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REPRESENTING IDEMPOTENTS AS A SUM OF TWO NILPOTENTS—AN APPROACH VIA MATRICES OVER DIVISION RINGS

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1. Introduction. It was proved in [3] that the Koethe conjecture is equivalent to the problem of determining whether a ring which is a sum of a nil subring and a nilpotent subring must be nil. A similar problem, whether a ring that is a sum of two locally nilpotent subrings must be nil, has a negative solution (see [8]). A simpler example of this type was then constructed in [11]. Therefore one may ask whether such a ring can contain a nonzero idempotent. This naturally leads to the following problem investigated in [4]: can a nonzero idempotent e be represented as a sum $e = x + y$ of two nilpotent elements x, y? It was proved there that this is impossible if the nilpotency degrees of x and y are ≤ 3 and ≤ 5 respectively (or ≤ 2 and any $n \in \mathbb{N}$) provided that the characteristic is equal to zero. If the characteristic is positive, examples of this type are easy to find (see [4]), whence in this paper we restrict our attention to algebras over a field of characteristic zero.

We show that idempotents of such type exist if the nilpotency degrees of x, y are both 4, or 3 and 6 respectively. This is done by investigating representations in matrices over division rings. In this context, the first Weyl algebra appears unexpectedly and unavoidably, as shown by our main results: Theorems 8 and 12. In particular, we prove that $M_4(D)$ contains a nonzero idempotent with zero diagonal if and only if D contains a copy of the first Weyl algebra.

It was shown in [4] that the identity element may be represented as a sum of four nilpotent elements of nilpotency degree 2. We prove that the identity element can also be a sum of three nilpotent elements of nilpotency degree 3. This is used to construct an example with $0 \neq e = e^2 = x + y$ and $x^3 = y^6 = 0$.

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Finally, we give an application to a problem closely connected to Kegel's theorem asserting that a ring which is a sum of two nilpotent subrings must be nilpotent (see [6, 7]).

It might seem possible that the diamond lemma (see [1]) can be applied to construct examples of the above types; however, it leads to very complex computations, which are not conclusive.

Throughout the paper D (K respectively) denotes a skew field (resp. a field) of characteristic zero. All spaces will be left spaces over D (resp. K). We denote by $v_1, \ldots, v_n$ the standard basis of D^n; $M_n(D) = \mathrm{End}_D(D^n)$ stands for the ring of $n \times n$ matrices over D, and I for the identity element of $M_n(D)$. The mappings $\pi_i \in \mathrm{End}_D(D^n)$ ($1 \le i \le n$) are defined by $\pi_i(v_i) = 0$ and $\pi_i(v_j) = v_j$ for $i \ne j$. If A is a K-algebra, then $\mathrm{GKdim}(A)$ and $\mathcal{J}(A)$ denote the Gelfand–Kirillov dimension and the Jacobson radical of A respectively.

2. Idempotents with zero diagonal in $M_4(D)$. As explained above, our approach to the problem proposed in [4] is based on matrix algebras $M_n(D)$. Their multiplicative structure was investigated in [10]. We describe all idempotents with zero diagonal in $M_4(D)$; clearly such an element is a sum of two nilpotents. If $n < 4$, then $M_n(D)$ does not contain such idempotents. A similar problem for $M_n(D)$, $n \ge 5$, seems to be difficult. First we need some preparatory results.

LEMMA 1. *Let V be a linear space over D. Assume that $W_1 \subseteq \ldots \subseteq W_n = V$ and $Z_1 \subseteq \ldots \subseteq Z_m = V$ are chains of subspaces of V. Then we can find subspaces $Y_{i,j}$ of V such that $W_i \cap Z_j = \bigoplus_{k \le i,\, l \le j} Y_{k,l}$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$.*

Proof. Choose subspaces $Y_{i,j}$ satisfying

$$Y_{i,j} \oplus (W_i \cap Z_{j-1} + W_{i-1} \cap Z_j) = W_i \cap Z_j \tag{1}$$

where $W_0 = Z_0 = 0$. Consider the order on the set of all pairs (i, j) defined by:

$$(i, j) \le (i', j') \quad \text{if and only if} \quad i \le i' \text{ and } j \le j'.$$

By induction we prove that $W_i \cap Z_j = \bigoplus_{k \le i,\, l \le j} Y_{k,l}$. By the induction hypothesis we get

$$W_i \cap Z_{j-1} = \bigoplus_{r \le i,\, s \le j-1} Y_{r,s}, \tag{2}$$

$$W_{i-1} \cap Z_j = \bigoplus_{p \le i-1,\, q \le j} Y_{p,q}. \tag{3}$$

Hence

$$\begin{aligned} W_i \cap Z_j &= Y_{i,j} \oplus (W_i \cap Z_{j-1} + W_{i-1} \cap Z_j) \\ &= Y_{i,j} \oplus \Big(\bigoplus_{r\le i,\, s\le j-1} Y_{r,s} + \bigoplus_{p\le i-1,\, q\le j} Y_{p,q} \Big) = \sum_{r\le i,\, s\le j} Y_{r,s}. \end{aligned}$$

Now we prove that this sum is direct. Let $y_{r,s} \in Y_{r,s}$, where $r \le i$ and $s \le j$, be such that $\sum_{r\le i,\, s\le j} y_{r,s} = 0$. By (1), $y_{i,j} = 0$, hence

$$\sum_{r\le i,\, s\le j-1} y_{r,s} = -\sum_{k\le i-1} y_{k,j} \in (W_i \cap Z_{j-1}) \cap (W_{i-1} \cap Z_j) = W_{i-1} \cap Z_{j-1}.$$

This implies

$$-\sum_{k\le i-1} y_{k,j} = \sum_{p\le i-1,\, q\le j-1} \overline{y}_{p,q} \quad \text{for some } \overline{y}_{p,q} \in Y_{p,q}.$$

By (3) we get $y_{k,j} = 0$ for $k \le i-1$. Hence

$$\sum_{r\le i,\, s\le j-1} y_{r,s} = -\sum_{k\le i-1} y_{k,j} = 0.$$

By (2), $y_{r,s} = 0$ for $r \le i$ and $s \le j-1$. So we have proved that $y_{r,s} = 0$ for all r, s. ■

LEMMA 2. *An element $e \in M_n(D)$ is a sum of two nilpotent elements if and only if e has zero diagonal in some basis of D^n.*

Proof. Assume that $e = x+y$ where $x^n = y^n = 0$. Define $W_i = \mathrm{Ker}(x^i)$ and $Z_j = \mathrm{Ker}(y^j)$, $1 \le i, j \le n$. Choose subspaces $Y_{i,j}$ as in Lemma 1 and take a basis which is the union of some bases of all nonzero $Y_{i,j}$. It is easy to see that e has zero diagonal in this basis.

Conversely, assume that the diagonal of e is zero. Then e can be represented as a sum of a strictly upper triangular and a strictly lower triangular matrices, which are clearly nilpotent. ■

LEMMA 3. *Every idempotent e of rank 1 in $M_n(D)$ has a nonzero diagonal.*

Proof. Suppose that e is an idempotent of rank 1 with zero diagonal. Changing the order of $v_1, \ldots, v_n$ we can assume that $v_1, \ldots, v_k \in \mathrm{Ker}(e)$, $v_{k+1}, \ldots, v_n \notin \mathrm{Ker}(e)$ for some $1 \le k \le n$. Let $\mathrm{Im}(e) = \mathrm{Lin}_D(v)$ for some $v \in D^n$. By the assumption $e(v_j) \in \mathrm{Lin}_D\{v_l : l \ne j\}$ for $j > k$, hence $v \in \mathrm{Lin}_D\{v_l : l \ne j\}$. Clearly $\bigcap_{j>k} \mathrm{Lin}_D\{v_l : l \ne j\} = \mathrm{Lin}_D(v_1, \ldots, v_k)$. This implies that $\mathrm{Im}(e) = \mathrm{Lin}_D(v) \subseteq \mathrm{Lin}_D(v_1, \ldots, v_k) \subseteq \mathrm{Ker}(e)$, a contradiction. ■

LEMMA 4. *Assume that $n > 1$. Then every idempotent of rank $n-1$ in $M_n(D)$ has a nonzero diagonal.*

P r o o f. Let $e = e^2 \in M_n(D)$ be an idempotent of rank $n-1$. Suppose e has zero diagonal. Let $f = I - e$ and $\mathrm{Im}(f) = \mathrm{Lin}_D(v)$ for some $v \in D^4$. Then $f(v_i) = \alpha_i v$ for some $\alpha_i \in D$ and $f(v_i) = v_i + w_i$ for some $w_i \in \mathrm{Lin}_D\{v_j : j \neq i\}$ by the assumptions on e. This implies that $v = \sum_i \alpha_i^{-1} v_i$ ($\alpha_i \neq 0$ in particular) and $f(v_i) = \alpha_i \sum_j \alpha_j^{-1} v_j$. Hence

$$f(v_i) = f(f(v_i)) = f\Big(\alpha_i \sum_j \alpha_j^{-1} v_j\Big) = \sum_j \alpha_i \alpha_j^{-1} f(v_j)$$
$$= \sum_j \alpha_i \alpha_j^{-1} \alpha_j v = n f(v_i).$$

So $f(v_i) = 0$ and $f = 0$, a contradiction. ∎

LEMMA 5. *Assume that* $e \in M_n(D)$ *is an idempotent and* $e(v_i) \neq 0$ *for some* $i \in \{1, \ldots, n\}$. *Then* $\pi_i(\mathrm{Ker}(e)) \cap \mathrm{Im}(e) \neq 0$ *if and only if* $e(v_i) \in \mathrm{Lin}_D\{v_j : j \neq i\}$.

P r o o f. (⇒) Assume that $\pi_i(v) = e(w) \neq 0$ and $e(v) = 0$ for some v, wD^n. Let $v = \alpha v_i + \pi_i(v)$ for some $\alpha \in D$. Then $0 = e(v) = \alpha e(v_i) + e(\pi_i(v))$. Hence $-\alpha e(v_i) = e(\pi_i(v)) = e(e(w)) = e(w) = \pi_i(v)$. If $\alpha = 0$, then $v = \pi_i(v) = e(w)$. Hence $0 = e(v) = e^2(w) = e(w)$, a contradiction. So $\alpha \neq 0$ and $e(v_i) = -\alpha^{-1}\pi_i(v) \in \mathrm{Lin}_D\{v_j : j \neq i\}$.

(⇐) Assume that $e(v_i) \in \mathrm{Lin}_D\{v_j : j \neq i\}$. We claim that

$$[\mathrm{Ker}(e) + \mathrm{Lin}_D(v_i)] \cap \mathrm{Im}(e) \subseteq \pi_i(\mathrm{Ker}(e)) \cap \mathrm{Im}(e).$$

Any vector of $[\mathrm{Ker}(e)+\mathrm{Lin}_D(v_i)]\cap\mathrm{Im}(e)$ can be written in the form $v+\alpha v_i = e(w)$, where $v, w \in D^n$, $e(v) = 0$ and $\alpha \in D$. Then $\pi_i(v) + \alpha\pi_i(v_i) = \pi_i(e(w))$. Hence

$$\pi_i(v) = \pi_i(e(e(w))) = \pi_i(e(v + \alpha v_i)) = \pi_i(e(\alpha v_i))$$
$$= \alpha\pi_i(e(v_i)) = \alpha e(v_i) = e(e(w) - v) = e(w).$$

This shows that $e(w) = \pi_i(v) \in \pi_i(\mathrm{Ker}(e)) \cap \mathrm{Im}(e)$, proving the claim.

Since $e(v_i) \neq 0$, we get $[\mathrm{Ker}(e)+\mathrm{Lin}_D(v_i)]\cap\mathrm{Im}(e) \neq 0$. Hence $\pi_i(\mathrm{Ker}(e))\cap \mathrm{Im}(e) \neq 0$, as desired. ∎

LEMMA 6. *Let* $e \in M_4(D)$ *be an idempotent of rank* 2 *with zero diagonal. Then* $\mathrm{Ker}(e) \cap \mathrm{Lin}_D(v_i, v_j) = 0$ *for any* $i \neq j$, $i, j \in \{1, 2, 3, 4\}$.

P r o o f. First suppose that there exist $\alpha, \beta \in D\setminus\{0\}$ such that $\alpha v_i + \beta v_j \in \mathrm{Ker}(e)$. Hence $\alpha e(v_i) + \beta e(v_j) = 0$. Since e has zero diagonal, $e(v_i), e(v_j) \in \mathrm{Lin}_D(v_k, v_l)$ whenever $\{i, j, k, l\} = \{1, 2, 3, 4\}$. Hence the diagonal of e is zero in the basis $\alpha v_i + \beta v_j, v_j, v_k, v_l$. Let $^- : D^4 \to D^4/\mathrm{Lin}_D(\alpha v_i + \beta v_j)$ denote the quotient map and $\overline{e} \in \mathrm{End}_D(D^4/\mathrm{Lin}_D(\alpha v_i + \beta v_j))$ be defined by $\overline{e}(\overline{v}) = \overline{e(v)}$. Then $\overline{e}$ is an idempotent of rank 2 (in $M_3(D)$) with zero diagonal in the basis $\overline{v}_j, \overline{v}_k, \overline{v}_l$. This contradicts Lemma 4.

It remains to consider the case when $v_i \in \mathrm{Ker}(e)$ or $v_j \in \mathrm{Ker}(e)$. Let for example $v_i \in \mathrm{Ker}(e)$. Then considering $^- : D^4 \to D^4/\operatorname{Lin}_D(v_i)$ and $\overline{e}$ we get a contradiction as above. ■

LEMMA 7. *Let* $V, W \subseteq D^4$ *be subspaces such that* $\dim V = \dim W = 2$ *and* $V \cap \operatorname{Lin}_D(v_i, v_j) = 0$ *for all* $i \neq j$. *If* $\pi_i(V) \cap W \neq 0$ *for all* i, *then* $V = W$ *or* $V \cap W = 0$.

Proof. Suppose that $V \cap W \neq 0$ and $V \neq W$. Fix some i. We claim that either $V \cap W \subseteq \operatorname{Lin}_D\{v_j : j \neq i\}$ or $W \subseteq \operatorname{Lin}_D(v_i) + V$.

Assume that $W \not\subseteq \operatorname{Lin}_D(v_i) + V$. Since $\dim(\operatorname{Lin}_D(v_i) + V) = 3$ and $\dim W = 2$ by hypothesis, $(\operatorname{Lin}_D(v_i) + V) \cap W \neq 0$. Hence $\dim(\operatorname{Lin}_D(v_i) + V) \cap W = 1$. Since $\pi_i(V) \subseteq \operatorname{Lin}_D(v_i) + V$, we have $0 \neq \pi_i(V) \cap W \subseteq (\operatorname{Lin}_D(v_i) + V) \cap W$. Therefore $(\operatorname{Lin}_D(v_i) + V) \cap W = \pi_i(V) \cap W$. Similarly $0 \neq V \cap W \subseteq (\operatorname{Lin}_D(v_i) + V) \cap W$ yields $V \cap W = (\operatorname{Lin}_D(v_i) + V) \cap W$. This implies $V \cap W = \pi_i(V) \cap W \subseteq \operatorname{Lin}_D\{v_j : j \neq i\}$, proving the claim.

If $W \subseteq \operatorname{Lin}_D(v_i) + V$ and $W \subseteq \operatorname{Lin}_D(v_j) + V$ for some $i \neq j$ then $V + W \subseteq (\operatorname{Lin}_D(v_i) + V) \cap (\operatorname{Lin}_D(v_j) + V)$. Since $\dim(V + W) = 3$, we get equality and so $\operatorname{Lin}_D(v_i) + V = \operatorname{Lin}_D(v_j) + V$. This contradicts the fact that $V \cap \operatorname{Lin}_D(v_i, v_j) = 0$. So by the initial remark $V \cap W \subseteq \operatorname{Lin}_D\{v_j : j \neq i\}$ for at least three values of i.

If this inclusion holds for $i = 1, 2, 3, 4$ we get $V \cap W = 0$, a contradiction. So it holds for exactly three values of i. This implies $V \cap W = \operatorname{Lin}_D(v_i)$ for some i. Since $v_i \in V$, we get $\dim \pi_i(V) = 1$. As $\pi_i(V) \cap W \neq 0$ we have $\pi_i(V) \subseteq W$. Clearly $\pi_i(V) \subseteq \operatorname{Lin}_D(v_i) + V = V$. This yields $\operatorname{Lin}_D(v_i) + \pi_i(V) \subseteq V \cap W$. Since $\dim(\operatorname{Lin}_D(v_i) + \pi_i(V)) = 2$, we conclude that $V = W$, a contradiction. ■

REMARK. It may be proved that if V, W are subspaces of D^4, $\dim V = \dim W = 2$ and $\pi_i(V) \cap W \neq 0$ for all i, then either $V = W$ or $V \cap W = 0$ or $0 \neq V \cap W \subseteq \operatorname{Lin}_D(v_i, v_j)$ for some $i \neq j$ or $V + W \subseteq \operatorname{Lin}_D(v_i, v_j, v_k)$ for some distinct i, j, k. Moreover, if the first, third or fourth possibility holds, then $\pi_i(V) \cap W \neq 0$ for all i.

Denote by $A_1 = K\langle x, y : xy - yx = 1\rangle$ the first Weyl algebra over K. It is well known that A_1 is a simple domain which has two-sided Ore fractions (see [9]). Hence the division ring D contains an isomorphic copy of $A_1A_1^{-1}$ if and only if D contains two elements x and y satisfying $xy - yx = 1$. We are now ready to prove our first main result.

THEOREM 8. *Let* D *be a division ring of characteristic zero. Then there exists a nonzero idempotent* $e \in M_4(D)$ *which is a sum of two nilpotent elements if and only if* D *contains a copy of* $A_1A_1^{-1}$.

Proof. By Lemma 2, $e^2 = e$ is a sum of two nilpotent elements if and only if e has zero diagonal in some basis. We can assume that v_1, v_2, v_3, v_4 is the appropriate basis. Our assertion may be reformulated as follows:

There exists a nonzero idempotent with zero diagonal in $M_4(D)$ if and only if there exist two-dimensional subspaces $V, W \subseteq D^4$ such that $V \cap \mathrm{Lin}_D(v_i, v_j) = 0$ for $i \neq j$, $\pi_i(V) \cap W \neq 0$ for all i and $V \neq W$.

Indeed, if e is a nonzero idempotent with zero diagonal, then by Lemmas 3 and 4, $\mathrm{rank}(e) = 2$. Put $V = \mathrm{Ker}(e)$ and $W = \mathrm{Im}(e)$. Then by Lemma 6, $V \cap \mathrm{Lin}_D(v_i, v_j) = 0$ for $i \neq j$. By Lemma 5($\Leftarrow$), $\pi_i(V) \cap W \neq 0$. Conversely, if V and W are subspaces satisfying the above conditions then by Lemma 7, $V \cap W = 0$. Define the idempotent e by $\mathrm{Ker}(e) = V$ and $\mathrm{Im}(e) = W$ ($V \oplus W = D^4$). Then by Lemma 5($\Rightarrow$), e has zero diagonal.

Assume that subspaces V, W are given. Then $V \not\subseteq \bigcup_i \mathrm{Lin}_D\{v_j : j \neq i\}$. Indeed, otherwise $V \subseteq \mathrm{Lin}_D\{v_j : j \neq i\}$ for some i, leading to $V \cap \mathrm{Lin}_D(v_k, v_l) \neq 0$ for some $k \neq l$ and contradicting the assumption on V. Hence we can find a vector $\alpha_1' v_1 + \ldots + \alpha_4' v_4 \in V$ with $\alpha_1', \ldots, \alpha_4' \in D \setminus \{0\}$. Replacing the basis $v_1, \ldots, v_4$ by $\alpha_1' v_1, \ldots, \alpha_4' v_4$ we can assume that $v_1 + \ldots + v_4 \in V$. Hence $V = \mathrm{Lin}_D(v_1 + \ldots + v_4, \alpha_1 v_1 + \ldots + \alpha_4 v_4)$ for some $\alpha_i \in D$. In this situation the condition $V \cap \mathrm{Lin}_D(v_i, v_j) = 0$ $(i \neq j)$ is equivalent to $\alpha_i \neq \alpha_j$ for $i \neq j$.

Assume now that V is given and we try to find a subspace W such that $\dim W = 2$ and $\pi_i(V) \cap W \neq 0$ for all i. Note first that $\pi_i(V) \cap \pi_j(V) = 0$ for $i \neq j$. Indeed, let for example $i = 1$ and $j = 2$. Take $w \in \pi_1(V) \cap \pi_2(V)$. Then $w = \pi_1(z_1) = \pi_2(z_2) \in \mathrm{Lin}_D(v_3, v_4)$ for some $z_1, z_2 \in V$. Moreover, $z_1 = \alpha v_1 + w$, $z_2 = \beta v_2 + w$ for some $\alpha, \beta \in D$. This gives $z_1 - z_2 \in \mathrm{Lin}_D(v_1, v_2) \cap V = 0$. Hence $\alpha = \beta = 0$ and $z_1 = w \in V \cap \mathrm{Lin}_D(v_3, v_4) = 0$. This proves the desired claim.

Let $w_i = p_i(\sum_{j \neq i} v_j) + q_i(\sum_{j \neq i} \alpha_j v_j) \in \pi_i(V) \cap W \setminus \{0\}$ for some $p_i, q_i \in D$, $i = 1, \ldots, 4$. Then, by the last paragraph, w_i, w_j are linearly independent for any $i \neq j$. Since $w_i \in \pi_i(V)$, existence of a subspace W with the desired properties is equivalent to $\dim \mathrm{Lin}_D\{w_i : i = 1, \ldots, 4\} = 2$. The latter is equivalent to $w_3 = rw_1 + sw_2$, $w_4 = tw_1 + uw_2$ for some $r, s, t, u \in D$ (of course $r, s, t, u \neq 0$). By the definition of w_i this can be written as

$$
\begin{cases}
p_3 + q_3\alpha_1 = sp_2 + sq_2\alpha_1, \\
p_3 + q_3\alpha_2 = rp_1 + rq_1\alpha_2, \\
0 = rp_1 + rq_1\alpha_3 + sp_2 + sq_2\alpha_3, \\
p_3 + q_3\alpha_4 = rp_1 + rq_1\alpha_4 + sp_2 + sq_2\alpha_4, \\
p_4 + q_4\alpha_1 = up_2 + uq_2\alpha_1, \\
p_4 + q_4\alpha_2 = tp_1 + tq_1\alpha_2, \\
p_4 + q_4\alpha_3 = tp_1 + tq_1\alpha_3 + up_2 + uq_2\alpha_3, \\
0 = tp_1 + tq_1\alpha_4 + up_2 + uq_2\alpha_4.
\end{cases}
\tag{4}
$$

Now we prove that the condition $w_1, \dots, w_4 \neq 0$ can be replaced by $w_1 \neq 0$, or equivalently $(p_1, q_1) \neq (0,0)$. Assume that $w_1 ot = 0$. If $w_2 = 0$ then $w_3 = rw_1 \neq 0$. Since $\pi_1(V) \cap \pi_3(V) \neq 0$, w_1 and w_3 are linearly independent, and we get a contradiction. Hence $w_1, w_2 \neq 0$ and w_1, w_2 are linearly independent. This implies that $w_3 = rw_1 + sw_2 \neq 0$ and $w_4 = tw_1 + uw_2 \neq 0$.

So our problem is reduced to solving (4) under the assumptions: $\alpha_i \neq \alpha_j$ for $i \neq j$, $r, s, t, u \neq 0$, $(p_1, q_1) \neq (0,0)$ and the solution corresponds to $V \neq W$.

First assume that such a solution is given. We prove that $D \supseteq A_1 A_1^{-1}$. From the first and fifth equations of (4) we get $p_3 = sp_2 + sq_2\alpha_1 - q_3\alpha_1$ and $p_4 = up_2 + uq_2\alpha_1 - q_4\alpha_1$. Now we can eliminate p_3 and p_4 from (4) passing to

$$\begin{cases} q_3\alpha_1 + rp_1 + rq_1\alpha_2 = q_3\alpha_2 + sp_2 + sq_2\alpha_1, \\ 0 = rp_1 + rq_1\alpha_3 + sp_2 + sq_2\alpha_3, \\ sp_2 + sq_2\alpha_1 - q_3\alpha_1 + q_3\alpha_4 = rp_1 + rq_1\alpha_4 + sp_2 + sq_2\alpha_4, \\ q_4\alpha_1 + tp_1 + tq_1\alpha_2 = q_4\alpha_2 + up_2 + uq_2\alpha_1, \\ up_2 + uq_2\alpha_1 - q_4\alpha_1 + q_4\alpha_3 = tp_1 + tq_1\alpha_3 + up_2 + uq_2\alpha_3, \\ 0 = tp_1 + tq_1\alpha_4 + up_2 + uq_2\alpha_4. \end{cases}$$

By the first and fourth equations we have

$$\text{(5)} \qquad \begin{cases} q_3 = (sp_2 - rp_1 + sq_2\alpha_1 - rq_1\alpha_2)(\alpha_1 - \alpha_2)^{-1}, \\ q_4 = (up_2 - tp_1 + uq_2\alpha_1 - tq_1\alpha_2)(\alpha_1 - \alpha_2)^{-1}. \end{cases}$$

So q_3 and q_4 may be eliminated:

$$\begin{cases} 0 = rp_1 + rq_1\alpha_3 + sp_2 + sq_2\alpha_3, \\ sp_2 + sq_2\alpha_1 + (sp_2 - rp_1 + sq_2\alpha_1 - rq_1\alpha_2)(\alpha_1 - \alpha_2)^{-1}(\alpha_4 - \alpha_1) \\ \qquad = rp_1 + rq_1\alpha_4 + sp_2 + sq_2\alpha_4, \\ up_2 + uq_2\alpha_1 + (up_2 - tp_1 + uq_2\alpha_1 - tq_1\alpha_2)(\alpha_1 - \alpha_2)^{-1}(\alpha_3 - \alpha_1) \\ \qquad = tp_1 + tq_1\alpha_3 + up_2 + uq_2\alpha_3, \\ 0 = tp_1 + tq_1\alpha_4 + up_2 + uq_2\alpha_4. \end{cases}$$

Multiplying the first and second (resp. third and fourth) equations on the left by s^{-1} (resp. u^{-1}) we can take $s^{-1}r$ to be "new s" (resp. $u^{-1}t$ to be "new t") and hence we can assume that $s = u = 1$. From the first and fourth equations we obtain $p_2 = -(rp_1 + rq_1\alpha_3 + q_2\alpha_3) = -(tp_1 + tq_1\alpha_4 + q_2\alpha_4)$. So p_2 can be eliminated:

$$\begin{cases} rp_1 + rq_1\alpha_3 + q_2\alpha_3 = tp_1 + tq_1\alpha_4 + q_2\alpha_4, \\ q_2\alpha_1 + (-rp_1 - rq_1\alpha_3 - q_2\alpha_3 - rp_1 + q_2\alpha_1 - rq_1\alpha_2)(\alpha_1 - \alpha_2)^{-1}(\alpha_4 - \alpha_1) \\ \qquad = rp_1 + rq_1\alpha_4 + q_2\alpha_4, \\ q_2\alpha_1 + (-tp_1 - tq_1\alpha_4 - q_2\alpha_4 - tp_1 + q_2\alpha_1 - tq_1\alpha_2)(\alpha_1 - \alpha_2)^{-1}(\alpha_3 - \alpha_1) \\ \qquad = tp_1 + tq_1\alpha_3 + q_2\alpha_3. \end{cases}$$

Define $A = (\alpha_1 - \alpha_2)^{-1}(\alpha_4 - \alpha_1)$ and $B = (\alpha_1 - \alpha_2)^{-1}(\alpha_3 - \alpha_1)$. Transforming the above equations we pass to

$$\begin{cases} rp_1 + rq_1\alpha_3 + q_2\alpha_3 = tp_1 + tq_1\alpha_4 + q_2\alpha_4, \\ q_2(\alpha_1 - \alpha_3 A + \alpha_1 A - \alpha_4) + rp_1(-2A - 1) + rq_1(-\alpha_3 A - \alpha_2 A - \alpha_4) = 0, \\ q_2(\alpha_1 - \alpha_4 B + \alpha_1 B - \alpha_3) + tp_1(-2B - 1) + tq_1(-\alpha_4 B - \alpha_2 B - \alpha_3) = 0. \end{cases}$$

From the first equation we get

$$q_2 = (tp_1 - rp_1 + tq_1\alpha_4 - rq_1\alpha_3)(\alpha_3 - \alpha_4)^{-1}. \tag{6}$$

Now q_2 can be eliminated:

$$\begin{cases} (tp_1 - rp_1 + tq_1\alpha_4 - rq_1\alpha_3)(\alpha_3 - \alpha_4)^{-1}(\alpha_1 - \alpha_4 + (\alpha_1 - \alpha_3)A) \\ \qquad + rp_1(-2A - 1) + rq_1(-(\alpha_2 + \alpha_3)A - \alpha_4) = 0, \\ (tp_1 - rp_1 + tq_1\alpha_4 - rq_1\alpha_3)(\alpha_3 - \alpha_4)^{-1}(\alpha_1 - \alpha_3 + (\alpha_1 - \alpha_4)B) \\ \qquad + tp_1(-2B - 1) + tq_1(-(\alpha_2 + \alpha_4)B - \alpha_3) = 0. \end{cases}$$

Define

$$\overline{A} = (\alpha_3 - \alpha_4)^{-1}(\alpha_1 - \alpha_4 + (\alpha_1 - \alpha_3)A),$$
$$\overline{B} = (\alpha_3 - \alpha_4)^{-1}(\alpha_1 - \alpha_3 + (\alpha_1 - \alpha_4)B).$$

After transformations we get

$$\begin{cases} t(p_1 + q_1\alpha_4)\overline{A} = r[p_1(\overline{A} + 2A + 1) + q_1(\alpha_3\overline{A} + (\alpha_2 + \alpha_3)A + \alpha_4)], \\ t[p_1(\overline{B} - 2B - 1) + q_1(\alpha_4\overline{B} - (\alpha_2 + \alpha_4)B - \alpha_3)] = r(p_1 + q_1\alpha_3)\overline{B}. \end{cases} \tag{7}$$

It is easy to see that $\overline{A} = 0$ implies $\alpha_2 = \alpha_3$, a contradiction. Hence $\overline{A} \neq 0$. Since $t, r \neq 0$, the elements

$$p_1 + q_1\alpha_4 \quad \text{and} \quad p_1(\overline{A} + 2A + 1) + q_1(\alpha_3\overline{A} + (\alpha_2 + \alpha_3)A + \alpha_4)$$

are either both zero or both nonzero. If both are zero, then by eliminating p_1 and q_1 ($(p_1, q_1) \neq (0, 0)$) we conclude that $\alpha_2 = \alpha_4$. This contradiction shows that the above two elements are nonzero. Similarly one can prove that both sides of the second equation of (7) are nonzero. Now r and t may be eliminated:

$$\begin{aligned} &[p_1(\overline{A} + 2A + 1) + q_1(\alpha_3\overline{A} + (\alpha_2 + \alpha_3)A + \alpha_4)]\overline{A}^{-1}(p_1 + q_1\alpha_4)^{-1} \\ &\qquad = (p_1 + q_1\alpha_3)\overline{B}[p_1(\overline{B} - 2B - 1) + q_1(\alpha_4\overline{B} - (\alpha_2 + \alpha_4)B - \alpha_3)]^{-1}. \end{aligned} \tag{8}$$

Define $X = p_1 + q_1\alpha_4$ and $Y = p_1 + q_1\alpha_3$. Then $p_1 = -X(\alpha_4 - \alpha_3)^{-1}\alpha_3 + Y(\alpha_4 - \alpha_3)^{-1}\alpha_4$ and $q_1 = (X - Y)(\alpha_4 - \alpha_3)^{-1}$. Put $\overline{X} = X(\alpha_4 - \alpha_3)^{-1}$ and $\overline{Y} = Y(\alpha_4 - \alpha_3)^{-1}$. Then $p_1 = -\overline{X}\alpha_3 + \overline{Y}\alpha_4$ and $q_1 = \overline{X} - \overline{Y}$. Substituting this to (8) we get

$$\begin{aligned} &[(-\overline{X}\alpha_3 + \overline{Y}\alpha_4)(\overline{A} + 2A + 1) \\ &\qquad + (\overline{X} - \overline{Y})(\alpha_3\overline{A} + (\alpha_2 + \alpha_3)A + \alpha_4)]\overline{A}^{-1}(\alpha_4 - \alpha_3)^{-1}\overline{X}^{-1} \\ &\quad = \overline{Y}(\alpha_4 - \alpha_3)\overline{B}[(-\overline{X}\alpha_3 + \overline{Y}\alpha_4)(\overline{B} - 2B - 1) \\ &\qquad + (\overline{X} - \overline{Y})(\alpha_4\overline{B} - (\alpha_2 + \alpha_4)B - \alpha_3)]^{-1}. \end{aligned}$$

By the definition of $\bar{A}$ and $\bar{B}$ we obtain

$$\begin{aligned}(9)\quad &\{\overline{X}[(\alpha_2-\alpha_3)A+(\alpha_4-\alpha_3)]\\ &\quad+\overline{Y}[(\alpha_4-\alpha_1)+(2\alpha_4-\alpha_1-\alpha_2)A]\}[(\alpha_4-\alpha_1)+(\alpha_3-\alpha_1)A]^{-1}\overline{X}^{-1}\\ &=\overline{Y}[(\alpha_3-\alpha_1)+(\alpha_4-\alpha_1)B]\{\overline{X}[(\alpha_3-\alpha_1)+(2\alpha_3-\alpha_1-\alpha_2)B]\\ &\quad+\overline{Y}[(\alpha_2-\alpha_4)B+(\alpha_3-\alpha_4)]\}^{-1}\end{aligned}$$

Define $\beta_2=\alpha_1-\alpha_2$, $\beta_3=\alpha_1-\alpha_3$ and $\beta_4=\alpha_1-\alpha_4$. Then $A=-\beta_2^{-1}\beta_4$ and $B=-\beta_2^{-1}\beta_3$. So $\alpha_1,\dots,\alpha_4,A,B$ can be eliminated:

$$\begin{aligned}(10)\quad &\{\overline{X}[(\beta_3-\beta_2)(-\beta_2^{-1}\beta_4)+(\beta_3-\beta_4)]\\ &\quad+\overline{Y}[-\beta_4+(\beta_2-2\beta_4)(-\beta_2^{-1}\beta_4)]\}[-\beta_4-\beta_3(-\beta_2^{-1}\beta_4)]^{-1}\overline{X}^{-1}\\ &=\overline{Y}[-\beta_3-\beta_4(-\beta_2\beta_3^{-1})]\{\overline{X}[-\beta_3+(\beta_2-2\beta_3)(-\beta_2^{-1}\beta_3)]\\ &\quad+\overline{Y}[(\beta_4-\beta_2)(-\beta_2^{-1}\beta_3)+(\beta_4-\beta_3)]\}^{-1}.\end{aligned}$$

Multiplying by $\overline{Y}^{-1}$ on the left and by $\overline{X}$ on the right and setting $T=\overline{X}^{-1}\overline{Y}$ we obtain $PQ^{-1}=RS^{-1}$, where

$$\begin{aligned}&P=T^{-1}(-\beta_3\beta_2^{-1}\beta_4+\beta_3)+(-2\beta_4+2\beta_4\beta_2^{-1}\beta_4),\\ &Q=-\beta_4+\beta_3\beta_2^{-1}\beta_4,\qquad R=-\beta_3+\beta_4\beta_2^{-1}\beta_3,\\ &S=(-2\beta_3+2\beta_3\beta_2^{-1}\beta_3)+T(-\beta_4\beta_2^{-1}\beta_3+\beta_4).\end{aligned}$$

Then clearly

$$(\beta_4^{-1}P\beta_4^{-1})(\beta_3^{-1}Q\beta_4^{-1})^{-1}=(\beta_4^{-1}R\beta_3^{-1})(\beta_3^{-1}S\beta_3^{-1})^{-1}.$$

But

$$\begin{aligned}&\beta_4^{-1}P\beta_4^{-1}=(\beta_4^{-1}T^{-1}\beta_3-2)(\beta_4^{-1}-\beta_2^{-1}),\qquad \beta_3^{-1}Q\beta_4^{-1}=-\beta_3^{-1}+\beta_2^{-1},\\ &\beta_4^{-1}R\beta_3^{-1}=-\beta_4^{-1}+\beta_2^{-1},\qquad \beta_3^{-1}S\beta_3^{-1}=(-2+\beta_3^{-1}T\beta_4)(\beta_3^{-1}-\beta_2^{-1}).\end{aligned}$$

Hence

$$\begin{aligned}&(\beta_4^{-1}T^{-1}\beta_3-2)(\beta_4^{-1}-\beta_2^{-1})(-\beta_3^{-1}+\beta_2^{-1})^{-1}\\ &\qquad=(-\beta_4^{-1}+\beta_2^{-1})(\beta_3^{-1}-\beta_2^{-1})^{-1}(\beta_3^{-1}T\beta_4-2)^{-1}.\end{aligned}$$

Define $Z=\beta_3^{-1}T\beta_4$ and $w_0=(\beta_4^{-1}-\beta_2^{-1})(\beta_2^{-1}-\beta_3^{-1})^{-1}$. Then we get the equation

$$(11)\qquad (z^{-1}-2)w_0=w_0(z-2)^{-1}.$$

This implies $(1-2z)w_0(z-2)=zw_0$. After substituting $z=t+1$ we get $2tw_0t-tw_0+w_0t=0$. Hence

$$(12)\qquad (2t)w_0(2t)-(2t)w_0+w_0(2t)=0.$$

Next we check that $t \neq 0$. If $t = 0$, then $z = 1$ and $T = \beta_3\beta_4^{-1}$. Hence $\overline{X}^{-1}\overline{Y} = \beta_3\beta_4^{-1}$. This implies

$$(\alpha_4 - \alpha_3)X^{-1}Y(\alpha_4 - \alpha_3)^{-1} = (\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4)^{-1}.$$

Therefore $X^{-1}Y = 1 + (\alpha_1 - \alpha_4)^{-1}(\alpha_4 - \alpha_3)$. Since $X = p_1 + q_1\alpha_4$ and $Y = p_1 + q_1\alpha_3$,

$$(p_1 + q_1\alpha_4)[1 + (\alpha_1 - \alpha_4)^{-1}(\alpha_4 - \alpha_3)] = p_1 + q_1\alpha_3.$$

This is equivalent to $p_1 + q_1\alpha_1 = 0$. In this case $0 \neq p_1(v_1 + \ldots + v_4) + q_1(\alpha_1 v_1 + \ldots + \alpha_4 v_4) = p_1(v_2 + v_3 + v_4) + q_1(\alpha_2 v_2 + \alpha_3 v_3 + \alpha_4 v_4) \in V \cap W$. By Lemma 7, $V = W$, a contradiction. Hence $2t \neq 0$.

Multiplying (12) by $(2tw_0)^{-1}$ on the left and by $(2t)^{-1}$ on the right we get $1 - (2t)^{-1} + w_0^{-1}(2t)^{-1}w_0 = 0$. Substituting $x = w_0$ and $y = w_0^{-1}(2t)^{-1}$ we have $1 = xy - yx$. Hence the division ring generated by x and y over K is isomorphic to $A_1A_1^{-1}$.

Now assume that D contains two elements x, y such that $xy - yx = 1$. Following the argument of the "if" part in reverse order one can construct the desired solution of (4). Namely, define $w_0 = x, t = \frac{1}{2}y^{-1}x^{-1}$ and $z = t + 1$. Then (11) is satisfied. Elements $\beta_2, \beta_3, \beta_4 \in D \setminus \{0\}$ such that $w_0 = (\beta_4^{-1} - \beta_2^{-1})(\beta_2^{-1} - \beta_3^{-1})^{-1}$ are easy to find. Define $T = \beta_3 Z \beta_4^{-1}$. Choose $\overline{X}, \overline{Y} \in D$ satisfying $T = \overline{X}^{-1}\overline{Y}$. Then (10) is true.

Define $A = -\beta_2^{-1}\beta_4$ and $B = -\beta_2^{-1}\beta_3$. Choose $\alpha_1, \ldots, \alpha_4 \in D$ such that $\beta_i = \alpha_1 - \alpha_i$ for $i = 2, 3, 4$. Then (9) holds.

Define also $\overline{A} = (\alpha_3 - \alpha_4)^{-1}(\alpha_1 - \alpha_4 + (\alpha_1 - \alpha_3)A)$, $\overline{B} = (\alpha_3 - \alpha_4)^{-1}(\alpha_1 - \alpha_3 + (\alpha_1 - \alpha_4)B)$, $X = \overline{X}(\alpha_4 - \alpha_3), Y = \overline{Y}(\alpha_4 - \alpha_3)$, $p_1 = -\overline{X}\alpha_3 + \overline{Y}\alpha_4$ and $q_1 = \overline{X} - \overline{Y}$. Then (8) is satisfied.

Now we can find $t, r \in D \setminus \{0\}$ satisfying (7). Next q_2 can be calculated from (6). Define $p_2 = -(rp_1 + rq_1\alpha_3 + q_2\alpha_3)$. Put $s = u = 1$. Then q_3 and q_4 are given by (5). Finally, define $p_3 = sp_2 + sq_2\alpha_1 - q_3\alpha_1$ and $p_4 = up_2 + uq_2\alpha_1 - q_4\alpha_1$. In this way a solution of (4) is obtained.

Now we have to check that our solution corresponds to $V \neq W$ (the remaining conditions are easy to verify). Suppose that $V = W$. Then

$$w_1 = p_1 \sum_{j \neq 1} v_j + q_1 \sum_{j \neq 1} \alpha_j v_j \in \pi_1(V) \cap V.$$

Hence we can find $g, h \in D$ such that

$$w_1 = g \sum_j v_j + h \sum_j \alpha_j v_j.$$

Comparing both expressions one can prove that $g = p_1$ and $h = q_1$. Hence $p_1 + q_1\alpha_1 = 0$. Repeating the arguments given at the end of the proof of the "if" part in reverse order, we can prove that $t = 0$. This contradicts the choice of t $(= \frac{1}{2}y^{-1}x^{-1})$. ∎

3. The identity as a sum of three nilpotents. Bokut' proved that each algebra can be embedded into a simple algebra which is a sum of three nilpotent algebras of degree 3 (see [2]). We show that the identity element of $M_3(D)$ can be represented as a sum of three nilpotent elements for certain D. The proof will be preceded by auxiliary lemmas. However, we start with a negative result concerning such representations.

PROPOSITION 9. *The equality* $1 = x+y+z$ *where* $x^2 = y^3 = z^5 = 0$ *does not hold for any* K*-algebra* A *with unit and* $x, y, z \in A$.

Proof. Consider the algebra $A = K\langle x, y, z : 1 = x + y + z, x^2 = y^3 = z^5 = 0\rangle$. Eliminating z we get $A = K\langle x, y : x^2 = y^3 = (1 - x - y)^5 = 0\rangle$. Since $x^2 = 0, y^3 = 0$, from $x(1 - x - y)^5 x = 0$ and $x(1 - x - y)^5 y = 0$ it follows that

$$xy^2xy^2x = -xyxyxyx + \ldots, \tag{13}$$

$$xy^2xyxy = -xyxy^2xy - xyxyxy^2 + \ldots, \tag{14}$$

where monomials of degrees ≤ 6 are not specified. Set $B = \{1, y, y^2\}$ and $E = \{1, x, xy^2, xy^2x, xy^2xy, xy^2xy^2, xy^2xyx\}$. Define also

$$V_m = \mathrm{Lin}_K\{\text{monomials of degree } \leq m\},$$

$$Z_m = \mathrm{Lin}_K\{b(xy)^n e' : b \in B,\ e' \in E,\ n \in \mathbb{N} \cup \{0\} \text{ and } b(xy)^n e' \text{ is of degree } \leq m\}.$$

Every nonzero monomial which cannot be written as in the definition of Z_m must be either (i) $b(xy)^n xy^2xy^2xa$ or (ii) $b(xy)^n xy^2xyxya$ for some $b \in B$, $n \geq 0$, and for a monomial a (maybe empty). In the first case, applying (13) we get

$$b(xy)^n xy^2xy^2xa = -b(xy)^n xyxyxyxa + \ldots = -b(xy)^{n+3}xa + \ldots$$

In the second case applying (14) we get

$$\begin{aligned} b(xy)^n xy^2xyxya &= -b(xy)^n xyxy^2xya - b(xy)^n xyxyxy^2a + \ldots \\ &= -b(xy)^{n+1}xy^2xya - b(xy)^{n+2}xy^2a + \ldots \end{aligned}$$

(monomials of smaller degrees are not specified). Repeating the above arguments for monomials of degree m we increase n. This allows us to prove that $V_m \subseteq Z_m + V_{m-1}$. Then

$$V_m \subseteq Z_m + V_{m-1} \subseteq Z_m + (Z_{m-1} + V_{m-1}) \subseteq \ldots \subseteq Z_m + \ldots + Z_1 = Z_m.$$

Hence $V_m \subseteq Z_m$. By the definition of Z_m we see that $\mathrm{GKdim}(A) \leq 1$. By [12] we know that A is PI. Hence $A/J(A)$ is a subdirect product of $M_{n_i}(D_i)$, $i \in I$, where D_i are finite-dimensional division algebras over their centers $Z(D_i)$ (see [5]). Now using the $Z(D_i)$-linear trace function on $M_{n_i}(D_i)$ we can prove that $A = \mathcal{J}(A)$, but $1 \in A$, a contradiction. ■

A similar method can be used to prove that if $e = e^2 = x + y$ and $x^3 = y^5 = 0$ then $e = 0$. This yields a simpler proof than that given in [4]. It is easy to see that $1 = x + y + z$ and $x^2 = y^2 = z^n = 0$, $n \in \mathbb{N}$, leads to a contradiction. The next cases to be considered are those where $x^2 = y^4 = z^4 = 0$ or $x^2 = y^3 = z^6 = 0$. We conjecture that examples of algebras of these types exist.

LEMMA 10. *Assume that*

$$g \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} g^{-1} = (a_{i,j})$$

for some $g \in M_3(D)$ *and* $a_{i,j} \in D$ *such that* $a_{i,i} = 0$. *Then* $a_{i,j} \neq 0$ *for all* $i \neq j$.

Proof. Suppose that $a_{i,j} = 0$ for some $i \neq j$. Conjugating by a permutation matrix we can assume that $a_{2,1} = 0$. Then

$$I = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_{3,1} & a_{3,2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & a_{1,2} & a_{1,3} \\ 0 & 0 & a_{2,3} \\ 0 & 0 & 0 \end{pmatrix} - g \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} g^{-1}.$$

This contradicts Proposition 9. ∎

LEMMA 11. *Under the assumptions of Lemma* 10, *we can find* (i,j,k) *such that* $\{1,2,3\} = \{i,j,k\}$, $a_{k,j}a_{j,k} \neq 1$ *and* $a_{i,j}a_{j,k} + a_{i,k} \neq 0$.

Proof. First assume that $a_{k,j}a_{j,k} \neq 1$ for some $k \neq j$. Then we can assume that $a_{i,j}a_{j,k} + a_{i,k} = 0$ for i such that $\{1,2,3\} = \{i,j,k\}$. In this case $a_{j,k}a_{k,j} \neq 1$ and $a_{i,k}a_{k,j} + a_{i,j} = (-a_{i,j}a_{j,k})a_{k,j} + a_{i,j} = a_{i,j}(1 - a_{j,k}a_{k,j}) \neq 0$ by Lemma 10. Hence the triple (i,k,j) satisfies the claim.

Now, suppose that $a_{i,j}a_{j,i} = 1$ for every $i \neq j$. Let

$$h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & a_{1,2} & 0 \\ 0 & 0 & a_{1,2}a_{2,3} \end{pmatrix}.$$

Then

$$(hg) \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} (hg)^{-1} = h(a_{i,j})h^{-1} = \begin{pmatrix} 0 & 1 & \lambda \\ 1 & 0 & 1 \\ \lambda^{-1} & 1 & 0 \end{pmatrix}$$

where $\lambda = a_{1,3}a_{2,3}^{-1}a_{1,2}^{-1} \neq 0$. Let

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = hg \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \neq 0.$$

Then

$$\begin{pmatrix} 0 & 1 & \lambda \\ 1 & 0 & 1 \\ \lambda^{-1} & 1 & 0 \end{pmatrix} \begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} p \\ q \\ r \end{pmatrix}.$$

This immediately implies that $\lambda = -1$. Then

$$2 = \operatorname{rank}\left[(hg)\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}(hg)^{-1} - I\right] = \operatorname{rank}\begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{pmatrix} = 1,$$

a contradiction. ∎

THEOREM 12. *Let D be a division ring of characteristic zero. Then $M_3(D)$ contains elements x, y, z such that $I = x+y+z$ and $x^3 = y^3 = z^3 = 0$ if and only if D contains a copy of $A_1 A_1^{-1}$.*

Proof. ($\Rightarrow$) By Proposition 9, x is nilpotent of index 3. So x is equal to $\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}$ in a certain basis. By Lemma 2, $y+z$ has zero diagonal in some basis. Since $I - x = y + z$, we can find an invertible $g \in M_3(D)$ such that

$$g\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} g^{-1} = \begin{pmatrix} 0 & p & q \\ s & 0 & r \\ t & u & 0 \end{pmatrix}$$

for some $p, q, r, s, t, u \in D$. By Lemma 10, $p, q, r, s, t, u \neq 0$. By Lemma 11, changing the order of v_1, v_2, v_3 if necessary, we can assume that $sp \neq 1$ and $tp + u \neq 0$. Denoting

$$g = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, \qquad a, b, \ldots \in D,$$

we have

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & p & q \\ s & 0 & r \\ t & u & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}. \tag{15}$$

Comparing the first columns on both sides of (15) we get

$$\begin{cases} a = pd + qg, \\ d = sa + rg, \\ g = ta + ud. \end{cases}$$

Eliminating $a = pd + qg$ we get

$$\begin{cases} (1 - sp)d = (sq + r)g, \\ (1 - tq)g = (tp + u)d. \end{cases}$$

If $g = 0$ then $d = 0$ and $a = 0$. This implies that g is not invertible, a contradiction. Hence $g \neq 0$ and $(1 - sp)^{-1}(sq + r) = (tp + u)^{-1}(1 - tq)$. Any solution of our system of equations with respect to a, d, g looks like

$$\begin{cases} g = g, \\ d = (1-sp)^{-1}(sq+r)g, \\ a = p(1-sp)^{-1}(sq+r)g + qg. \end{cases}$$

Define $X = 1 - tq$, $Y = 1 - sp$, $P = sq + r$ and $Q = tp + u$. Since we have $(tq)(sq)^{-1}(sp)(tp)^{-1} = 1$ it follows that

$$(1 - X)(P - r)^{-1}(1 - Y)(Q - u)^{-1} = 1. \tag{16}$$

The previous equations can be written in the form

$$Y^{-1}P = Q^{-1}X \quad \text{and} \quad \begin{cases} g = g, \\ d = Y^{-1}Pg, \\ a = (pY^{-1}P + q)g. \end{cases} \tag{17}$$

Next we consider further equations derived from (15) (second columns):

$$\begin{cases} a + b = pe + qh, \\ d + e = sb + rh, \\ g + h = tb + ue. \end{cases}$$

By eliminating $b = pe + qh - a$ we obtain

$$\begin{cases} d + sa = (sp-1)e + (sq+r)h, \\ g + ta = (tp+u)e + (tq-1)h. \end{cases}$$

After eliminating e, h vanishes and we get

$$(sp-1)^{-1}(d + sa) = (tp+u)^{-1}(g + ta).$$

By (17) we have

$$\begin{aligned} d + sa &= (Y^{-1}P + spY^{-1}P + sq)g \\ &= (Y^{-1}P + (1-Y)Y^{-1}P + (P-r))g = (2Y^{-1}P - r)g \end{aligned}$$

and

$$g + ta = (1 + tpY^{-1}P + tq)g = (1 + (Q-u)Y^{-1}P + 1 - X)g = (2 - uY^{-1}P)g.$$

Therefore

$$-Y^{-1}(2Y^{-1}P - r) = Q^{-1}(2 - uY^{-1}P). \tag{18}$$

Any solution of the last system of equations with respect to b, e, h has the form

$$\begin{cases} h = h, \\ e = -Y^{-1}((2Y^{-1}P - r)g - Ph), \\ b = -pY^{-1}((2Y^{-1}P - r)g - Ph) + qh - (pY^{-1}P + q)g \\ \quad = -pY^{-1}((2Y^{-1}P - r + P)g - Ph) + q(h - g). \end{cases} \tag{19}$$

Now, consider the remaining equations coming from (15) (third columns):

$$\begin{cases} b + c = pf + qi, \\ e + f = sc + ri, \\ h + i = tc + uf. \end{cases}$$

By eliminating $c = pf + qi - b$ we obtain

$$\begin{cases} e + sb = (sp-1)f + (sq+r)i, \\ h + tb = (tp+u)f + (tq-1)i. \end{cases}$$

After eliminating f, i vanishes and we get

$$(sp-1)^{-1}(e+sb) = (tp+u)^{-1}(h+tb).$$

By (19) we have

$$\begin{aligned} e + sb = {} & -Y^{-1}((2Y^{-1}P - r)g - Ph) \\ & + (-sp)Y^{-1}((2Y^{-1}P - r + P)g - Ph) + sq(h-g) \\ = {} & -Y^{-1}((2Y^{-1}P - r)g - Ph) \\ & + (Y-1)Y^{-1}((2Y^{-1}P - r + P)g - Ph) + (P-r)(h-g) \\ = {} & Y^{-1}(-2(2Y^{-1}P - r) + P)g + (2Y^{-1}P - r)h \end{aligned}$$

and

$$\begin{aligned} h + tb &= h + (-tp)Y^{-1}((2Y^{-1}P - r + P)g - Ph) + tq(h-g) \\ &= h + (u-Q)Y^{-1}((2Y^{-1}P - r + P)g - Ph) + (1-x)(h-g) \\ &= ((u-Q)Y^{-1}(2Y^{-1}P - r) + uY^{-1}P - QY^{-1}P - 1 + X)g \\ &\quad + (2 - (u-Q)Y^{-1}P - X)h \\ &= ((u-Q)Y^{-1}(2Y^{-1}P - r) + (-2 + uY^{-1}P) + 1)g \\ &\quad + (2 - (u-Q)Y^{-1}P - X)h \qquad \text{by (17)} \\ &= ((u-Q)Y^{-1}(2Y^{-1}P - r) + QY^{-1}(2Y^{-1}P - r) + 1)g \\ &\quad + (2 - (u-Q)Y^{-1}P - X)h \qquad \text{by (18)} \\ &= (uY^{-1}(2Y^{-1}P - r) + 1)g + (2 - (u-Q)Y^{-1}P - X)h. \end{aligned}$$

Hence

$$\begin{aligned} & -Y^{-1}[Y^{-1}(-2(2Y^{-1}P - r) + P)g + (2Y^{-1}P - r)h] \\ & \qquad = Q^{-1}[uY^{-1}(2Y^{-1}P - r + 1)g + (2 - (u-Q)Y^{-1}P - X)h]. \end{aligned}$$

By (17) and (18) we have $-Y^{-1}(2Y^{-1}P - r)h = Q^{-1}(2 - (u-Q)Y^{-1}P - X)h$. Hence $-Y^{-2}(-2(2Y^{-1}P - r) + P) = Q^{-1}(uY^{-1}(2Y^{-1}P - r) + 1)$. This implies

$$(2 - YQ^{-1}u)Y^{-1}(2Y^{-1}P - r) = Y^{-1}P + YQ^{-1}. \tag{20}$$

Set $\overline{P} = Y^{-1}P$ and $\overline{Q} = YQ^{-1}$. Then (17) becomes $X = \overline{Q}^{-1}Y\overline{P}$, and (16), (18) and (20) can be rewritten as

$$\begin{cases} (1 - \overline{Q}^{-1}Y\overline{P})(Y\overline{P} - r)^{-1}(1 - Y)(\overline{Q}^{-1}Y - u)^{-1} = 1, \\ -(2\overline{P} - r) = \overline{Q}(2 - u\overline{P}), \\ (2 - \overline{Q}u)Y^{-1}(2\overline{P} - r) = \overline{P} + \overline{Q}. \end{cases}$$

Next we will prove that $\overline{P} \neq 0$. If $\overline{P} = 0$ then

$$\begin{cases} (-r)^{-1}(1-Y)(\overline{Q}^{-1}Y-u)^{-1} = 1, \\ r = 2\overline{Q}, \\ (2-\overline{Q}u)Y^{-1}(-r) = \overline{Q}. \end{cases}$$

Eliminating r we get

$$\begin{cases} Y-1 = 2\overline{Q}(\overline{Q}^{-1}Y-u), \\ (2-\overline{Q}u)Y^{-1}(-2\overline{Q}) = \overline{Q}. \end{cases}$$

Hence $3 = 0$, a contradiction.

Our system of equations is equivalent to

$$\begin{cases} (\overline{Q}\,\overline{P}^{-1}-Y)(Y-r\overline{P}^{-1})(1-Y)(Y-\overline{Q}u)^{-1} = 1, \\ r\overline{P}^{-1}-2 = 2\overline{Q}\,\overline{P}^{-1}-\overline{Q}u, \\ (2-\overline{Q}u)Y^{-1}(2-r\overline{P}^{-1}) = 1+\overline{Q}\,\overline{P}^{-1}. \end{cases}$$

Define $A = \overline{Q}\,\overline{P}^{-1}$, $B = r\overline{P}^{-1}$ and $C = \overline{Q}u$. Then

$$\begin{cases} (A-Y)(Y-B)^{-1}(1-Y)(Y-C)^{-1} = 1, \\ -2+B = 2A-C, \\ (2-C)Y^{-1}(2-B) = 1+A. \end{cases}$$

Substituting $A = \frac{1}{2}B + \frac{1}{2}C - 1$ we obtain

$$\begin{cases} Y-\frac{1}{2}B-\frac{1}{2}C+1 = (Y-C)(Y-1)^{-1}(Y-B), \\ (2-C)Y^{-1}(2-B) = \frac{1}{2}B+\frac{1}{2}C. \end{cases}$$

If $B+C = 0$ then $B = 2$ or $C = 2$. Let for example $B = 2$. Then $C = -2$. Hence $(Y+1) = (Y-2)(Y-1)^{-1}(Y-2)$. This is equivalent to $0 = 3$, a contradiction. Hence $B+C \neq 0$. Set $S = (B+C)^{-1}$, $T = B(B+C)^{-1}$, $(1-T = C(B+C)^{-1})$ and $\overline{Y} = Y(B+C)^{-1}$. Then

$$\begin{cases} \overline{Y}+S-\frac{1}{2} = (\overline{Y}-(1-T))(\overline{Y}-S)^{-1}(\overline{Y}-T), \\ (2S-(1-T))\overline{Y}^{-1}(2S-T) = \frac{1}{2}. \end{cases}$$

Hence

$$\begin{cases} \big(\overline{Y}+S-\frac{1}{2}\big)(\overline{Y}-T)^{-1} = (\overline{Y}+T-1)(\overline{Y}-S)^{-1}, \\ \overline{Y} = \frac{1}{2}(2S-T)(2S+T-1). \end{cases}$$

The first equation will be transformed equivalently:

$$\big((\overline{Y}-T)+\big(T+S-\tfrac{1}{2}\big)\big)(\overline{Y}-T)^{-1} = ((\overline{Y}-S)+(T+S-1))(\overline{Y}-S)^{-1},$$

$$\big(T+S-\tfrac{1}{2}\big)(\overline{Y}-T)^{-1} = (T+S-1)(\overline{Y}-S)^{-1},$$

$$(\overline{Y}-T)\big(T+S-\tfrac{1}{2}\big)^{-1} = (\overline{Y}-S)(T+S-1)^{-1},$$

$$\overline{Y}\big(\big(T+S-\tfrac{1}{2}\big)^{-1}-(T+S-1)^{-1}\big) = T\big(T+S-\tfrac{1}{2}\big)^{-1}-S(T+S-1)^{-1},$$

$$\overline{Y}\big(\big(T+S-\tfrac{1}{2}\big)^{-1}(T+S-1)-1\big) = T\big(T+S-\tfrac{1}{2}\big)^{-1}(T+S-1)-S,$$

$$\overline{Y}\big[\big(T+S-\tfrac{1}{2}\big)^{-1}\big(\big(T+S-\tfrac{1}{2}\big)-\tfrac{1}{2}\big)-1\big] = T\big(T+S-\tfrac{1}{2}\big)^{-1}\big(\big(T+S-\tfrac{1}{2}\big)-\tfrac{1}{2}\big)-S,$$

$$\overline{Y}\left(-\tfrac{1}{2}\right)\left(T+S-\tfrac{1}{2}\right)^{-1}=T\left(1-\tfrac{1}{2}\left(T+S-\tfrac{1}{2}\right)^{-1}\right)-S,$$
$$\left(-\tfrac{1}{2}\overline{Y}+\tfrac{1}{2}T\right)\left(T+S-\tfrac{1}{2}\right)^{-1}=T-S.$$

Substituting $\overline{Y}=\frac{1}{2}(2S-T)(2S+T-1)$ we have

$$-\tfrac{1}{4}(2S-T)(2S+T-1)+\tfrac{1}{2}T=(T-S)\left(T+S-\tfrac{1}{2}\right),$$
$$-\tfrac{3}{4}T^2+\tfrac{1}{2}ST-\tfrac{1}{2}TS+\tfrac{3}{4}T=0,$$
$$-\tfrac{3}{4}T+\tfrac{1}{2}(ST^{-1})T-\tfrac{1}{2}T(ST^{-1})+\tfrac{3}{4}=0,$$
$$\left(\tfrac{2}{3}ST^{-1}\right)(T-1)-(T-1)\left(\tfrac{2}{3}ST^{-1}\right)=T-1.$$

If $T=1$ then $0=C=\overline{Q}u=YQ^{-1}u$, a contradiction. Hence $T-1\neq 0$ and we have

$$\left[\left(\tfrac{2}{3}ST^{-1}\right)(T-1)^{-1}\right](T-1)-(T-1)\left[\left(\tfrac{2}{3}ST^{-1}\right)(T-1)^{-1}\right]=1$$

Define $M=\left(\frac{2}{3}ST^{-1}\right)(T-1)^{-1}$ and $N=T-1$. Then $MN-NM=1$. Hence D contains a copy of $A_1A_1^{-1}$.

($\Leftarrow$) Let $M,N\in D$ be such that $MN-NM=1$. We will find a solution of the system of equations arising from (15). This may be done by following the argument of the proof of ($\Rightarrow$) in reverse order. If $M=\frac{2}{3}ST^{-1}(T-1)^{-1}$ and $N=T-1$ then $T=N+1$ and $S=\frac{3}{2}MN(N+1)$. Hence $\overline{Y}=\frac{1}{2}(2S-T)(2S+T-1)$. Now we get $B=TS^{-1}, C=(1-T)S^{-1}$ and $A=\frac{1}{2}B+\frac{1}{2}C-1$. Put $\overline{P}=1$. Then $\overline{Q}=A, r=B, u=A^{-1}C$. Moreover, $P=Y\overline{P}$, $Q=\overline{Q}^{-1}Y$ and $X=\overline{Q}^{-1}Y\overline{P}$. We have defined X,Y,P,Q,r,u such that (16), (18) and (20) are satisfied.

Next we use (16) to define p,q,s,t. Put $p=1$. Then $s=1-Y$, $q=s^{-1}(P-r)$ and $t=Q-u$. Now, the proof of ($\Rightarrow$) yields a solution of the system of equations arising from (15), with p,q,r,s,t,u given. For example let $g=1$, $d=Y^{-1}Pg$ and $a=(pY^{-1}P+q)g$. From (19) we can read h,e,b (h can be chosen arbitrary), and so forth.

Now we have to prove that the solution just constructed leads to an invertible matrix g. Let

$$W=\left\{\begin{pmatrix}p\\q\\r\end{pmatrix}: p,q,r\in D,\ g\begin{pmatrix}p\\q\\r\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}.$$

Then by (15), $\left(\begin{smallmatrix}1&1&0\\0&1&1\\0&0&1\end{smallmatrix}\right)W\subseteq W$. Hence if $W\neq 0$ then $\left(\begin{smallmatrix}1\\0\\0\end{smallmatrix}\right)\in W$ and the first column of g is zero, a contradiction. Therefore $W=0$ and g is invertible. Hence

$$I=g\begin{pmatrix}0&-1&0\\0&0&-1\\0&0&0\end{pmatrix}g^{-1}+\begin{pmatrix}0&0&0\\s&0&0\\t&u&0\end{pmatrix}+\begin{pmatrix}0&p&q\\0&0&r\\0&0&0\end{pmatrix}$$

is the desired decomposition. ∎

It is easy to see that the assertions of Theorems 8 and 12 remain true, with the same proofs, if D is any division ring of characteristic $\neq 2, 3$.

Now we can give an example of an idempotent e such that $e = x + y$ and $x^3 = y^6 = 0$. The assertion of Theorem 12 will be used in the construction.

Example 13. Let T, T^1 be the free semigroup, respectively monoid, generated by $1, 1', 2, 2', 3, 3'$. Put $X = \{1, 2, 3\}$, $X' = \{1, 1', 2, 2', 3, 3'\}$ and define an order $<$ on X' by $1 < 1' < 2 < 2' < 3 < 3'$. In a vector space over K consider the following system of equations in unknowns v_r for $r \in T$:

$$\begin{cases} \sum_{i\in X'} v_{r_1 i r_2} = v_{r_1 r_2} & \text{where } r_1, r_2 \in T, \\ v_{r_1 i i r_2} = 0 & \text{where } r_1, r_2 \in T^1, i \in X', \\ v_{r_1 i i' r_2} = 0 & \text{where } r_1, r_2 \in T^1, i \in X. \end{cases} \tag{21}$$

We will find a solution of (21) which satisfies the condition

$$\sum_{i\in X'} v_{1i} \neq 0. \tag{22}$$

Assume that D is a division ring such that $A_1 A_1^{-1} \subseteq D$. From Theorem 12 we know that there exist $b_1, b_2, b_3 \in M_3(D)$ such that $I = b_1 + b_2 + b_3$ and $b_1^3 = b_2^3 = b_3^3 = 0$. Each b_i can be written as

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in some basis of D^3. Hence we can find $a_i, a_{i'} \in M_3(D)$ such that $b_i = a_i + a_{i'}$ and $a_i^2 = a_i a_{i'} = a_{i'}^2 = 0$. Let $v \in D^3 \setminus \{0\}$ be such that $a_1 v \neq 0$. Define $v_{i_k \ldots i_1} = a_{i_k} \ldots a_{i_1} v$ for $i_1, \ldots, i_k \in X'$. If $r_1 = i_j i_{j-1} \ldots i_{k+1}$ and $r_2 = i_k i_{k-1} \ldots i_1$ for $i_1, \ldots, i_j \in X'$, then

$$\begin{aligned} \sum_{i\in X'} v_{r_1 i r_2} &= \sum_{i\in X'} a_{i_j} \ldots a_{i_{k+1}} a_i a_{i_k} \ldots a_{i_1} v = a_{i_j} \ldots a_{i_{k+1}} \Big(\sum_{i\in X'} a_i\Big) a_{i_k} \ldots a_{i_1} v \\ &= a_{i_j} \ldots a_{i_{k+1}} a_{i_k} \ldots a_{i_1} v = v_{r_1 r_2}, \\ v_{r_1 i i r_2} &= a_{i_j} \ldots a_{i_{k+1}} a_i a_i a_{i_k} \ldots a_{i_1} v = 0, \\ v_{r_1 i i' r_2} &= 0 \quad \text{similarly.} \end{aligned}$$

Hence (21) is satisfied. Since $\sum_{i\in X'} v_{1i} = \sum_{i\in X'} a_1 a_i v = a_1 v \neq 0$, (22) holds. Let $\overline{V}$ be a linear space (over K) spanned by the vectors v_r ($r \in T$), subject to the relations given in (21). Define $e, x, y \in \mathrm{End}_K(\overline{V})$ by

$$x(v_{ri}) = \sum_{j\in X' : j > i} v_{rij}, \quad y(v_{ri}) = \sum_{j\in X' : j < i} v_{rij} \quad \text{for } r \in T^1, \quad e = x + y.$$

Then x, y are indeed well defined since they preserve the relations defining

$\overline{V}$. For example: if $r_2 = r_2'k$ for $r_2' \in T^1$ and $k \in X'$, then

$$x\Big(\sum_{i\in X'} v_{r_1 i r_2' k}\Big) - x v_{r_1 r_2' k} = \Big(\sum_{i,j\in X':j>k} v_{r_1 i r_2' k j}\Big) - \Big(\sum_{j\in X':j>k} v_{r_1 r_2' k j}\Big)$$
$$= \sum_{j\in X':j>k} \Big[\Big(\sum_{i\in X'} v_{r_1 i (r_2' k j)}\Big) v_{r_1 (r_2' k j)}\Big] = 0.$$

Moreover, e is a nonzero idempotent because

$$e(ev_r) = e\Big(\sum_{i\in X'} v_{ri}\Big) = \sum_{i,j\in X'} v_{rij} = \sum_{j\in X'}\Big(\sum_{i\in X'} v_{rij}\Big) = \sum_{j\in X'} v_{rj} = ev_r$$

for $r \in T$ and $e(v_1) = \sum_{i\in X'} v_{1i} \neq 0$. Let $r' \in T^1$ and $i \in X'$. It is easy to see that

$$x^3(v_{r'i}) = \sum_{j_1,j_2,j_3\in X':i<j_1<j_2<j_3} v_{r' i j_1 j_2 j_3}.$$

Since certain neighbouring elements of the sequence i, j_1, j_2, j_3 are equal to k, k' for some $k \in X$, from (21) it follows that $v_{r' i j_1 j_2 j_3} = 0$. This proves that $x^3 = 0$. Similarly one can show that $y^6 = 0$.

Examples of the following two types were constructed in [4]: an identity element which is a sum of four nilpotent elements of degree 2, and a nonzero idempotent which is a sum of three nilpotent elements of degree 2. Another construction of this type can be obtained from [1, Prop. 2.2.1]. Here we give new examples by considering $M_2(D)$. As in the preceding constructions, this leads to the first Weyl algebra.

PROPOSITION 14. *There exists an idempotent $e \in M_2(D)\setminus\{0\}$ which is a sum of three nilpotent elements if and only if D contains a copy of $A_1A_1^{-1}$.*

Proof. ($\Rightarrow$) Let $e = x+y+z$ where $x, y, z \in M_2(D)$ are nilpotent. Then $x^2 = y^2 = z^2 = 0$. Since $e-z$ is a sum of two nilpotent elements, by Lemma 2 we can assume (changing the basis) that $e - z = \left(\begin{smallmatrix} 0 & q\\ p & 0\end{smallmatrix}\right)$ for some $p, q \in D$. Let $z = \left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)$, $a, b, c, d \in D$. Since $z^2 = 0$ and $z \neq 0$, either $b \neq 0$ or $c \neq 0$.

Let for example $b\neq 0$. From $z^2 = 0$ we get $a^2 + bc = 0$. Hence $c = -b^{-1}a^2$. Also $ab + bd = 0$, so that $d = -b^{-1}ab$. (It is easy to see that

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} a & b\\ -b^{-1}a^2 & -b^{-1}ab\end{pmatrix}$$

is nilpotent). Therefore

$$e = \begin{pmatrix} a & b+q\\ p - b^{-1}a^2 & -b^{-1}ab\end{pmatrix}.$$

Define $\overline{p} = p - b^{-1}a^2$ and $\overline{q} = b + q$. Then

$$e = \begin{pmatrix} a & \overline{q} \\ \overline{p} & -b^{-1}ab \end{pmatrix}.$$

If $\overline{p} = \overline{q} = 0$ then from $e^2 = e$ we get $a^2 = a$ and $b^{-1}a^2b = -b^{-1}ab$. Hence $a = 0$ and $e = 0$, a contradiction.

So suppose for example that $\overline{p} \neq 0$. Then $e^2 = e$ implies $\overline{p}a + (-b^{-1}ab)\overline{p} = \overline{p}$. Hence $(ab\overline{p})(\overline{p}^{-1}b^{-1}) - (\overline{p}^{-1}b^{-1})(ab\overline{p}) = 1$. Defining $x = ab\overline{p}$ and $y = \overline{p}^{-1}b^{-1}$ we obtain $xy - yx = 1$. Therefore D contains a copy of $A_1A_1^{-1}$.

($\Leftarrow$) Assume that $x, y \in D$ such that $xy - yx = 1$ are given. Let $a = xy$, $b = y^{-1}$, $\overline{p} = 1$ and $\overline{q} = xy - (xy)^2$. Then

$$e = \begin{pmatrix} a & \overline{q} \\ \overline{p} & -b^{-1}ab \end{pmatrix}$$

is a nonzero idempotent. Put $p = \overline{p} + b^{-1}a^2 = 1 + yxyxy$ and $q = \overline{q} - b = xy - (xy)^2 - y^{-1}$. Then

$$e = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} + \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a & b \\ -b^{-1}a^2 & -b^{-1}ab \end{pmatrix}$$

is a sum of three nilpotent elements. ∎

PROPOSITION 15. *$I \in M_2(D)$ can be represented as a sum of four nilpotent elements if and only if D contains a copy of $A_1A_1^{-1}$.*

Proof. ($\Rightarrow$) Let $I = x + y + z + t$ where $x, y, z, t \in M_2(D)$ are nilpotent. By Lemma 2 we can assume that $z + t = \left(\begin{smallmatrix} 0 & q \\ p & 0 \end{smallmatrix}\right)$ for some $p, q \in D$. Similarly there exists an invertible $A \in M_2(D)$ and $r, s \in D$ such that

$$x + y = A \begin{pmatrix} 0 & s \\ r & 0 \end{pmatrix} A^{-1}.$$

Hence

$$A = A \begin{pmatrix} 0 & s \\ r & 0 \end{pmatrix} + \begin{pmatrix} 0 & q \\ p & 0 \end{pmatrix} A.$$

Assume that $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, $a, b, c, d \in D$. Then our equality is equivalent to

$$\begin{cases} a = br + qc, \\ b = as + qd, \\ c = dr + pa, \\ d = cs + pb. \end{cases} \tag{23}$$

If one of p, q, r, s is zero then I can be represented as a sum of three nilpotent elements of degree 2. This contradicts [4]. Hence $p, q, r, s \neq 0$.

Next we show that $a, b, c, d \neq 0$. Let for example $a = 0$. Then we get

$$\begin{cases} 0 = br + qc, \\ b = qd, \\ c = dr, \\ d = cs + pb. \end{cases}$$

Substituting $b = qd$ and $c = dr$ to the first equation we get $0 = 2qdr$. Hence $d = 0$ and $b = qd = 0$, $c = dr = 0$, a contradiction (A is invertible). The proof that $b, c, d \neq 0$ is similar.

Now we eliminate q from the first and second equations of (23) and p from the third and fourth equations of (23):

$$\begin{cases} (a - br)c^{-1} = (b - as)d^{-1}, \\ (c - dr)a^{-1} = (d - cs)b^{-1}. \end{cases}$$

Defining $f = b^{-1}a$ and $g = d^{-1}c$ we get

$$\begin{cases} f - r = (1 - fs)g, \\ g - r = (1 - gs)f. \end{cases}$$

After eliminating r we obtain $-f + g - fsg = -g + f - gsf$. Define h by $g = f + h$ and substitute it to the previous equation. We get $2h = fsh - hsf$. If $h = 0$ then $b^{-1}a = d^{-1}c$ and A is not invertible. Hence $h \neq 0$. Therefore $1 = \left(\frac{1}{2}h^{-1}f\right)(sh) - (sh)\left(\frac{1}{2}h^{-1}f\right)$. Put $x = \frac{1}{2}h^{-1}f$ and $y = sh$. Then $1 = xy - yx$. Hence $D \supseteq A_1 A_1^{-1}$.

($\Leftarrow$) Assume that $x, y \in D$ such that $1 = xy - yx$ are given. Put $h = 1$. Then reading the proof of ($\Rightarrow$) in reverse direction we have $f = 2hx = 2x$, $s = yh^{-1} = y$, $g = f + h = 2x + 1$ and $r = f - (1 - fs)g = -1 + 2xy + 4xyx$. Put also $b = d = 1$. Then $a = bf = 2x$, $c = dg = 2x + 1$, $q = (a - br)c^{-1} = 1 - 2xy$ and $p = (c - dr)a^{-1} = 1 + x^{-1} - xyx^{-1} + 2xy$. The elements a, b, c, d, p, q, r, s just found satisfy (23) and A is invertible because $h \neq 0$. This gives the desired decomposition of I. ∎

4. An application. In [6, 7] Kegel proved that a ring R which is a sum of two nilpotent subrings R_1 and R_2 must be nilpotent. If R is an algebra over a field K, then we may assume that $R \subseteq \mathrm{End}_K(V)$, where V is a K-linear space. Define $W_i = \{v \in V : R_1^i v = 0\}$ and $Z_j = \{v \in V : R_2^j v = 0\}$ and $i, j = 1, 2, \ldots$ Then $W_1 \subseteq \ldots \subseteq W_n = V$ and $Z_1 \subseteq \ldots \subseteq Z_m = V$ where n, m are the nilpotency degrees of R_1, R_2 respectively. By Lemma 1 we can find subspaces $Y_{i,j} \subseteq V$ such that $W_k = \bigoplus_{i \leq k} Y_{i,j}$ and $Z_l = \bigoplus_{j \leq l} Y_{i,j}$. Since $R_1 W_i \subseteq W_{i-1}$, $R_2 Z_j \subseteq Z_{j-1}$ for $i, j = 1, 2, \ldots$ ($W_0 = Z_0 = 0$) and $R = R_1 + R_2$ we have

$$R(Y_{i,j}) \subseteq W_{i-1} + Z_{j-1} \subseteq \bigoplus_{(k,l) \neq (i,j)} Y_{k,l}.$$

So it is natural to consider the following problem. Let $V=\bigoplus_{i=1}^n V_i$ where V_i are subspaces of a K-linear space V and let $R\subseteq \mathrm{End}_K(V)$ be a subalgebra satisfying $R(V_i)\subseteq\bigoplus_{j\neq i}V_j$. Is R necessarily nilpotent?

The answer is negative in general by Theorem 8 (take $R=\mathrm{Lin}_K(e)$). Hence we shall discuss the case where V is finite-dimensional. Then R must be nilpotent since $\mathrm{tr}(R)=0$. The natural question that arises here is whether the nilpotency degree of R is bounded by a function depending on n only (as in Kegel's theorem). We answer this question in the more convenient setting of a semigroup R (clearly, if R satisfies the desired conditions, then the linear span of R also satisfies them).

PROPOSITION 16. *Let* $V=\bigoplus_{i=1}^n V_i$, $\dim_K(V)<\infty$ *and let* $S\subseteq\mathrm{End}_K(V)$ *be a semigroup satisfying* $S(V_i)\subseteq\bigoplus_{j\neq i}V_j$. *Then*

(a) *if* $n=2$ *then* $S^2=0$,
(b) *if* $n=3$ *then* $S^4=0$.

On the other hand, *if* $n=4$ *then* S *may have an arbitrarily large nilpotency degree.*

Proof. (a) For any $s\in S$ we have $s(V_1)\subseteq V_2$ and $s(V_2)\subseteq V_1$. Let $s_1,s_2\in S$. Then $s_1s_2(V_1)\subseteq s_1(V_2)\subseteq V_1$ and $s_1s_2(V_2)\subseteq s_1(V_1)\subseteq V_2$. Since $s_1s_2\in S$, we have $s_1s_2(V_1)\subseteq V_1\cap V_2=0$ and $s_1s_2(V_2)\subseteq V_2\cap V_1=0$. This implies $s_1s_2=0$. Hence $S^2=0$.

(b) First we show that it is sufficient to prove the claim for semigroups S generated by one element. Let $s_1,\ldots,s_4\in S$. Consider $\overline{s}\in\mathrm{End}_K(V^5)$ defined by $\overline{s}(v_1,v_2,v_3,v_4,v_5)=(s_1v_2,s_2v_3,s_3v_4,s_4v_5,0)$ for any $v_i\in V$. Then the condition $s_1\ldots s_4=0$ is equivalent to $\overline{s}^4=0$. Moreover, $V^5=\bigoplus_{i=1}^4 V_i^5$ and $\overline{s}^k(V_i^5)\subseteq\bigoplus_{j\neq i}V_j^5$ for $k=1,2,\ldots$ Hence it is enough to consider the semigroup generated by $\overline{s}$. So we can indeed assume that S is generated by some $s\in S$.

Let $v_1\in V_1$. Then $s(v_1)=v_2+v_3$ for some $v_2\in V_2$ and $v_3\in V_3$. Similarly $s(v_2)=v_1'+v_3'$, $s(v_3)=v_1''+v_2'$ for some $v_1',v_1''\in V_1$, $v_2'\in V_2$ and $v_3'\in V_3$. Since

$$s^2(v_1)=s(v_2+v_3)=(v_1'+v_1'')+v_2'+v_3'\in V_2\oplus V_3,$$

we get $v_1''=-v_1'$. Moreover, $s(v_1')=\overline{v}_2+\overline{v}_3$, $s(v_2')=\overline{v}_1+\overline{\overline{v}}_3$ and $s(v_3')=\overline{\overline{v}}_1+\overline{\overline{v}}_2$ for some $\overline{v}_1,\overline{\overline{v}}_1\in V_1$, $\overline{v}_2,\overline{\overline{v}}_2\in V_2$ and $\overline{v}_3,\overline{\overline{v}}_3\in V_3$. Now

$$s^2(v_2)=s(v_1'+v_3')=\overline{\overline{v}}_1+(\overline{v}_2+\overline{\overline{v}}_2)+\overline{v}_3\in V_1\oplus V_3$$

implies that $\overline{\overline{v}}_2=-\overline{v}_2$. Since

$$s^2(v_3)=s(-v_1'+v_2')=\overline{v}_1-\overline{v}_2+(-\overline{v}_3+\overline{\overline{v}}_3)\in V_1\oplus V_2,$$

we get $\overline{\overline{v}}_3=\overline{v}_3$. Similarly

$$s^3(v_1)=s(v_2'+v_3')=(\overline{v}_1+\overline{\overline{v}}_1)+\overline{\overline{v}}_2+\overline{\overline{v}}_3\in V_2\oplus V_3$$

implies that $\overline{\overline{v}}_1 = -\overline{v}_1$. We also have $s(\overline{v}_1) = \widetilde{v}_2 + \widetilde{v}_3$, $s(\overline{v}_2) = \widetilde{v}_1 + \widehat{v}_3$ and $s(\overline{v}_3) = \widehat{v}_1 + \widehat{v}_2$ for some $\widetilde{v}_i, \widehat{v}_i \in V_i$, $i = 1, 2, 3$. Since

$$s^2(v_1') = s(\overline{v}_2 + \overline{v}_3) = (\widetilde{v}_1 + \widehat{v}_1) + \widehat{v}_2 + \widehat{v}_3 \in V_2 \oplus V_3,$$

it follows that $\widehat{v}_1 = -\widetilde{v}_1$. From

$$s^2(v_2') = s(\overline{v}_1 + \overline{v}_3) = \widehat{v}_1 + (\widetilde{v}_2 + \widehat{v}_2) + \widetilde{v}_3 \in V_1 \oplus V_3$$

it follows that $\widehat{v}_2 = -\widetilde{v}_2$. Similarly

$$s^2(v_3') = s(-\overline{v}_1 - \overline{v}_2) = -\widetilde{v}_1 - \widetilde{v}_2 + (-\widetilde{v}_3 - \widehat{v}_3) \in V_1 \oplus V_2$$

leads to $\widehat{v}_3 = -\widetilde{v}_3$.

Next note that

$$s^3(v_2) = s(-\overline{v}_1 + \overline{v}_3) = \widehat{v}_1 + (-\widetilde{v}_2 + \widehat{v}_2) - \widetilde{v}_3 = -\widetilde{v}_1 - 2\widetilde{v}_2 - \widetilde{v}_3 \in V_1 \oplus V_3,$$

and consequently $\widetilde{v}_2 = 0$. Since

$$s^3(v_3) = s(\overline{v}_1 - \overline{v}_2) = -\widetilde{v}_1 + \widetilde{v}_2 + (\widetilde{v}_3 - \widehat{v}_3) = -\widetilde{v}_1 + \widetilde{v}_2 + 2\widetilde{v}_3 \in V_1 \oplus V_2,$$

we also get $\widetilde{v}_3 = 0$. Finally,

$$s^4(v_1) = s(-\overline{v}_2 + \overline{v}_3) = (-\widetilde{v}_1 + \widehat{v}_1) + \widehat{v}_2 - \widehat{v}_3 = -2\widetilde{v}_1 - \widetilde{v}_2 + \widetilde{v}_3 \in V_2 \oplus V_3,$$

so that $\widetilde{v}_1 = 0$. This implies $s^4(v_1) = 0$ because $\widetilde{v}_i = 0$ for $i = 1, 2, 3$. Similarly one can prove that $s^4|_{V_i} = 0$ for $i = 2, 3$. Hence $s^4 = 0$, as desired.

To prove the remaining assertion, fix some $n \in \mathbb{N}$. Let T be the free monoid generated by $1, 2, 3, 4$. We denote by $|w|$ the length of a word $w \in T$. Put $X = \{1, 2, 3, 4\}$. Consider the system of linear equations with unknown vectors v_r, where $r \in T$, and $|r| \le n$,

$$\sum_{r \in T : |r| = k} v_{r_1 i r i r_2} = 0 \quad \text{for } r_1, r_2 \in T,\ i \in X,\ k \in \mathbb{N} \cup \{0\} \text{ such that } |r_1 i r i r_2| \le n. \tag{24}$$

We show that there exists a solution of (24) satisfying

$$\sum_{r \in T : |r| = n} v_r \neq 0. \tag{25}$$

Let $e \in M_4(D)$ be an idempotent with zero diagonal arising from Theorem 8. Let $e_i \in M_4(D)$ denote the projection on the ith coordinate in D^4, $i = 1, 2, 3, 4$. Define $v_1 = (1, 0, 0, 0)$, $v_2 = v_3 = v_4 = 0 \in D^4$ and $v_{i_1 \dots i_k} = (e_{i_k} e) \dots (e_{i_2} e) v_{i_1}$ for $k \ge 2, i_1, \dots, i_k \in X$. We check that (24) is satisfied.

Let $r_1 = i_1 \dots i_p$ and $r_2 = j_1 \dots j_q$. Then

$$\sum_{r \in T : |r| = k} v_{r_1 i r i r_2}$$
$$= \sum_{z_1, \dots, z_k \in X} (e_{j_q} e) \dots (e_{j_1} e)(e_i e)[(e_{z_k} e) \dots (e_{z_1} e)](e_i e)(e_{i_p} e) \dots (e_{i_2} e) v_{i_1}$$

$$= (e_{j_q}e)\dots(e_{j_1}e)(e_ie)\Big(\sum_{i=1}^{4}e_ie\Big)^k(e_ie)(e_{i_p}e)\dots(e_{i_2}e)v_{i_1}$$

$$= (e_{j_q}e)\dots(e_{j_1}e)(e_iee_ie)(e_{i_p}e)\dots(e_{i_2}e)v_{i_1} = 0$$

because $e_iee_i = 0$ and $e_iev_i = 0$ if $|r_1| = 0$. Moreover,

$$\sum_{r\in T:|r|=n} v_r = \sum_{i_1,\dots,i_n\in X}(e_{i_n}e)\dots(e_{i_2}e)v_{i_1} = \Big(\sum_{i=1}^{4}(e_ie)\Big)^{n-1}v_1 = ev_1 \neq 0.$$

This yields (25).

Let $\overline{V}_n$ be the K-linear space spanned by the vectors $v_t, (t \in T)$ subject to relations (24) and $v_t = 0$ for $|t| \geq n+1$. Define $s \in \mathrm{End}_K(\overline{V}_n)$ by $s(v_r) = \sum_{i=1}^{n} v_{ri}$ for $|r| \leq n-1$ and $s(v_r) = 0$ for $|r| \geq n$. Then s is indeed well defined since it preserves the defining relations of $\overline{V}_n$. Namely,

$$s\Big(\sum_{r\in T:|r|=k} v_{r_1irir_2}\Big) = \sum_{r\in T:|r|=k}\sum_{j=1}^{4} v_{r_1irir_2j} = \sum_{j=1}^{4}\Big(\sum_{r\in T:|r|=k} v_{r_1irir_2j}\Big) = 0$$

and $s(v_t) = \sum_{i=1}^{4} v_{ti} = 0$ for $t \in T$ such that $|t| \geq n+1$. Let $\overline{V}_{n,i} = \mathrm{Lin}_K\{v_{ri} : r \in T\}, i \in X$. Since the defining relations of $\overline{V}_n$ are homogeneous with respect to the last letter of the index $r \in T$ of v_r, it follows that $\overline{V}_n = \bigoplus_{i=1}^{4}\overline{V}_{n,i}$.

We check that $s^m(\overline{V}_{n,i}) \subseteq \bigoplus_{j\neq i}\overline{V}_{n,j}$. Let $r_1 \in T$. Then

$$\begin{aligned} s^m(v_{r_1i}) &= \sum_{i_1,\dots,i_m\in X} v_{r_1ii_1\dots i_m} \\ &= \sum_{i_1,\dots,i_{m-1}} v_{r_1ii_1\dots i_{m-1}i} + \sum_{i_1,\dots,i_m:i_m\neq i} v_{r_1ii_1\dots i_{m-1}i_m} \\ &= \sum_{i_1,\dots,i_m:i_m\neq i} v_{r_1ii_1\dots i_{m-1}i_m} \in \bigoplus_{j\neq i}\overline{V}_{n,j}. \end{aligned}$$

Moreover, $s^n(v_{r_1}) = \sum_{i_1,\dots,i_n\in X} v_{r_1i_1\dots i_n} = 0$ and

$$s^{n-1}(v_1 + \dots + v_4) = \sum_{r\in T:|r|=n} v_{i_1\dots i_n} \neq 0$$

because there exists a solution of (24) satisfying (25). Hence s is nilpotent of degree n and the semigroup generated by s has the desired properties. ∎

Our final example shows that the bound on the nilpotency degree of the semigroup S in Proposition 16(b) cannot be improved.

EXAMPLE 17. Let $v_1, \dots, v_9$ be a basis of a K-linear space V. Define $s \in \mathrm{End}_K(V)$ by $s(v_1) = v_4 + v_7$, $s(v_2) = v_6 + v_9$, $s(v_3) = 0$, $s(v_4) = v_2 + v_8$, $s(v_5) = v_3 + v_9$, $s(v_6) = 0$, $s(v_7) = -v_2 + v_5$, $s(v_8) = -v_3 - v_6$

and $s(v_9) = 0$. Set also $V_1 = \mathrm{Lin}_K(v_1, v_2, v_3)$, $V_2 = \mathrm{Lin}_K(v_4, v_5, v_6)$ and $V_3 = \mathrm{Lin}_K(v_7, v_8, v_9)$. Then $V = V_1 \oplus V_2 \oplus V_3$. It is easy to check that $s^k(V_i) \subseteq \bigoplus_{j \neq i} V_j$ for $k = 1, 2, \ldots$, $i = 1, 2, 3$ and $s^3 \neq 0$.

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