(\lambda_0)$ is a Fredholm operator of index zero.

Remarks. 1. A bounded linear operator $L : X \to Y$ is *Fredholm of index zero* if the dimension of the null space is finite and equal to the codimension of the range; i.e., $\dim N(L) = \dim(Y/R(L)) < \infty$.

2. Hypothesis (H5)(b) is a transversality condition which was also assumed by Alexander and Fitzpatrick [3]. Fitzpatrick [10] has shown that it is equivalent to: there exists $\varepsilon > 0$ such that

$$\|L(\lambda_0 + r\widehat{\lambda})x\| \geq \varepsilon|r| \cdot \|x\|$$

for all $r \in (0, \varrho]$ and all $x \in X$. From this we see that $\lambda_0 + r\widehat{\lambda} \notin C(L)$ for all $r \in (0, \varrho]$. See the remark below, following the statement of Lemma 1.1, for additional comments on this well known condition.

We can now define a global bifurcation point.

Definition. Let λ_0 satisfy hypothesis (H5). Denote by $\mathcal{C}_S$ the component, in $X \times \mathbb{R}^k$, of S' containing the point $(0, \lambda_0)$. We say that λ_0 is a *global bifurcation point* of (0.1) provided that $\mathcal{C}_S$ satisfies at least one of the following:

(i) $\mathcal{C}_S$ is an unbounded set in $X \times \mathbb{R}^k$;

(ii) $(0, \widehat{\lambda}_0) \in \mathcal{C}_S$ for some element $\widehat{\lambda}_0 \in C(L)$ where $\widehat{\lambda}_0 \notin \mathcal{C}$;

(iii) If G has a boundary, then $\inf\{\operatorname{dist}(\lambda, \partial G) : (x, \lambda) \in \mathcal{C}_S$ for some $x \in X\} = 0$.

The author has proved the following theorem [27]:

Theorem. *Assume that hypotheses* (H1)–(H5) *are satisfied. Then* λ_0 *is a global bifurcation point of* (0.1) *if*

$$\operatorname{Deg}(L(\lambda_0 - r\widehat{\lambda}), W, 0) \neq \operatorname{Deg}(L(\lambda_0 + r\widehat{\lambda}), W, 0)$$

for sufficiently small $r > 0$, *where* $W \subset X$ *is an arbitrary open bounded set containing zero.*

We shall see that (H6) plays a crucial role in proving the degree condition of the Theorem. The following lemmas are required.

LEMMA 1.1. *Set* $\lambda = \lambda_0 + r\widehat{\lambda}$. *Then*

$$X = N(L(\lambda_0)) \oplus (L(\lambda))^{-1}R(L(\lambda_0))$$

and

$$Y = L(\lambda)N(L(\lambda_0)) \oplus R(L(\lambda_0)),$$

for each r *in the interval* $(0, \varrho]$. *Moreover*, $(L(\lambda))^{-1}R(L(\lambda_0))$ *is closed.*

REMARK. The direct sum decompositions described by Lemma 1.1 are direct consequences of hypothesis (H5)(b). It is clear from the decomposition of Y that (H5)(b) is equivalent to a well-known transversality condition that was originally defined by Crandall and Rabinowitz [6] and Westreich [28]. This transversality condition has been generalized by many authors including Magnus [17] and Esquinas and López-Gómez [7]. A multiparameter version is applied by Alexander and Fitzpatrick [3].

Proof (of Lemma 1.1). Since $L(\lambda_0)$ is A-proper, the set $R(L(\lambda_0))$ is closed [18] and, therefore, so is $(L(\lambda))^{-1}R(L(\lambda_0))$.

By our choice of λ, $L(\lambda)$ is a homeomorphism [18] and, trivially,

$$N[L(\lambda_0)(L(\lambda))^{-1}] = L(\lambda)N(L(\lambda_0)).$$

Also, if $x \in N[(L(\lambda_0)(L(\lambda))^{-1})^2]$, then setting $w = L(\lambda_0)(L(\lambda))^{-1}x$, it follows from (H5)(b) that $w = 0$. Thus,

$$N[(L(\lambda_0)(L(\lambda))^{-1})^2] \subset N[L(\lambda_0)(L(\lambda))^{-1}].$$

The reverse inclusion always holds, so the ascent of $L(\lambda_0)(L(\lambda))^{-1}$ is equal to one. By (H6), the descent of $L(\lambda_0)(L(\lambda))^{-1}$ is also equal to one. Therefore, by Theorem 6.2 of [23],

$$Y = N[L(\lambda_0)(L(\lambda))^{-1}] \oplus R[L(\lambda_0)(L(\lambda))^{-1}] = L(\lambda)N(L(\lambda_0)) \oplus R(L(\lambda_0))$$

and

$$X = N(L(\lambda_0)) \oplus (L(\lambda))^{-1}R(L(\lambda_0)).$$

In the next lemma, by utilising the direct sum decomposition of Lemma 1.1, we write $L(\lambda_0)$ as a compact perturbation of a homeomorphism.

LEMMA 1.2. $L(\lambda_0) = H - C$, *where* $C : X \to L(\lambda)N(L(\lambda_0))$ *is defined to be the linear operator* $Cx = C(x_1 + x_2) = -L(\lambda)x_1$, *where* $x_1 \in N(L(\lambda_0))$ *and* $x_2 \in (L(\lambda))^{-1}R(L(\lambda_0))$ *are uniquely defined with respect to the direct sum decomposition from Lemma* 1.1 *where* $\lambda = \lambda_0 + r\widehat{\lambda}$ *with* $r \in (0, \varrho]$. $H : X \to Y$ *is then defined to be the linear operator* $Hx = (L(\lambda_0) + C)x$, *for each* $x \in X$. *Then* C *is compact and* H *is a homeomorphism.*

Proof. Since $L(\lambda_0)$ is A-proper, $N(L(\lambda_0))$ is finite-dimensional [18], which implies that C has finite-dimensional range and is, therefore, compact.

So, H is A-proper, being a compact perturbation of an A-proper operator [21]. Since a bounded, linear, injective, A-proper operator is a homeomorphism [18], we need only show that H is injective to complete the proof.

Suppose that $Hx=0$. Then $(L(\lambda_0)+C)x = 0$ or, equivalently, $L(\lambda_0)x_2 = L(\lambda)x_1$ where $x_1 \in N(L(\lambda_0))$ and $x_2 \in (L(\lambda))^{-1}R(L(\lambda_0))$. By Lemma 1.1, $x_1 = 0$, from which we see that

$$x_2 \in N(L(\lambda_0)) \cap (L(\lambda))^{-1}R(L(\lambda_0)) = \{0\}.$$

Hence, H is a homeomorphism as was to be shown.

Lemma 1.2 enables us to transform (0.1) into the equivalent form

$$\overline{F}(y,\lambda) = \overline{L}(\lambda)y + \overline{R}(y,\lambda) = 0, \tag{1.1}$$

where: $\overline{F}, \overline{R} : Y \times \mathbb{R}^k \to Y$ and $\overline{L}(\lambda) : Y \to Y$ for each $\lambda \in G$ with $\overline{L}(\lambda) = I - CH^{-1} - T(\lambda - \lambda_0)H^{-1}$ and $\overline{R}(\cdot,\lambda) = R(H^{-1}(\cdot),\lambda)$.

Remark. (1.1) satisfies the corresponding hypotheses (H1)–(H6) with A-properness defined relative to the admissible scheme $\Gamma_H = \{H(X_n), Y_n, Q_n\}$ for mappings from Y into Y.

The Theorem guarantees that λ_0 will be a global bifurcation point of (1.1) provided that

$$\mathrm{Deg}(I - CH^{-1} - T(-r\widehat{\lambda})H^{-1}, W, 0) \neq \mathrm{Deg}(I - CH^{-1} - T(r\widehat{\lambda})H^{-1}, W, 0)$$

for sufficiently small $r > 0$.

In order to prove that this topological degree result holds, we require two additional lemmas.

Lemma 1.3.

$$\dim \bigcup_{n=1}^{\infty} N((I - CH^{-1})^n) = \dim N(L(\lambda_0)).$$

Proof. Suppose that $y \in N(I - CH^{-1}) \cap R(I - CH^{-1})$. Then $0 = (H - C)(H^{-1}y) = L(\lambda_0)(H^{-1}y)$, implying that $H^{-1}y \in N(L(\lambda_0))$. From Lemma 1.2,

$$y \in CN(L(\lambda_0)) = L(\lambda)N(L(\lambda_0)) \tag{1.2}$$

for $\lambda \in \mathbb{R}^k$ satisfying the conditions of Lemma 1.1.

Also, there exists $\widehat{y} \in Y$ such that $(I - CH^{-1})\widehat{y} = y$. So, $L(\lambda_0)(H^{-1}\widehat{y}) = y$. Equation (1.2) and Lemma 1.2 yield that $y = 0$ and so $N(I - CH^{-1}) \cap R(I - CH^{-1}) = \{0\}$, which easily implies that $N[(I - CH^{-1})^2] = N(I - CH^{-1})$. Hence, $\dim \bigcup_{n=1}^{\infty} N[(I - CH^{-1})^n] = \dim N(I - CH^{-1}) = \dim N(L(\lambda_0))$, which completes the proof.

Lemma 1.4. *There exist constants* $K_1, K_2 > 0$ *and* $\delta_0 > 0$ *such that* $\|L(\lambda_0)x\| \geq K_1$ *for all* $x \in (L(\lambda))^{-1}R(L(\lambda_0))$ *with* $\lambda \notin C(L)$ *and* $\|x\| = 1$;

$L(\lambda_0) + B$ is A-proper with respect to the admissible scheme Γ for all bounded linear operators $B : X \to Y$ with $\|B\| < K_2$; and $\|T(\lambda)\| < (1/2)\min\{K_1, K_2\}$ whenever $|\lambda| \le \delta_0$.

REMARK. In Lemma 1.4, K_1 and δ_0 are dependent upon λ. However, by the time we come to use this lemma in Theorem 1.5, we will have chosen and fixed λ.

Proof (of Lemma 1.4). K_2 is guaranteed by a result of Petryshyn [20].

Suppose that K_1 does not exist. Then there exists a sequence $\{x_n\}$ in $(L(\lambda))^{-1}R(L(\lambda_0))$ with $\|x_n\| = 1$ for each positive integer n such that $\|L(\lambda_0)x_n\| \le 1/n$. But $L(\lambda_0)$ is A-proper and is, therefore, proper on closed bounded subsets of X [19]. (Recall that a function $f : X \to Y$ is said to be *proper* on the closed bounded set $F \subset X$ if for any compact set $K \subset Y$, the nonempty set $F \cap f^{-1}(K)$ is compact in X.) So, without loss of generality, there exists $x \in X$ with $\|x\| = 1$ such that $x_n \to x$ as $n \to \infty$ and $L(\lambda_0)x = 0$. It is shown in [18] that, by the A-properness of $L(\lambda_0)$, $R(L(\lambda_0))$ is closed. Hence, from Lemma 1.1, $x \in (L(\lambda))^{-1}R(L(\lambda_0)) \cap N(L(\lambda_0)) = \{0\}$. This contradiction proves that K_1 exists as claimed.

The existence of δ_0 is trivial.

THEOREM 1.5. *Let* (0.1) *satisfy hypotheses* (H1)–(H6). *Then λ_0 is a global bifurcation point of* (0.1) *provided that* $\dim N(L(\lambda_0))$ *is an odd number.*

Proof. Set $\delta = (1/2)\min\{1, \delta_0, \varrho\}$, where δ_0 and ϱ are defined in Lemma 1.4 and hypotheses (H5), respectively.

By the Theorem, λ_0 is a global bifurcation point of (0.1) if for each r in the interval $0 < r \le \delta$,

$$\mathrm{Deg}(I - CH^{-1} - T(-r\widehat{\lambda})H^{-1}, W, 0) \ne \mathrm{Deg}(I - CH^{-1} - T(r\widehat{\lambda})H^{-1}, W, 0),$$

where $W \subset Y$ is an arbitrary open bounded set containing zero. Recall that $\widehat{\lambda}$ was defined in (H5).

First we prove that

$$\deg_{\mathrm{LS}}(I - (1-r)CH^{-1}, W, 0) = -\deg_{\mathrm{LS}}(I - (1+r)CH^{-1}, W, 0) \tag{1.3}$$

for $r \in (0, \delta]$.

In Lemmas 1.1–1.4, for the direct sum decompositions of X and Y and the corresponding definition of the operators C and H, we assume that $\lambda = \lambda_0 + r\widehat{\lambda}$ has been chosen and fixed with $r \in (0, \delta]$.

Trivially, 1 is a characteristic value of CH^{-1}. Suppose that for some $t \ne 1$, there exists $y \in Y$ with $\|y\| = 1$ such that $y - tCH^{-1}y = 0$. Then

$$[L(\lambda_0) - (t-1)C](x_1 + x_2) = 0$$

with $H^{-1}y = x_1 + x_2$, where $x_1 \in N(L(\lambda_0))$ and $x_2 \in (L(\lambda))^{-1}R(L(\lambda_0))$ represent the unique decomposition of $H^{-1}y$ guaranteed by Lemma 1.1. Then, by our choice of C in Lemma 1.2, $L(\lambda_0)x_2 = -(t-1)L(\lambda)x_1 = 0$, from which we conclude that $x_1 = 0$ and, therefore, $x_2 \in N(L(\lambda_0)) \cap (L(\lambda))^{-1}R(L(\lambda_0)) = \{0\}$. This yields the contradiction that $y = 0$ and tells us that 1 is the only characteristic value of CH^{-1}.

By the Leray–Schauder formula ([22] or [24]),

$$\deg_{\mathrm{LS}}(I-(1-r)CH^{-1},W,0) = (-1)^0 = 1,$$

and by Lemma 1.3,

$$\begin{aligned}\deg_{\mathrm{LS}}(I-(1+r)CH^{-1},W,0) &= (-1)^{\dim \bigcup_{n=1}^{\infty} N[(I-CH^{-1})^n]}\\ &= (-1)^{\dim N(L(\lambda_0))} = -1.\end{aligned}$$

Hence, (1.3) is satisfied.

To complete the proof, we apply the homotopy property for generalized degree.

Define $\overline{H} : \overline{W} \times [0,1] \to Y$ by

$$\overline{H}(y,s) = y - s(1+r)CH^{-1}y - (1-s)CH^{-1}y - T((1-s)r\widehat{\lambda})H^{-1}y. \tag{1.4}$$

We must show that $\overline{H}(\cdot,s) : Y \to Y$ is A-proper with respect to the admissible scheme Γ_H for all $s \in [0,1]$. Since we can write

$$\begin{aligned}\overline{H}(y,s) &= y - CH^{-1}y - T((1-s)r\widehat{\lambda})H^{-1}y - srCH^{-1}y\\ &= [H - C - T((1-s)r\widehat{\lambda}) - srC]H^{-1}y,\end{aligned}$$

and C is compact, we need only show that $[H-C-T((1-s)r\widehat{\lambda})]$ is A-proper with respect to Γ. However, $H - C = L(\lambda_0)$ and $\|T((1-s)r\widehat{\lambda})\| < K_2$ by the result of Lemma 1.4, which implies that $H(\cdot,s)$ is A-proper with respect to Γ_H for all $s \in [0,1]$.

By (H2), $\overline{H}(y,\cdot) : [0,1] \to Y$ is easily seen to be continuous, uniformly for y in closed bounded subsets of Y.

All that remains to be shown before applying the homotopy property is that $\overline{H}(\partial W,s) \neq 0$ for all $s \in [0,1]$.

Suppose that $\overline{H}(y,s) = 0$ for some $s \in [0,1]$ with $y \neq 0$. If $s = 1$, then $1+r$ is a characteristic value of CH^{-1}, which is not possible. If $s = 0$, then $\lambda_0 + r\widehat{\lambda} \in C(L)$, which is also a contradiction. So,

$$\overline{H}(y,s) = [H - C - T((1-s)r\widehat{\lambda}) - srC]H^{-1}y = 0$$

with $s \in (0,1)$. Setting $H^{-1}y = x_1 + x_2$, where $x_1 \in N(L(\lambda_0))$ and $x_2 \in (L(\lambda_0 + r\widehat{\lambda}))^{-1}R(L(\lambda_0))$, we obtain

$$L(\lambda_0)x_2 - T((1-s)r\widehat{\lambda})x_2 = T((1-s)r\widehat{\lambda})x_1 - srAx_1 + T(sr(\lambda_0 + r\widehat{\lambda}))x_1. \tag{1.5}$$

We can rewrite the left-hand side of (1.5) in the form $(1-s)L(\lambda_0 + r\widehat{\lambda})x_2 + sL(\lambda_0)x_2$, which clearly lies in $R(L(\lambda_0))$. Similarly, the right-hand side of (1.5) may be written as $-(1-s(1-r))L(\lambda_0 + r\widehat{\lambda})x_1$, which lies in $L(\lambda_0 + r\widehat{\lambda})N(L(\lambda_0))$. It follows by Lemma 1.1 with $\lambda = \lambda_0 + r\widehat{\lambda}$ that each side of (1.5) is equal to zero. Since $s \in (0,1)$, we have $(1-s(1-r)) \neq 0$, from which we conclude that $x_1 = 0$ and, therefore, $x = x_2 = H^{-1}y \neq 0$. Thus, the left-hand side of (1.5), being zero, is equivalent to the equation

$$\left\| L(\lambda_0)\left(\frac{x_2}{\|x_2\|}\right)\right\| = \left\| T((1-s)r\widehat{\lambda})\left(\frac{x_2}{\|x_2\|}\right)\right\|.$$

From Lemma 1.4, we conclude that $x_2 = 0$.

So, we arrive at the contradiction that $y = 0$ from which we conclude that $\overline{H}(\partial W, s) \neq 0$ for all $s \in [0,1]$. Hence, all the requirements of the homotopy property are met and we may set $s = 1$, respectively, $s = 0$ in (1.4) to obtain the degree result

$$\begin{aligned} -1 &= \deg_{\mathrm{LS}}(I - (1+r)CH^{-1}, W, 0) \\ &= \mathrm{Deg}(I - (1+r)CH^{-1}, W, 0) \\ &= \mathrm{Deg}(I - CH^{-1} - T(r\widehat{\lambda})H^{-1}, W, 0). \end{aligned} \tag{1.6}$$

Using an analogous homotopy where we replace r by $-r$ in (1.4), we can show similarly that

$$\begin{aligned} 1 &= \deg_{\mathrm{LS}}(I - (1-r)CH^{-1}, W, 0) \\ &= \mathrm{Deg}(I - (1-r)CH^{-1}, W, 0) \\ &= \mathrm{Deg}(I - CH^{-1} - T(-r\widehat{\lambda})H^{-1}, W, 0). \end{aligned} \tag{1.7}$$

In the analysis leading to (1.7), at the stage corresponding to (1.5), the left-hand side of the resulting equation may be written in the form $(s-1)L(\lambda_0 + r\widehat{\lambda})x_2 + (2-s)L(\lambda_0)x_2$, which is, again, in $R(L(\lambda_0))$ as required.

An application of the Theorem, using (1.6) and (1.7), completes the proof.

2. Example. We conclude with an elementary example which illustrates the type of problem to which our results can be applied.

Consider the ordinary differential equation

$$x'' + (\lambda_1 + \ldots + \lambda_k)x = g(x, x', x'', \lambda), \tag{2.1}$$

where $x(0) = x(\pi/2) = 0$; $\lambda = (\lambda_1, \ldots, \lambda_k)$ with $\lambda_1, \ldots, \lambda_k \geq 0$; and g is a continuous function.

Define the spaces

$$X = \{x \in C^2[0, \pi/2] : x(0) = x(\pi/2) = 0\}, \qquad Y = C[0, \pi/2].$$

Let the operators $L(\lambda),\ R(\cdot,\lambda) : X \to Y$ be defined by

$$L(\lambda)x(t) = x''(t) + (\lambda_1 + \ldots + \lambda_k)x(t),$$

and

$$R(x(t),\lambda) = -g(x(t), x'(t), x''(t), \lambda),$$

for $t \in [0, \pi/2]$ and $\lambda \in \mathbb{R}^k$.

Then (2.1) can be written in operator form

$$F(x,\lambda) = L(\lambda)x + R(x,\lambda) = 0.$$

It follows easily that λ is a characteristic value of $L(\lambda)$ if $\lambda_1 + \ldots + \lambda_k = 4n^2$ for some integer $n > 1$. We can, therefore, see that there are no *isolated* characteristic values of $L(\lambda)$ in $\mathbb{R}^k$. Notice also that each component of $C(L)$ is bounded.

The A-properness requirements on $R(\cdot,\lambda)$ can be guaranteed by the methods outlined in [21].

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