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EQUIMORPHY IN VARIETIES OF DOUBLE HEYTING ALGEBRAS

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We show that any finitely generated variety $\mathbf{V}$ of double Heyting algebras is *finitely determined*, meaning that for some finite cardinal $n(\mathbf{V})$, any class $\mathcal{S} \subseteq \mathbf{V}$ consisting of algebras with pairwise isomorphic endomorphism monoids has fewer than $n(\mathbf{V})$ pairwise non-isomorphic members. This result complements the earlier established fact of categorical universality of the variety of all double Heyting algebras, and contrasts with categorical results concerning finitely generated varieties of distributive double p-algebras.

A *double Heyting algebra* $A = (X; \vee, \wedge, \to, \leftarrow, 0, 1)$ is an algebra of type $(2, 2, 2, 2, 0, 0)$ such that $L = (X; \vee, \wedge, 0, 1)$ is a distributive $(0, 1)$-lattice that admits a binary operation $\to$ determined by the requirement that $t \le (x \to y)$ exactly when $t \wedge x \le y$, and also the dually defined binary operation $\leftarrow$. All double Heyting algebras form a variety which we denote by $\mathbf{2H}$.

Regarded as a category, the variety $\mathbf{2H}$ is *universal* [3]. This means that any full category of algebras is isomorphic to a full subcategory of $\mathbf{2H}$ (see [12]) and implies that for every monoid M there exists a proper class $\mathcal{S} \subseteq \mathbf{2H}$ of pairwise non-isomorphic algebras such that the endomorphism monoid $\mathrm{End}(D)$ is isomorphic to M for every $D \in \mathcal{S}$. Results of [4] and [3] show that this is already the case for a certain subvariety of $\mathbf{2H}$ generated by finitely many subdirectly irreducible algebras, and hence it seems natural to ask about the existence of *finitely generated* subvarieties of $\mathbf{2H}$ with the same property.

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Our present result implies that no such subvariety exists.

To state this result precisely, we say that a variety $\mathbf{V}$ is *α-determined* if α is the least cardinal such that any class $\mathcal{S} \subseteq \mathbf{V}$ of pairwise non-isomorphic and *equimorphic* algebras—that is, algebras whose endomorphism monoids are isomorphic—is a set with fewer than α members.

Main Theorem. *Any finitely generated variety $\mathbf{V}$ of double Heyting algebras is n-determined for some finite cardinal $n = n(\mathbf{V})$. On the other hand, for every finite cardinal m there exists a finitely generated variety $\mathbf{V}$ of double Heyting algebras with $n(\mathbf{V}) \geq m$.*

When the operation $\to$ of a double Heyting algebra is restricted to a unary operation given by $x \mapsto x^* = x \to 0$ and its operation $\leftarrow$ to $x \mapsto x^+ = x \leftarrow 1$, a distributive double p-algebra is obtained. Hence a comparison of the present results to those about the variety $\mathbf{2P}$ of all distributive double p-algebras is of some interest.

With the usual notion of a homomorphism, infinitely many finitely generated subvarieties (called almost regular varieties in [6]) of $\mathbf{2P}$ are n-determined for some finite n (see [6]) while infinitely many other finitely generated subvarieties of $\mathbf{2P}$ are universal [5] and hence not α-determined for any cardinal α. On the other hand, there is a marked similarity between results on double Heyting algebras and those about distributive double p-algebras that are *regular* (that is, forming a variety $\mathbf{R} \subset \mathbf{2P}$ determined by the requirement that $y \wedge y^+ \leq x \vee x^*$). As in the case of double Heyting algebras, the variety $\mathbf{R}$ is universal (and this also shows that $\mathbf{2H}$ is; see [3]), while all finitely generated subvarieties of $\mathbf{R}$ are n-determined for some finite n, and the set of these integers is also unbounded. In this sense, the present paper extends [3] and [6].

Other related results include the fact that Boolean algebras are 2-determined [7, 8, 13], that distributive $(0,1)$-lattices are 3-determined [9], and so are certain finitely generated varieties of Heyting algebras [2], and that those varieties of distributive p-algebras which are α-determined for some cardinal α are already 2-determined or 3-determined [1].

The general approach and the actual method used here are based on our earlier paper [6] on equimorphy in finitely generated almost regular subvarieties of $\mathbf{2P}$. We work entirely within the framework of Priestley's duality appropriate for double Heyting algebras. We describe this framework first, and then apply it to construct families of idempotent structure-reflecting endomorphisms of our double Heyting algebras that are recognizable within any monoid and preserved by any isomorphism $\Psi : \mathrm{End}(D) \to \mathrm{End}(D')$. Then we arrange these idempotents into blocks that reflect certain global features of their underlying algebras. On any given collection $\mathcal{S}$ of equimorphic algebras, we then define three progressively finer equivalences that have only finitely many classes, and conclude by showing that any class of the finest equivalence consists entirely of pairwise isomorphic algebras.

1. Preliminaries. We begin with a brief review of Priestley's duality.

Let $(X, \leq, \tau)$ be an *ordered topological space*, that is, let (X, τ) be a topological space and $(X, \leq)$ a partially ordered set. For any $Z \subseteq X$ write

$$(Z] = \{x \in X \mid \exists z \in Z \ x \leq z\} \quad \text{and} \quad [Z) = \{x \in X \mid \exists z \in Z \ z \leq x\}.$$

We say that a subset Z of X is *decreasing* if $(Z] = Z$, *increasing* if $[Z) = Z$, and *convex* if $(Z] \cap [Z) = Z$. The set Z is *clopen* if it is both τ-open and τ-closed. Any compact ordered topological space $(X, \leq, \tau)$ possessing a clopen decreasing set D such that $x \in D$ and $y \notin D$ for any $x, y \in X$ with $x \not\geq y$ is called a *Priestley space.*

The following is a well-known and useful property of Priestley spaces.

Lemma 1.1. *If F_0 is a closed subset of a Priestley space $(X, \leq, \tau)$, then $[F_0)$ and $(F_0]$ are closed. If $F_1 \subseteq X$ is also closed and $F_0 \cap (F_1] = \emptyset$, then there is a clopen decreasing set $D \subseteq X$ such that $F_1 \subseteq D$ and $F_0 \cap D = \emptyset$.* ■

Let $\mathbf{P}$ denote the category of all Priestley spaces and all their continuous order preserving mappings. Clopen decreasing sets of any Priestley space form a distributive (0,1)-lattice, and the inverse image map f^{-1} of any $\mathbf{P}$-morphism f is a (0,1)-homomorphism of these lattices. This gives rise to a contravariant functor $D : \mathbf{P} \to \mathbf{D}$ into the category $\mathbf{D}$ of all distributive (0,1)-lattices and all their (0,1)-homomorphisms. Conversely, for any lattice $L \in \mathbf{D}$, let $P(L) = (P(L), \leq, \tau)$ be the ordered topological space on the set $P(L)$ of all prime filters of L ordered by the reversed inclusion, and such that the sets $\{x \in P(L) \mid A \in x\}$ and $\{x \in P(L) \mid A \notin x\}$ with $A \in L$ form an open subbasis of τ. If $h : L \to L'$ is a morphism in $\mathbf{D}$ then h^{-1} maps $P(L')$ into $P(L)$ and, according to [10], this determines a contravariant functor $P : \mathbf{D} \to \mathbf{P}$.

Theorem 1.2 [10, 11]. *The composite functors $P \circ D : \mathbf{P} \to \mathbf{P}$ and $D \circ P : \mathbf{D} \to \mathbf{D}$ are naturally equivalent to the identity functors of their respective domains. Therefore $\mathbf{D}$ is a category dually isomorphic to $\mathbf{P}$.* ■

Now we turn to Priestley spaces representing double Heyting algebras.

Definition. A Priestley space $(X, \leq, \tau)$ is called a *dh-space* if for every convex clopen set $Z \subseteq X$ the sets $[Z)$ and $(Z]$ are also clopen. We say that a mapping $f : (X, \leq) \to (Y, \leq)$ between posets has the *dh-property* if $[h(x)) = h([x))$ and $(h(x)] = h((x])$ for every $x \in X$. Any continuous order-preserving mapping between *dh*-spaces that has the *dh*-property will be called a *dh-map.*

In what follows, **DH** will denote the category of all *dh*-spaces and all *dh*-maps. Thus **DH** is the Priestley dual of the category **2H** of all double Heyting algebras and all their homomorphisms.

THEOREM 1.3 (folklore). *A Priestley space* $(X, \leq, \tau)$ *is the Priestley dual of a double Heyting algebra if and only if it is a dh-space. A continuous order-preserving map* $f : (X, \leq, \tau) \to (Y, \leq, \sigma)$ *of dh-spaces is the Priestley dual of a double Heyting algebra homomorphism if and only if it is a dh-map.* ∎

From [4], for any *dh*-space $(X, \leq, \tau)$ and *any* clopen set $A \subseteq X$ (convex or not), the sets $[A)$ and $(A]$ are always clopen.

DEFINITION. A subset A of a poset $(X, \leq)$ is called a *diset* if it is both decreasing and increasing, that is, if $A = [A) = (A]$. Any diset A which is minimal with respect to the inclusion is called a *component* of $(X, \leq)$. Thus any diset is the disjoint union of its components, and each component is a connected poset.

Theorem 1.3 implies that the image $\mathrm{Im}(f)$ of any *dh*-map $f : (X, \leq, \tau) \to (Y, \leq, \sigma)$ is a closed diset, and that $f(C)$ is a component of $(Y, \leq)$ for any component C of $(X, \leq)$.

NOTATION. Let $X = (X, \leq, \tau)$ be a *dh*-space. Then

- $\mathbb{C}(X)$ denotes the set of all components of X,
- $\mathbb{C}_P(X)$ denotes the set of all components of X isomorphic to a poset P,
- $\chi_X(P) = |\mathbb{C}_P(X)|$ for a given poset P,
- $K(x)$ denotes the component of X containing a given $x \in X$.

Given a variety **V** of double Heyting algebras, let **PV** denote its dual, that is, the full subcategory of **DH** determined by all those *dh*-spaces which are the Priestley duals of algebras from **V**.

PROPOSITION 1.4 (folklore). *The following properties are equivalent for any dh-space* X:

(1) X *is the dual of a finite subdirectly irreducible double Heyting algebra*,
(2) X *is the dual of a finite simple double Heyting algebra*,
(3) X *is finite and order connected.* ∎

The description below of the dual of a finitely generated variety of double Heyting algebras follows easily from Jónsson's Lemma, Proposition 1.4, and the results of [4].

THEOREM 1.5 [4]. *For any finitely generated variety* **V** *of double Heyting algebras there exists a finite set* $\mathcal{P}_{\mathbf{V}}$ *of finite connected posets that contains all dh-quotients of any of its members, and with the property that a dh-space* X *belongs to* **PV** *if and only if* $P \in \mathcal{P}_{\mathbf{V}}$ *for every poset* P *with* $\mathbb{C}_P(X) \neq \emptyset$. ∎

This characterization leads to the definition below.

DEFINITION. Let **FG** denote the full subcategory of **DH** formed by all dh-spaces X for which $\mathbb{C}_P(X) \neq \emptyset$ only for finitely many finite posets P.

Thus **FG** is the union of the duals **PV** of all finitely generated varieties $\mathbf{V} \subseteq \mathbf{2H}$.

2. Idempotents with finite images. To build a supply of idempotent endomorphisms of a given dh-space $X \in \mathbf{FG}$, we begin with the following claim about its partial dh-maps. In algebraic terms, it says that any subdirectly irreducible homomorphic image of any double Heyting algebra D from a finitely generated variety **V** must be a retract of some direct factor of D.

LEMMA 2.1. *For every $X \in \mathbf{FG}$ and for every $C \in \mathbb{C}(X)$ there exists a clopen diset $Y \subseteq X$ with $C \subseteq Y$ and a dh-map $f : Y \to C$ such that $f(c) = c$ for every $c \in C$.*

Proof. Since C is finite, for every $x \in C$ there exists a clopen set A_x such that $A_x \cap C = \{x\}$ and $A_x \cap A_y = \emptyset$ whenever $x, y \in X$ are distinct. Then $C \subseteq A$ and $A = \bigcup_{x \in C} A_x$ is clopen because C is finite. Thus $X \setminus A$ is clopen. Since $X \in \mathbf{FG}$, and because the sets $[W)$ and $(W]$ are clopen for any clopen $W \subseteq X$, the set $B = K(X \setminus A)$ is also clopen.

Set $B_x = A_x \setminus B$. Then B_x is clopen, $B_x \cap C = \{x\}$ for every $x \in C$ and $B_x \cap B_y = \emptyset$ whenever $x, y \in C$ are distinct. We claim that $K(\bigcup_{z \in C} B_z) = \bigcup_{z \in C} B_z$. Indeed, if $a \in K(\bigcup_{z \in C} B_z)$, then $a \notin B$ and so $K(a) \subseteq A = \bigcup_{z \in C} A_z$. Thus $a \in A_x$ for some $x \in C$ and hence $a \in B_x$. Thus $\bigcup_{z \in C} B_z$ is a clopen diset.

Next we set $D = \bigcup_{x \in C}(X \setminus K(B_x))$ and $D_x = B_x \setminus D$. Then D is clopen and $D \cap C = \emptyset$. It is clear that D_x is clopen, $D_x \cap C = \{x\}$ for every $x \in C$, and $D_x \cap D_y = \emptyset$ whenever $x, y \in C$ are distinct. Suppose that $a \in D_z$ for some $z \in C$. Then $a \notin D$ and, since $D \subseteq X$ is a diset, this means that $a \in K(B_x)$, and therefore $a \in K(D_x)$ for any $x \in C$. But then $K(D_x) = \bigcup_{z \in C} D_z$ for any $x \in C$.

For every $x \in C$ we now define $L_x = ([D_x) \cap (D_x]) \setminus D_x$. Then L_x is clopen and $L_x \cap C = \emptyset$ for every $x \in C$. Moreover, the diset $L = \bigcup_{x \in C} K(L_x)$ is clopen and $L \cap C = \emptyset$. For any $x \in C$, write $E_x = D_x \setminus L$. Then the set E_x is clopen, $E_x \cap C = \{x\}$ and $K(E_x) = \bigcup_{z \in C} E_z$ for every $x \in C$, and $E_x \cap E_y = \emptyset$ for $x, y \in C$ with $x \neq y$. We claim that E_x is convex for any $x \in C$. Indeed, if $a, b \in E_x$ and $a \leq d \leq b$ then $a, b \in D_x$ and thus $a, b, d \in [D_x) \cap (D_x]$. Further, $a, b \notin L$ and hence $d \notin L$. So $d \in D_x$, and $d \in E_x$ follows.

For any $x, y \in C$ with $x \leq y$ we now set $S(x, y) = (E_x \setminus (E_y]) \cup (E_y \setminus [E_x))$. Since X is a dh-space, the set $S(x, y)$ is clopen and hence the diset $S =$

$\bigcup\{K(S(x,y)) \mid x,y \in C \text{ and } x \le y\}$ is clopen as well. For any $x \le y$ we have $C \cap S(x,y) = \emptyset$, and hence also $S \cap C = \emptyset$. Write $F_x = E_x \setminus S$ for each $x \in C$. Then $F_x \cap C = \{x\}$, the set F_x is clopen and convex, and $K(F_x) = \bigcup_{z\in C} F_z$ for all $x \in C$. It is also clear that $F_x \cap F_y = \emptyset$ for all $x, y \in C$ with $x \ne y$.

Let $x, y \in C$ be such that $x \le y$. Let $a \in F_x$. Thus $a \in E_x$ and, since $a \notin S$, also $a \notin S(x,y)$. This means that $a \le b$ for some $b \in E_y$. Since S is a diset, it follows that $b \notin S$, and hence $b \in F_y$. Analogously, for every $b \in F_y$ there exists some $a \in F_x$ with $a \le b$. Thus $F_x \subseteq (F_y]$ and $F_y \subseteq [F_x)$.

For $x, y \in C$ with $x \not\le y$ define $T(x,y) = [F_x) \cap F_y$. Then $T(x,y)$ is clopen and $T(x,y) \cap C = \emptyset$. Set $T = \bigcup\{K(T(x,y)) \mid x,y \in C \text{ and } x \not\le y\}$. Clearly, the diset T is clopen and $T \cap C = \emptyset$. Write $G_x = F_x \setminus T$ for every $x \in C$. Again, $G_x \cap C = \{x\}$, the set G_x is clopen and convex, satisfies $K(G_x) = \bigcup_{z\in C} G_z$ for all $x \in C$, and $G_x \cap G_y = \emptyset$ for all $x, y \in C$ with $x \ne y$. As before, $G_x \subseteq (G_y]$ and $G_y \subseteq [G_x)$ for all $x, y \in C$ with $x \le y$. Suppose that $a \in G_x$ and $b \in G_y$ with $x, y \in C$ are such that $a \le b$. Then $b \notin T$. Since $b \in [F_x) \cap F_y$, the set $T(x,y)$ is not defined, for else $b \in T$. Thus $x \le y$. This shows that $[G_x) \cap G_y \ne \emptyset$ for some $x, y \in C$ only when $x \le y$.

Finally, write $Y = \bigcup_{x\in C} G_x$. Then Y is a clopen diset. For every $y \in Y$ there exists an $x \in C$ with $y \in G_x$. Since the sets G_x with $x \in C$ are pairwise disjoint, setting $f(y) = x$ produces a surjective mapping $f : Y \to C$. Each of the finitely many sets G_x is clopen, and hence f is continuous. If $a \le b$ in Y, then $a \in G_{f(a)}$ and $b \in G_{f(b)}$ and $b \in [G_{f(a)}) \cap G_{f(b)}$. Hence $f(a) \le f(b)$, and f preserves the order.

To show that f has the *dh*-property, let $a \in Y$ and $x = f(a)$. Then $f((a]) \subseteq (f(a)]$ and $f([a)) \subseteq [f(a))$ because f preserves order. Suppose that $y \in (x] = (f(a)]$. Then $G_x \subseteq [G_y)$, and hence there exists some $b \in G_y$ with $b \le a$. But then $f(b) = y$, and $f((a]) = (f(a)]$ follows. Analogously, if $y \in [x) = [f(a))$, then $G_x \subseteq (G_y]$ and therefore there exists some $b \in G_y$ with $a \le b$. Now $f(b) = y$ implies $f([a)) = [f(a))$.

Altogether, $f : Y \to C$ is a surjective *dh*-map, and $f(x) = x$ follows from $G_x \cap C = \{x\}$ for every $x \in C$. ∎

Definition. We say that a diset $Y \subseteq X$ in a *dh*-space $X \in \mathbf{FG}$ is a *plot* of X if for every poset P,

$$\mathbb{C}_P(Y) \ne \emptyset \quad \text{if and only if} \quad \mathbb{C}_P(X) \ne \emptyset.$$

We say that a plot Y of X is *minimal* when

$$\chi_P(Y) = 1 \quad \text{if and only if} \quad \mathbb{C}_P(X) \ne \emptyset.$$

It is clear that any $X \in \mathbf{FG}$ has a finite plot, and hence also a minimal plot, and that any two minimal plots of X are finite and isomorphic.

Theorem 2.2. *Let Y be a finite plot of $X \in \mathbf{FG}$. Then for any finite diset $Z \subseteq X$ disjoint from Y, and for any dh-map $\varphi : Z \to Y$, there exists an idempotent $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = Y$ and $f{\restriction}Z = \varphi$.*

Proof. Define $\mathcal{D} = \mathbb{C}(Y)$ and $\mathcal{E} = \mathbb{C}(Z)$. Then $\mathcal{C} = \mathcal{D} \cup \mathcal{E} \subseteq \mathbb{C}(X)$ is finite, and Lemma 2.1 implies the existence of a family $\{Z_C \mid C \in \mathcal{C}\}$ of disjoint clopen disets such that $C \subseteq Z_C$ for every $C \in \mathcal{C}$, and of idempotent dh-maps $g_C \in \mathrm{End}(Z_C)$ with $\mathrm{Im}(g_C) = C$ for every $C \in \mathcal{C}$.

For any $C \in \mathcal{E}$, let $\varphi_C = \varphi{\restriction}C$.

The diset $W = X \setminus (\bigcup\{Z_C \mid C \in \mathcal{C}\})$ is clopen in X, and hence compact. Thus $W \in \mathbf{FG}$ and, again by Lemma 2.1, for every $D \in \mathbb{C}(W)$ there exists an idempotent dh-map $g_D : Z_D \to Z_D$ with $\mathrm{Im}(g_D) = D$ defined on a clopen diset $Z_D \subseteq W$. Since the set $W = \bigcup\{Z_D \mid D \in \mathbb{C}(W)\}$ is closed and hence compact, we may assume that $W = \bigcup\{Z_C \mid C \in \mathcal{F}\}$ for some finite $\mathcal{F} \subseteq \mathbb{C}(W) \subseteq \mathbb{C}(X)$. Clearly $\mathcal{F} \cap \mathcal{C} = \emptyset$. Since the sets Z_C with $C \in \mathcal{F}$ are clopen, we may also assume that they are pairwise disjoint. Since Y is a plot of X, for each $C \in \mathcal{F}$ we may choose some dh-map $\varphi_C : C \to Y$. Then a mapping $f : X \to X$ defined by

$$f(y) = \begin{cases} g_C(y) & \text{for all } y \in Z_C \text{ with } C \in \mathcal{D}, \\ \varphi_C g_C(y) & \text{for all } y \in Z_C \text{ with } C \in \mathcal{E} \cup \mathcal{F}, \end{cases}$$

is the required idempotent dh-map. ∎

We conclude with a simple but useful observation.

Observation 2.3. *Let $X \in \mathbf{FG}$, and let $f \in \mathrm{End}(X)$ be idempotent. If $C \in \mathbb{C}(X)$, then $\mathrm{Im}(f) \cap C \neq \emptyset$ exactly when $C \subseteq \mathrm{Im}(f)$.*

Proof. If $x \in \mathrm{Im}(f) \cap C$ and $y \in C$, then there exists a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that x_i is comparable to x_{i+1} for every $i = 0, \ldots, n-1$. The dh-property of f implies that $x_i \in \mathrm{Im}(f)$ for every such i, and hence also $y \in \mathrm{Im}(f)$. ∎

3. Recognizable idempotents

Notation. For idempotent maps $f, g \in \mathrm{End}(X)$ we write $f \leq g$ if and only if $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$; it is clear that this is also equivalent to $gf = f$. We write $f \in [g]$ if $f \leq g \leq f$; this means that $\mathrm{Im}(f) = \mathrm{Im}(g)$, and we say that f and g are *equivalent*. If $f \notin [g]$ then we say that f and g are *non-equivalent*. If $f \leq g$ and $f \notin [g]$ then we write $f < g$; this means that $\mathrm{Im}(f)$ is properly contained in $\mathrm{Im}(g)$.

Definition. Let $X \in \mathbf{FG}$. An idempotent $f \in \mathrm{End}(X)$ is

- an *rh-map* of X if, for any poset P,

$$\chi_{\mathrm{Im}(f)}(P) = \begin{cases} 1 & \text{if } \mathbb{C}_P(X) \neq \emptyset, \\ 0 & \text{if } \mathbb{C}_P(X) = \emptyset, \end{cases} \tag{r}$$

• a $2rh$-*map* of X if there exists a poset P_f such that $\chi_{\mathrm{Im}(f)}(P_f) = 2$, and (r) holds for any poset $P \ncong P_f$.

We note that, for any $X \in \mathbf{FG}$, the image of any rh-map $f \in \mathrm{End}(X)$ is a minimal plot of X, and that the image of any $2rh$-map contains exactly two minimal plots of X.

The claim below shows that any minimal plot of X is, in fact, the image of an rh-map, and gives an abstract characterization of these maps.

STATEMENT 3.1. *Let* $X \in \mathbf{FG}$. *Then*:

(1) *if* $\mathcal{C} \subseteq \mathbb{C}(X)$ *is any set such that* $\mathcal{C} \cap \mathbb{C}_P(X)$ *is a single component for every poset* P *with* $\mathbb{C}_P(X) \neq \emptyset$, *then there exists an* rh-*map* $f \in \mathrm{End}(X)$ *with* $\mathrm{Im}(f) = \bigcup \mathcal{C}$,

(2) *an idempotent* $f \in \mathrm{End}(X)$ *is an* rh-*map if and only if it is a maximal one with the property that any idempotents* $g_0, g_1 \in \mathrm{End}(X)$ *with* $g_0, g_1 \leq f$ *and* $g_i g_{1-i} = g_i$ *for* $i = 0, 1$ *must coincide.*

P r o o f. By the hypothesis, the set $Y = \bigcup \mathcal{C}$ with $\mathcal{C} \subseteq \mathbb{C}(X)$ from (1) is a finite plot of X. Hence, by Theorem 2.2, there is an idempotent $f \in \mathrm{End}(X)$ with $\mathrm{Im}(f) = Y$. It is clear that f is an rh-map.

To prove (2), suppose first that $f \in \mathrm{End}(X)$ is an rh-map, and let $g_0, g_1 \leq f$ be idempotents satisfying $g_0 g_1 = g_0$ and $g_1 g_0 = g_1$. Choose an $x \in X$ arbitrarily, and define $C_i = K(g_i(x))$ for $i = 0, 1$. Then $C_i \subseteq \mathrm{Im}(g_i) \subseteq \mathrm{Im}(f)$ for $i = 0, 1$ because of Observation 2.3, and the hypothesis $g_0, g_1 \leq f$. From $g_i g_{1-i} = g_i$ it follows that $g_i(C_{1-i}) = C_i$ for $i = 0, 1$. Since no two distinct components of $\mathrm{Im}(f)$ are isomorphic, it follows that $C_0 = C_1$ and hence also $g_0(x) = g_1(x)$. Thus $g_0 = g_1$ as was to be shown. It is clear that the rh-map $f \in \mathrm{End}(X)$ is a maximal idempotent with this property.

For the converse implication in (2), assume that an idempotent $f \in \mathrm{End}(X)$ is not an rh-map. There are two possible reasons for this. Either $\mathrm{Im}(f)$ contains distinct and isomorphic components $C_0, C_1 \in \mathbb{C}(X)$, or else there exists a component $C_2 \in \mathbb{C}(X)$ not isomorphic to any component of $\mathrm{Im}(f)$.

In the first case, select a finite plot S of $\mathrm{Im}(f)$ containing C_0 and C_1. Then, by Theorem 2.2, there is an idempotent $g \in \mathrm{End}(\mathrm{Im}(f))$ with $\mathrm{Im}(g) = S$. Let φ_1 be an isomorphism of C_0 onto C_1 and let φ_0 be the inverse of φ_1. Then, for $i = 0, 1$, the maps

$$g_i(x) = \begin{cases} \varphi_i g f(x) & \text{for } x \in (gf)^{-1}(C_{1-i}), \\ gf(x) & \text{for } x \in X \setminus (gf)^{-1}(C_{1-i}), \end{cases}$$

are distinct idempotent endomorphisms of X satisfying $g_i \leq f$ and $g_i g_{1-i} = g_i$. We may thus assume that no two components contained in $\mathrm{Im}(f)$ are isomorphic, and that there exists a component $C_2 \in \mathbb{C}(X)$ not isomorphic to any component of $\mathrm{Im}(f)$. It follows that $\mathrm{Im}(f)$ can be extended to a minimal finite plot T of X containing C_2. By (1), there is an rh-map h with $\mathrm{Im}(h) = T$, and this violates the maximality of f. ∎

STATEMENT 3.2. *Let $X \in \mathbf{FG}$. Then*

(1) *for every rh-map $f \in \mathrm{End}(X)$ and every component $C \in \mathbb{C}(X)$ with $C \cap \mathrm{Im}(f) = \emptyset$ there exists a $2rh$-map $g \in \mathrm{End}(X)$ with $\mathrm{Im}(g) = \mathrm{Im}(f) \cup C$,*

(2) *an idempotent $g \in \mathrm{End}(X)$ is a $2rh$-map if and only if there exist exactly two non-equivalent rh-maps $f_0, f_1 < g$.*

Proof. Since $\mathrm{Im}(f) \cup C$ is a finite plot of X, the first claim follows immediately from Theorem 2.2.

Let $g \in \mathrm{End}(X)$ be a $2rh$-map. From the definitions of $2rh$-maps and rh-maps, and from Statement 3.1(1), it follows that there are exactly two non-equivalent rh-maps $f_0, f_1 \le g$.

To prove the converse implication in (2), let f_0 and f_1 be two non-equivalent rh-maps such that $f_0, f_1 < g$. Then $\mathrm{Im}(f_0) \neq \mathrm{Im}(f_1)$ are minimal plots of X, and hence there exist $C_i \in \mathbb{C}(X)$ such that $C_i \subseteq \mathrm{Im}(f_i) \setminus \mathrm{Im}(f_{1-i})$ for $i = 0, 1$, and also components $C'_i \subseteq \mathrm{Im}(f_{1-i})$ isomorphic to C_i for $i = 0, 1$. Since the plot $\mathrm{Im}(f_i)$ is minimal, we have $C'_i \subseteq \mathrm{Im}(f_{1-i}) \setminus \mathrm{Im}(f_i)$ for $i = 0, 1$. Then $S = (\mathrm{Im}(f_0) \setminus C'_1) \cup C_1$ is a minimal plot of X, and hence there is an rh-map f_2 with $\mathrm{Im}(f_2) = S$, by Statement 3.1(1). Clearly, $f_2 < g$ and $f_2 \notin [f_0]$. But then $f_2 \in [f_1]$ by the hypothesis, and hence $\mathrm{Im}(f_1) = (\mathrm{Im}(f_0) \setminus C'_1) \cup C_1$. Therefore $C_0 = C'_1$.

Altogether, $C_0 \cong C_1$, $\mathrm{Im}(f_1) = (\mathrm{Im}(f_0) \setminus C_0) \cup C_1$ and $\mathrm{Im}(f_0) = (\mathrm{Im}(f_1) \setminus C_1) \cup C_0$. If $\mathrm{Im}(g)$ properly contains $\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$, then $C \subseteq \mathrm{Im}(g) \setminus (\mathrm{Im}(f_0) \cup \mathrm{Im}(f_1))$ for some component C and, regardless of whether $C \cong C_0$ or not, there exists an rh-map $f_3 \notin [f_0] \cup [f_1]$ with $\mathrm{Im}(f_3) \subseteq \mathrm{Im}(g)$. Therefore $\mathrm{Im}(g) = \mathrm{Im}(f_0) \cup \mathrm{Im}(f_1)$, and hence g is a $2rh$-map. ∎

STATEMENT 3.3. *For $X, Y \in \mathbf{FG}$, let $\Psi : \mathrm{End}(X) \to \mathrm{End}(Y)$ be an isomorphism, and let $f \in \mathrm{End}(X)$. Then*

(1) *f is an rh-map if and only if $\Psi(f)$ is an rh-map;*

(2) *f is a $2rh$-map if and only if $\Psi(f)$ is a $2rh$-map.*

Proof. These are immediate consequences of Statements 3.1(2) and 3.2(2). ∎

The following is a crucial separating property of the collection of all rh- and $2rh$-maps.

THEOREM 3.4. *For any $X \in \mathbf{FG}$ and any two distinct points $x, y \in X$ there exists an rh-map or a $2rh$-map $f \in \mathrm{End}(X)$ with $f(x) \neq f(y)$.*

Proof. If $K(x) = K(y)$ or $K(x) \ncong K(y)$, then there exists an rh-map $f \in \mathrm{End}(X)$ with $x, y \in \mathrm{Im}(f)$; see Statement 3.1(1). If the components

$K(x)$ and $K(y)$ are distinct and isomorphic, then there is an rh-map $g \in \mathrm{End}(X)$ such that $K(x) \subseteq \mathrm{Im}(g)$ and $K(y) \cap \mathrm{Im}(g) = \emptyset$, by Statement 3.1(1) again. But then, by Statement 3.2(1), there is a $2rh$-map $f \in \mathrm{End}(X)$ such that $K(x) \cup K(y) \subseteq \mathrm{Im}(f)$. ∎

4. Blocks

Definition. Any non-trivial maximal collection $\mathbb{G}$ of equivalence classes of rh-maps such that for any two distinct $[g_0], [g_1] \in \mathbb{G}$ there exists a $2rh$-map $f \in \mathrm{End}(X)$ with $g_0, g_1 < f$ is called a *block*.

Statement 4.1. *Let $\mathbb{G}$ be a block in $X \in \mathbf{FG}$. Then*

(1) *if $[g_0], [g_1], [g_2] \in \mathbb{G}$ are distinct, then $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \setminus \mathrm{Im}(g_2)$ is a component of X and $\mathrm{Im}(g_0) \cap \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \cap \mathrm{Im}(g_2) = \mathrm{Im}(g_1) \cap \mathrm{Im}(g_2)$,*

(2) *if $[g_0], [g_1], [g_2], [g_3] \in \mathbb{G}$ then $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) \cong \mathrm{Im}(g_2) \setminus \mathrm{Im}(g_3)$ whenever $[g_0] \neq [g_1]$ and $[g_2] \neq [g_3]$,*

(3) *there exists a finite poset P with $\chi_X(P) > 1$ such that the map $\beta : \mathbb{G} \to \mathbb{C}_P(X)$ defined by $\beta([g]) = \mathrm{Im}(g) \setminus \mathrm{Im}(g')$ for $[g'] \in \mathbb{G} \setminus \{[g]\}$ is a bijection,*

(4) *if $[g] \in \mathbb{G}$ and if $\mathbb{G}_1$ is a block in X such that $[g] \in \mathbb{G}_1 \neq \mathbb{G}$, then for every $[g_0] \in \mathbb{G} \setminus \{[g]\}$ and $[g_1] \in \mathbb{G}_1 \setminus \{[g]\}$ we have $(\mathrm{Im}(g) \setminus \mathrm{Im}(g_0)) \cap (\mathrm{Im}(g) \setminus \mathrm{Im}(g_1)) = \emptyset$.*

For any rh-map $g \in \mathrm{End}(X)$ and for any finite poset P with $\chi_X(P) > 1$ set $S = \mathrm{Im}(g) \setminus (\bigcup\{C \mid C \in \mathbb{C}_P(X)\})$. For every $C \in \mathbb{C}_P(X)$, let $g_C \in \mathrm{End}(X)$ be an rh-map with $\mathrm{Im}(g_C) = S \cup C$. Then $\mathbb{G} = \{[g_C] \mid C \in \mathbb{C}_P(X)\}$ is a block.

Proof. To prove (1), let $[g_0], [g_1], [g_2] \in \mathbb{G}$ be distinct. If the components $C_0 = \mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1)$, $D_0 = \mathrm{Im}(g_0) \setminus \mathrm{Im}(g_2)$ are distinct, then they are not isomorphic because g_0 is an rh-map. Since there exists a $2rh$-map h_1 with $\mathrm{Im}(h_1) = \mathrm{Im}(g_0) \cup \mathrm{Im}(g_1)$, from Statement 3.2 it follows that $C_1 = \mathrm{Im}(g_1) \setminus \mathrm{Im}(g_0) \cong C_0 \subseteq \mathrm{Im}(g_2)$ and $D_2 = \mathrm{Im}(g_2) \setminus \mathrm{Im}(g_0) \cong D_0 \subseteq \mathrm{Im}(g_1)$. Thus, by Observation 2.3, the components C_0, C_1, D_0 and D_2 are contained in the image of any idempotent $f > g_1, g_2$. Hence there is no $2rh$-map $f > g_1, g_2$—a contradiction, and $C_0 = D_0$ follows. But then $C_0 \cong D_2$ as well, and hence $\mathrm{Im}(g_0) \cap \mathrm{Im}(g_1) = \mathrm{Im}(g_0) \cap \mathrm{Im}(g_2) = \mathrm{Im}(g_1) \cap \mathrm{Im}(g_2)$. This proves (1), and implies that (2) holds as well.

Let P denote a poset isomorphic to the component C_0. From (2) it follows that $\chi_X(P) > 1$, and that the map $\beta : \mathbb{G} \to \mathbb{C}_P(X)$ from (3) is well-defined and injective. The map β is surjective because of the maximal property of $\mathbb{G}$. This proves (3).

Now we turn to (4). If $\mathbb{G}, \mathbb{G}_1 \ni [g]$ are blocks and if $I = (\mathrm{Im}(g) \setminus \mathrm{Im}(g_0)) \cap (\mathrm{Im}(g) \setminus \mathrm{Im}(g_1)) \neq \emptyset$ for some $[g_0] \in \mathbb{G} \setminus \{[g]\}$ and $[g_1] \in \mathbb{G}_1 \setminus \{[g]\}$, then $I \in \mathbb{C}(X)$ is isomorphic to $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g)$ and $\mathrm{Im}(g_1) \setminus \mathrm{Im}(g)$. Therefore the latter two components are isomorphic (or equal), and hence $\mathbb{G} = \mathbb{G}_1$ by the maximality of $\mathbb{G}$.

To prove the final claim, it suffices to note that $S \cup C$ is a minimal plot of X for every $C \in \mathbb{C}_P(X)$, apply Statement 3.1(1) to obtain an rh-map g_C with $\mathrm{Im}(g_C) = S \cup C$, and then use Statement 3.2(1) to obtain the requisite $2rh$-maps. The maximality of $\mathbb{G} = \{g_C \mid C \in \mathbb{C}_P(X)\}$ is obvious. ∎

Definition. Let P be a finite connected poset. We say that a block $\mathbb{G}$ is a *P-block* if $\mathrm{Im}(g_0) \setminus \mathrm{Im}(g_1) \in \mathbb{C}_P(X)$ for any two distinct $[g_0], [g_1] \in \mathbb{G}$.

Corollary 4.2. *For any P-block $\mathbb{G}$, the map $\beta : \mathbb{G} \to \mathbb{C}_P(X)$ from Statement 4.1(3) is a bijection. For every rh-map $g \in \mathrm{End}(X)$ and for every finite poset P with $\chi_X(P) > 1$, there exists exactly one P-block $\mathbb{G}$ with $[g] \in \mathbb{G}$.* ∎

An immediate consequence of Statement 3.3 and of the definition of a block is

Statement 4.3. *If $X, Y \in \mathbf{FG}$ and $\Psi : \mathrm{End}(X) \to \mathrm{End}(Y)$ is an isomorphism, then $\mathbb{G}$ is a block in X exactly when $\Psi(\mathbb{G}) = \{[\Psi(g)] \mid [g] \in \mathbb{G}\}$ is a block in Y.* ∎

Lemma 4.4. *Let $\mathbb{G}$ be a P-block in X. For every $C \in \mathbb{C}_P(X)$, let $g_C \in \mathrm{End}(X)$ denote an rh-map with $\mathrm{Im}(g_C) \cap C \neq \emptyset$ and $[g_C] \in \mathbb{G}$. Let $f \in \mathrm{End}(X)$ be any rh-map, let $[g] \in \mathbb{G}$, and let $C \in \mathbb{C}_P(X)$ be such that $\mathrm{Im}(g) \cap C = \emptyset$. Then $C \subseteq \mathrm{Im}(f)$ if and only if $kf \neq g'f$ for every $2rh$-map $k > g, g_C$ and every $g' \in [g]$.*

Proof. If $C \subseteq \mathrm{Im}(f)$, then $kf \neq g'f$ because $C \subseteq \mathrm{Im}(g_C) \subseteq \mathrm{Im}(k)$ and $\mathrm{Im}(g') \cap C = \mathrm{Im}(g) \cap C = \emptyset$. To prove the converse, suppose that $C \setminus \mathrm{Im}(f) \neq \emptyset$. Then $\mathrm{Im}(f) \cap C = \emptyset$, by Observation 2.3. By Theorem 2.2, there exists a $2rh$-map $k \in \mathrm{End}(X)$ with $\mathrm{Im}(k) = \mathrm{Im}(g) \cup C$ and $k(\mathrm{Im}(f)) \subseteq \mathrm{Im}(g)$. But then $g' = gk \in [g]$ and $kf = g'f$, as was to be shown. ∎

5. Three equivalences. Let $\mathcal{S}$ be a class of equimorphic objects in $\mathbf{FG}$. For any $X, Y \in \mathcal{S}$ choose an isomorphism $\Psi_{XY} : \mathrm{End}(X) \to \mathrm{End}(Y)$ so that

(C1) for any $X, Y, Z \in \mathcal{S}$ we have $\Psi_{XZ} = \Psi_{YZ}\Psi_{XY}$,

(C2) for any $X, Y \in \mathcal{S}$, the composite $\Psi_{YX}\Psi_{XY}$ is the identity on $\mathrm{End}(X)$.

Definition. For any finitely generated variety $\mathbf{V}$ of double Heyting algebras, let $R(\mathbf{V})$ denote the class of all dh-spaces $X \in \mathbf{PV}$ such that

$\chi_X(P) \leq 1$ for every poset P. In other words, members of $R(\mathbf{V})$ are exactly all dh-spaces from $\mathbf{PV}$ that are their own minimal plots. We define

$n_1(\mathbf{V})$ to be the number of non-isomorphic dh-spaces in $R(\mathbf{V})$,
$n_2(\mathbf{V}) = \max\{|\mathrm{Aut}(\mathrm{End}(X))| \mid X \in R(\mathbf{V})\}$,
$n_3(\mathbf{V}) = \max\{|\mathbb{C}(X)| \mid X \in R(\mathbf{V})\}$.

The claim below is an immediate consequence of Theorem 1.5.

LEMMA 5.1. *For any finitely generated variety* $\mathbf{V}$ *of double Heyting algebras, the cardinals* $n_1(\mathbf{V})$, $n_2(\mathbf{V})$, *and* $n_3(\mathbf{V})$ *are finite.* ■

On the class $\mathcal{S}$ we now define the *first equivalence* $\sim_1$ by the requirement that

$Y \sim_1 Z$ if and only if the images of rh-maps in Y and Z are isomorphic.

In other words, for $Y, Z \in \mathcal{S}$ we write $Y \sim_1 Z$ if and only if these dh-spaces have isomorphic minimal plots. The claim below follows immediately from Lemma 5.1.

LEMMA 5.2. *The equivalence* $\sim_1$ *has at most* $n_1(\mathbf{V})$ *classes.* ■

We need the following observation from [2].

LEMMA 5.3 [2]. *If* X *is a dh-space and* $r \in \mathrm{End}(X)$ *is idempotent, then the map* $\xi : \mathrm{End}(\mathrm{Im}(r)) \to r\,\mathrm{End}(X)r$ *defined by* $\xi(k) = kr$ *is an isomorphism of* $\mathrm{End}(\mathrm{Im}(r))$ *onto* $r\,\mathrm{End}(X)r$ *with inverse* $\xi^{-1}(h) = rh\restriction \mathrm{Im}(r)$. ■

For any class $\mathcal{S}_1$ of the equivalence $\sim_1$ select some $X \in \mathcal{S}_1$. Choose an rh-map $r_X \in \mathrm{End}(X)$, and for every $Y \in \mathcal{S}_1$ set $r_Y = \Psi_{XY}(r_X)$. Then $r_Y \in \mathrm{End}(Y)$ is an rh-map and, because $X \sim_1 Y$, the dh-spaces $\mathrm{Im}(r_X)$ and $\mathrm{Im}(r_Y)$ are isomorphic. Thus for any $Y \in \mathcal{S}_1$ there exists a dh-isomorphism $\varphi_Y : \mathrm{Im}(r_X) \to \mathrm{Im}(r_Y)$. For any $h \in \mathrm{End}(\mathrm{Im}(r_X))$, write $\tau_Y(h) = \varphi_Y h \varphi_Y^{-1}$. Then $\tau_Y : \mathrm{End}(\mathrm{Im}(r_X)) \to \mathrm{End}(\mathrm{Im}(r_Y))$ is a monoid isomorphism such that $\varphi_Y h = \tau_Y(h)\varphi_Y$ for every $h \in \mathrm{End}(\mathrm{Im}(r_X))$, or, equivalently, for any $g \in \mathrm{End}(\mathrm{Im}(r_Y))$ we have $\tau_Y^{-1}(g)\varphi_Y^{-1} = \varphi_Y^{-1} g$.

By Lemma 5.3, for any $Y \in \mathcal{S}$, the map $\xi_Y : \mathrm{End}(\mathrm{Im}(r_Y)) \to r_Y\,\mathrm{End}(Y)r_Y$ given by $\xi_Y(h) = hr_Y$ is an isomorphism whose inverse ξ_Y^{-1} is given by $\xi_Y^{-1}(k) = k\restriction \mathrm{Im}(r_Y)$ for every $k \in r_Y\,\mathrm{End}(Y)r_Y$. Furthermore, the domain-range restriction of Ψ_{YZ} maps $r_Y\,\mathrm{End}(Y)r_Y \subseteq \mathrm{End}(Y)$ bijectively onto $r_Z\,\mathrm{End}(Z)r_Z \subseteq \mathrm{End}(Z)$ because $\Psi_{YZ}(r_Y) = r_Z$.

We are now prepared to define the *second equivalence* $\sim_2$ on $\mathcal{S}$ by setting

$$Y \sim_2 Z \quad \text{if and only if} \quad Y \sim_1 Z \text{ and } \Psi_{YX}\xi_Y\tau_Y = \Psi_{ZX}\xi_Z\tau_Z.$$

For any $Y \sim_2 Z$, we write $\phi_{YZ} = \varphi_Z \varphi_Y^{-1}$.

LEMMA 5.4. *If the equivalence* $\sim_1$ *has* s_1 *classes, then the equivalence* $\sim_2$ *has at most* $s_1 n_2(\mathbf{V})$ *classes. Furthermore, if* $Y \sim_2 Z$, *then* ϕ_{YZ} :

$\mathrm{Im}(r_Y) \to \mathrm{Im}(r_Z)$ *is a dh-isomorphism such that*, *for any* $y \in \mathrm{Im}(r_Y)$ *and any* $f \in \mathrm{End}(Y)$,

$$\phi_{YZ} r_Y f(y) = r_Z \Psi_{YZ}(f) \phi_{YZ}(y).$$

If also $U \sim_2 Z$, *then*

$$\phi_{ZU}\phi_{YZ} = \phi_{YU} \quad \textit{and}$$
$$\phi_{UY}\phi_{YU} = \phi_{YY} \textit{ is the identity map on } \mathrm{Im}(r_Y).$$

Proof. From Lemma 5.3 it follows that the composite $\xi_X^{-1}\Psi_{YX}\xi_Y\tau_Y$ is an automorphism of $\mathrm{End}(\mathrm{Im}(r_X))$ for every Y. Thus if the equivalence $\sim_1$ has s_1 classes, then the equivalence $\sim_2$ has at most $s_1 n_2(\mathbf{V})$ classes.

Suppose that $Y \sim_2 Z$. Then $\Psi_{YX}\xi_Y\tau_Y = \Psi_{ZX}\xi_Z\tau_Z$ and hence

$$\xi_Z^{-1}\Psi_{XZ}\Psi_{YX}\xi_Y\tau_Y\tau_Y^{-1} = \xi_Z^{-1}\Psi_{XZ}\Psi_{ZX}\xi_Z\tau_Z\tau_Y^{-1}.$$

Using (C1) and (C2), it then follows that $\xi_Z^{-1}\Psi_{YZ}\xi_Y = \tau_Z\tau_Y^{-1}$. From the latter fact and all appropriate definitions, for any $f \in \mathrm{End}(Y)$ and $y \in \mathrm{Im}(r_Y)$, we obtain

$$\begin{aligned}
\phi_{YZ} r_Y f(y) &= \varphi_Z\varphi_Y^{-1} r_Y f(y) = \varphi_Z(\tau_Y^{-1}(r_Y f \restriction \mathrm{Im}(r_Y))\varphi_Y^{-1}(y) \\
&= \tau_Z(\tau_Y^{-1}(r_Y f \restriction \mathrm{Im}(r_Y)))\varphi_Z\varphi_Y^{-1}(y) \\
&= \xi_Z^{-1}\Psi_{YZ}\xi_Y(r_Y f \restriction \mathrm{Im}(r_Y))\phi_{YZ}(y) \\
&= \xi_Z^{-1}\Psi_{YZ}(r_Y f r_Y)\phi_{YZ}(y) = \xi_Z^{-1}(r_Z\Psi_{YZ}(f) r_Z)\phi_{YZ}(y) \\
&= r_Z\Psi_{YZ}(f)\phi_{YZ}(y).
\end{aligned}$$

The remaining two equalities are obvious. ∎

For each class $\mathcal{S}_2$ of the second equivalence, select and fix some $X \in \mathcal{S}_2$. Define

$$\mathbb{T} = \{P \mid P \text{ is a poset}, \chi_X(P) > 1\}, \quad \text{and}$$
$$\mathbb{P} = \{P \mid P \text{ is a poset}, \chi_X(P) \geq 1\}.$$

For each $P \in \mathbb{T}$, let $\mathbb{G}_P$ denote the P-block in X that contains $[r_X]$.

Let $Y \sim_2 X$. Then, by Statement 4.3, for every $P \in \mathbb{T}$ there is a poset Q such that the collection $\Psi_{XY}(\mathbb{G}_P)$ is a Q-block in Y. Define $\gamma_Y(P) = Q$. Then $\gamma_Y(P) \in \mathbb{P}$ because $\mathrm{Im}(r_Y)$ is isomorphic to $\mathrm{Im}(r_X)$, and the mapping $\gamma_Y : \mathbb{T} \to \mathbb{P}$ thus defined is one-to-one because Ψ_{XY} is an isomorphism and r_Y is an rh-map.

We now define the *third equivalence* $\sim_3$ on $\mathcal{S}$ by requiring that

$$Y \sim_3 Z \quad \text{if and only if} \quad Y \sim_2 Z \text{ and } \gamma_Y = \gamma_Z.$$

Lemma 5.5. *If the equivalence* $\sim_2$ *has* s_2 *equivalence classes then the equivalence* $\sim_3$ *has fewer than* $3s_2 n_3(\mathbf{V})!$ *equivalence classes. If* $Y \sim_3 Z$ *and* P *is a poset*, *then* $\mathbb{G}$ *is a* P*-block in* Y *exactly when* $\Psi_{YZ}(\mathbb{G})$ *is a* P*-block in* Z.

P r o o f. Let $Y \sim_3 Z$. Since $\mathbb{T} \subseteq \mathbb{P}$, the map $\gamma_Y^{-1}\gamma_Z$ is a partial permutation of $\mathbb{P}$, and the first claim follows immediately. The second claim is a direct consequence of the definition of the third equivalence. ■

Let $Y \sim_3 Z$. For each $C \in \mathbb{C}(Y)$ with $\mathrm{Im}(r_Y) \cap C = \emptyset$, select some rh-map $g_C \in \mathrm{End}(Y)$ with $C \cap \mathrm{Im}(g_C) \neq \emptyset$ for which $[g_C]$ belongs to the C-block containing $[r_Y]$. Then the component $C = \mathrm{Im}(g_C) \setminus \mathrm{Im}(r_Y)$ is isomorphic to $\mathrm{Im}(r_Y) \setminus \mathrm{Im}(g_C) \in \mathbb{C}(Y)$.

Define a mapping $\varepsilon_{YZ} : \mathbb{C}(Y) \to \mathbb{C}(Z)$ by setting, for every $C \in \mathbb{C}(Y)$,

$$\varepsilon_{YZ}(C) = \begin{cases} \phi_{YZ}(C) & \text{if } C \cap \mathrm{Im}(r_Y) \neq \emptyset, \\ \mathrm{Im}(\Psi_{YZ}(g_C)) \setminus \mathrm{Im}(r_Z) & \text{if } C \cap \mathrm{Im}(r_Y) = \emptyset. \end{cases}$$

The map ε_{YZ} is well-defined since ϕ_{YZ} maps $\mathrm{Im}(r_Y)$ isomorphically onto $\mathrm{Im}(r_Z)$ and because Ψ_{YZ} maps any block $\mathbb{G}$ containing $[r_Y]$ bijectively onto a block containing $[r_Z]$. It is also clear that the definition of ε_{YZ} does not depend on the particular choice of the rh-maps g_C with $C \cap \mathrm{Im}(r_Y) = \emptyset$.

LEMMA 5.6. *Let $Y \sim_3 Z \sim_3 U$. Then*

(1) $C \cap \mathrm{Im}(r_Y) \neq \emptyset$ *if and only if* $\varepsilon_{YZ}(C) \cap \mathrm{Im}(r_Z) \neq \emptyset$, *for every* $C \in \mathbb{C}(Y)$,

(2) $\varepsilon_{YZ}(C) \cong C$ *for every* $C \in \mathbb{C}(Y)$,

(3) $\varepsilon_{YU} = \varepsilon_{ZU}\varepsilon_{YZ}$,

(4) $\varepsilon_{ZY}\varepsilon_{YZ}$ *is the identity of* $\mathbb{C}(Y)$,

(5) *if* $f \in \mathrm{End}(Y)$ *is an rh-map and* $C \in \mathbb{C}(Y)$, *then* $C \subseteq \mathrm{Im}(f)$ *if and only if* $\varepsilon_{YZ}(C) \subseteq \mathrm{Im}(\Psi_{YZ}(f))$.

P r o o f. Assume first that $C \cap \mathrm{Im}(r_Y) \neq \emptyset$. Lemma 5.4 gives $\phi_{YZ} r_Y = r_Z \phi_{YZ}$, and hence $\varepsilon_{YZ}(C) = \phi_{YZ}(C)$ intersects $\mathrm{Im}(r_Z)$. But then C is a component of $\mathrm{Im}(r_Y)$, and $\phi_{YZ}(C)$ is a component of $\mathrm{Im}(r_Z)$ because $\phi_{YZ} : \mathrm{Im}(r_Y) \to \mathrm{Im}(r_Z)$ is an isomorphism. This proves (2) for any C with $C \cap \mathrm{Im}(g_C) \neq \emptyset$, and one implication in (1). From Lemma 5.4 it also follows that (3) and (4) hold for any C with $C \cap \mathrm{Im}(r_Y) \neq \emptyset$.

Secondly, assume that $C \cap \mathrm{Im}(g_C) = \emptyset$. Then there is a C-block $\mathbb{G}$ in Y containing both $[r_Y]$ and $[g_C] \neq [r_Y]$. The block $\Psi_{YZ}(\mathbb{G})$ then contains $[\Psi_{YZ}(g_C)] \neq [r_Z]$, and $\mathrm{Im}(\Psi_{YZ}(g_C)) \setminus \mathrm{Im}(r_Z) \cong C$ because of Lemma 5.5. This completes the proof of (1) and (2). The remainder of (3) and (4) follows from (C1), (C2) and the definition of ε_{YZ}.

Finally, (5) follows from Lemma 4.4 and the fact that the isomorphism Ψ_{YZ} preserves rh-maps, $2rh$-maps and blocks. ■

Let $\mathcal{S}_3$ be a class of the third equivalence. Select and fix some dh-space $X \in \mathcal{S}_3$. Let $C \in \mathbb{C}(X)$ be such that $C \cap \mathrm{Im}(r_X) = \emptyset$. Then, as we already know, there is a C-block $\mathbb{G}_C$ such that $[r_X], [g'_C] \in \mathbb{G}_C$ for an rh-map g'_C with $\mathrm{Im}(g'_C) \setminus \mathrm{Im}(r_X) = C$.

To continue the argument, we need a more specific choice of rh-maps respectively equivalent to r_X and g'_C. By Theorem 2.2, for every $C \in \mathbb{C}(X)$ with $C \cap \mathrm{Im}(r_X) = \emptyset$, there exist rh-maps g_C with $\mathrm{Im}(g_C) \setminus \mathrm{Im}(r_X) = C$ and $h_C \in [r_X]$ such that $g_C h_C = g_C$ and $h_C g_C = h_C$. For any $C \in \mathbb{C}(X)$ contained in $\mathrm{Im}(r_X)$, we select $g_C = h_C = r_X$.

For any $Y \in \mathcal{S}_3$ and any component $D \in \mathbb{C}(Y)$ define $C = \varepsilon_{YX}(D)$ and $g_D = \Psi_{XY}(g_C)$, $h_D = \Psi_{XY}(h_C)$. Clearly, $g_D h_D = g_D$, $h_D g_D = h_D$, $h_D \in [r_Y]$, and $[g_D] \in \mathbb{G}_D$ in Y.

For any $Y \sim_3 Z$ we now define a mapping $\sigma_{YZ} : Y \to Z$ by

$$\sigma_{YZ}(x) = \begin{cases} \phi_{YZ}(x) & \text{if } x \in \mathrm{Im}(r_Y), \\ \Psi_{YZ}(g_{K(x)})\phi_{YZ}h_{K(x)}(x) & \text{if } x \in Y \setminus \mathrm{Im}(r_Y). \end{cases}$$

Since $g_C = h_C = r_Y$ for any $C \in \mathbb{C}(Y)$ contained in $\mathrm{Im}(r_Y)$, and because $\Psi_{YZ}(r_Y) = r_Z$, we may simply write

$$\sigma_{YZ}(x) = \Psi_{YZ}(g_{K(x)})\phi_{YZ}h_{K(x)}(x) \quad \text{for all } x \in Y.$$

STATEMENT 5.7. *Let $Y \sim_3 Z \sim_3 U$. Then*

(1) *$\sigma_{YZ} : Y \to Z$ is a bijection,*

(2) *σ_{YZ} preserves order and has the dh-property,*

(3) *$\sigma_{YU} = \sigma_{ZU}\sigma_{YZ}$,*

(4) *$\sigma_{ZY}\sigma_{YZ}$ is the identity map of Y,*

(5) *$\sigma_{YZ}(\mathrm{Im}(f)) = \bigcup\{\varepsilon_{YZ}(C) \mid C \in \mathbb{C}(Y)$ and $C \subseteq \mathrm{Im}(f)\} = \mathrm{Im}(\Psi_{YZ}(f))$ for every rh-map $f \in \mathrm{End}(Y)$.*

Proof. First, the restriction $\sigma_{YZ} \restriction \mathrm{Im}(r_Y) = \phi_{YZ}$ satisfies the first four claims because $\phi_{YZ} : \mathrm{Im}(r_Y) \to \mathrm{Im}(r_Z)$ is a dh-isomorphism for which the second and the third claims of Lemma 5.4 hold.

Let $D \in \mathbb{C}(Y)$ be such that $D \cap \mathrm{Im}(r_Y) = \emptyset$. Then $\sigma_{YZ}(D)$ is a component of $\mathbb{C}(Z)$ disjoint from $\mathrm{Im}(r_Z)$, by Lemma 5.6 and the definition of σ_{YZ}. Also, the restriction of σ_{YZ} to any such D is a dh-isomorphism, so that (2) holds. The composition properties (3) and (4) hold on $Y \setminus \mathrm{Im}(r_Y)$ because of the definition of σ_{YZ} and similar composition properties of ε_{YZ} and ϕ_{YZ}; see Lemmas 5.4 and 5.6. The fact that $\varepsilon_{YZ} : \mathbb{C}(Y) \to \mathbb{C}(Z)$ is a bijection, and (1), follow from what was already noted.

Finally, (5) is a consequence of the definition of σ_{YZ} and Lemma 5.6(5). ∎

LEMMA 5.8. *Let $Y \sim_3 Z$. If $f \in \mathrm{End}(Y)$ is an rh-map or a 2rh-map, then*

$$\sigma_{YZ} f = \Psi_{YZ}(f)\sigma_{YZ}. \tag{e}$$

Proof. Our proof has three steps.

STEP 1. Suppose that $f \in \mathrm{End}(Y)$ is such that $\mathrm{Im}(f) \subseteq \mathrm{Im}(r_Y)$. Since this assumption is equivalent to $r_Y f = f$ and because $\Psi_{YZ}(r_Y) = r_Z$, we also have $\mathrm{Im}(\Psi_{YZ}(f)) \subseteq \mathrm{Im}(r_Z)$. Thus for every $y \in \mathrm{Im}(r_Y)$ we obtain, using Lemma 5.4, $\Psi_{YZ}(f)\sigma_{YZ}(y) = r_Z\Psi_{YZ}(f)\phi_{YZ}(y) = \phi_{YZ}r_Y f(y) = \sigma_{YZ}f(y)$. If $y \in Y\backslash\mathrm{Im}(r_Y)$, then for $C = K(y)$ we have $g_C(y) = y$ and $h_C(y) \in \mathrm{Im}(r_Y)$. Therefore

$$\begin{aligned}\Psi_{YZ}(f)\sigma_{YZ}(y) &= \Psi_{YZ}(f)\Psi_{YZ}(g_C)\phi_{YZ}h_C(y)\\ &= r_Z\Psi_{YZ}(f)\Psi_{YZ}(g_C)\phi_{YZ}h_C(y)\\ &= r_Z\Psi_{YZ}(r_Y f g_C)\phi_{YZ}h_C(y) = \phi_{YZ}r_Y f g_C h_C(y)\\ &= \phi_{YZ} f g_C(y) = \phi_{YZ} f(y),\end{aligned}$$

using the definition of σ_{YZ} and Lemma 5.4. This shows that (e) holds for any $f \in \mathrm{End}(Y)$ with $\mathrm{Im}(f) \subseteq \mathrm{Im}(r_Y)$.

STEP 2. Let f be any rh-map of Y. Let $g_0 \in [r_Y]$ and $g_1 \in [f]$ be rh-maps satisfying $g_0g_1 = g_0$ and $g_1g_0 = g_1$. First we show that

$$\text{(a)} \qquad \Psi_{YZ}(g_1)\sigma_{YZ}(y) = \sigma_{YZ}g_1(y) \quad \text{for every } y \in \mathrm{Im}(r_Y).$$

Using $g_0g_1 = g_0$ and Step 1, we obtain

$$\text{(b)} \qquad \begin{aligned}\Psi_{YZ}(g_0)\Psi_{YZ}(g_1)\sigma_{YZ} &= \Psi_{YZ}(g_0)\sigma_{YZ} = \sigma_{YZ}g_0\\ &= \sigma_{YZ}g_0g_1 = \Psi_{YZ}(g_0)\sigma_{YZ}g_1.\end{aligned}$$

Since $g_1(\mathrm{Im}(r_Y)) = \mathrm{Im}(g_1)$ and $\mathrm{Im}(\Psi_{YZ}(g_1)) = \Psi_{YZ}(g_1)(\mathrm{Im}(r_Z)) = \Psi_{YZ}(g_1)\sigma_{YZ}(\mathrm{Im}(r_Y))$, from Statement 5.7(5) applied to g_1 it now follows that $\sigma_{YZ}g_1(\mathrm{Im}(r_Y)) = \sigma_{YZ}(\mathrm{Im}(g_1)) = \mathrm{Im}(\Psi_{YZ}(g_1)) = \Psi_{YZ}(g_1)\sigma_{YZ}(\mathrm{Im}(r_Y))$. For every $y \in \mathrm{Im}(r_Y)$ we thus have

$$\text{(c)} \qquad \Psi_{YZ}(g_0)\Psi_{YZ}(g_1)\sigma_{YZ}(y) = \Psi_{YZ}(g_0)\sigma_{YZ}g_1(y).$$

But $\Psi_{YZ}(g_0)$ is injective on $\mathrm{Im}(\Psi_{YZ}(g_1))$ because $\Psi_{YZ}(g_1)\Psi_{YZ}(g_0) = \Psi_{YZ}(g_1)$, and (a) follows from (b) and (c).

Since $g_1g_0f = f$ and $g_0 \in [r_Y]$, from Step 1 and (a) we now deduce that, for any $x \in Y$,

$$\begin{aligned}\sigma_{YZ}f(x) &= \sigma_{YZ}g_1g_0f(x) = \Psi_{YZ}(g_1)\sigma_{YZ}g_0f(x)\\ &= \Psi_{YZ}(g_1)\Psi_{YZ}(g_0f)\sigma_{YZ}(x)\\ &= \Psi_{YZ}(g_1g_0f)\sigma_{YZ}(x) = \Psi_{YZ}(f)\sigma_{YZ}(x).\end{aligned}$$

Hence (e) holds for any rh-map $f \in \mathrm{End}(Y)$.

STEP 3. Suppose that $f \in \mathrm{End}(Y)$ is a $2rh$-map. Select and fix some $y \in Y$. Then there exists a plot W of Y containing $K(y)$, whose all other components are contained in $\mathrm{Im}(f)$, and which contains $K(f(y))$ in case when $K(f(y)) \ncong K(y)$. Let g_1 be an rh-map with $\mathrm{Im}(g_1) = W$.

Then $\mathrm{Im}(fg_1) \subseteq \mathrm{Im}(g_2)$ for some rh-map g_2. We may assume that g_2 is one-to-one on $\mathrm{Im}(r_Y)$; see Theorem 2.2. Then there exists some $g_3 \in [r_Y]$ so that $g_3g_2 = g_3$ and $g_2g_3 = g_2$. Set $h = g_3fg_1$. Then $\mathrm{Im}(h) \subseteq \mathrm{Im}(r_Y)$ and $g_2h = g_2fg_1 = fg_1$. Since $g_1(y) = y$, using Steps 1 and 2, we then obtain

$$\begin{aligned}\sigma_{YZ}f(y) &= \sigma_{YZ}fg_1(y) = \sigma_{YZ}g_2h(y) = \Psi_{YZ}(g_2)\sigma_{YZ}h(y)\\ &= \Psi_{YZ}(g_2)\Psi_{YZ}(h)\sigma_{YZ}(y) = \Psi_{YZ}(f)\Psi_{YZ}(g_1)\sigma_{YZ}(y)\\ &= \Psi_{YZ}(f)\sigma_{YZ}g_1(y) = \Psi_{YZ}(f)\sigma_{YZ}(y).\end{aligned}$$

Therefore (e) holds also for any $2rh$-map. ∎

STATEMENT 5.9. *If $Y \sim_3 Z$, then σ_{YZ} is a dh-isomorphism.*

Proof. In view of Statement 5.7, we need only prove that σ_{YZ} is continuous.

First we note that, for any compact 0-dimensional space (X, τ),

(A) any collection $\mathcal{U}$ of τ-clopen subsets that separates points of X is a subbase of τ.

Indeed, if σ is the coarsest topology on X for which every $U \in \mathcal{U}$ is σ-clopen, then (X, σ) is Hausdorff, and $\mathrm{id}_X : (X, \tau) \to (X, \sigma)$ is continuous. Since (X, τ) is compact, both (X, τ) and (X, σ) are compact Hausdorff spaces, and hence $\sigma = \tau$.

Clearly, for any two topological spaces (X, τ) and (Y, σ),

(B) a map $f : X \to Y$ is continuous whenever $f^{-1}(U)$ is τ-open for every $U \in \mathcal{U}$ from some subbase $\mathcal{U}$ of σ.

Now, by Theorem 3.4 and (A), the collection

$$\mathcal{U} = \{f^{-1}\{z\} \mid z \in \mathrm{Im}(f),\ f \in \mathrm{End}(Z) \text{ is an } rh\text{-map or a } 2rh\text{-map}\}$$

is a subbase of the topology on Z. By Statement 3.3, an endomorphism f of Z is an rh-map (or a $2rh$-map) if and only if $\Psi_{ZY}(f)$ is an rh-map (or a $2rh$-map, respectively). By Lemma 5.8, we have $\sigma_{ZY}f = \Psi_{ZY}(f)\sigma_{ZY}$ for any $f \in \mathrm{End}(Z)$ which is an rh-map or a $2rh$-map. Since σ_{YZ} is a bijection and $\sigma_{ZY} = \sigma_{YZ}^{-1}$, for any such f and each $z \in \mathrm{Im}(f)$ we have $\sigma_{YZ}^{-1}(f^{-1}\{z\}) = \Psi_{ZY}(f)^{-1}(\sigma_{YZ}^{-1}\{z\}) = \Psi_{ZY}(f)^{-1}(\sigma_{ZY}\{z\})$. Thus $\sigma_{YZ}^{-1}(f^{-1}\{z\})$ is clopen in Y, and hence σ_{YZ} is continuous, by (B). ∎

Now we are ready to complete the proof of our result.

Proof of Main Theorem. The first claim follows from Lemmas 5.1, 5.2, 5.4, 5.5, and from Statement 5.9.

Turning to the second claim, for any integer $n > 0$ we consider the dh-space P_n on the set $\{0, 1, \dots, 2n+1\}$ whose order is given by $2i < 2i+1 > 2i+2$ for $i = 0, 1, \dots, n-1$ and $2n < 2n+1$.

We claim that the double Heyting algebra $D(P_n)$ dual to any such P_n has only the trivial endomorphism. Since P_n is order connected, the algebra $D(P_n)$ is simple (see Proposition 1.4) and hence every $f \in \mathrm{End}(P_n)$ is invertible. If $a, b \in P_n$ then $|[a)| = 2$ for $a = 0$ alone, and $|(b]| = 2$ only for $b = 2n+1$. Since the unique order path connecting 0 to $2n+1$ passes through all other elements of P_n, the identity map is the only dh-endomorphism of P_n.

The duals $D(P_i)$ of the posets P_i with $0 < i \leq n$ generate a finitely generated variety $\mathbf{V}_n$ of double Heyting algebras that has more than n non-isomorphic members with isomorphic endomorphism monoids. ∎

COROLLARY 5.10. *For any finitely generated variety* $\mathbf{V}$ *of double Heyting algebras there is an integer*

$$n < 3n_1(\mathbf{V})n_2(\mathbf{V})n_3(\mathbf{V})!$$

for which $\mathbf{V}$ *is* n*-determined.* ∎

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