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FURTHER PROPERTIES OF AN EXTREMAL SET OF UNIQUENESS

BY

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Let $\mathbb{T}$ denote the group $[0,1)$ with addition modulo one. In [4] we presented an elementary construction of a countable, compact subset S of $\mathbb{T}$ which could not be expressed as the union of two H-sets, and conjectured that S is not expressible as the union of finitely many H-sets. Here we use a descriptive set theory result of S. Kahane [6] to help show that S cannot be expressed as the union of finitely many Dirichlet sets. For the connection of this problem with that of characterizing sets of uniqueness for trigonometric series on $\mathbb{T}$, see [7] and [4].

Let $\mathbb{Z}$ denote the integers and $\mathbb{N}$ the nonnegative integers. If x and y are real numbers then by $x \equiv y$ we shall mean $x - y \in \mathbb{Z}$, and in this case we identify x and y with a single point in $\mathbb{T}$. A subset E of $\mathbb{T}$ is a *set of uniqueness* if the only trigonometric series $\sum_{n=-\infty}^{\infty} c(n)e^{2\pi inx}$ on $\mathbb{T}$ which converges to zero for all x outside E is the zero series: $c(n) = 0$ for all n. A compact subset E of $\mathbb{T}$ is an *H-set* if there exists a nonempty open interval I in $\mathbb{T}$ such that

$$N(E; I) = \{n \in \mathbb{Z} : nx \notin I \text{ for all } x \in E\}$$

is infinite; E is a *Dirichlet set* if $N(E; (\varepsilon, 1-\varepsilon))$ is infinite for all $\varepsilon > 0$. The families of all H-sets and Dirichlet sets in $\mathbb{T}$ will be denoted by H and D, respectively. Every finite subset of $\mathbb{T}$ is a Dirichlet set [3], every Dirichlet set is clearly an H-set, and every H-set is a set of uniqueness [8]. Indeed, any countable union of (compact) H-sets is a set of uniqueness [1].

A family B of compact subsets of $\mathbb{T}$ is *hereditary* if $E \in B$ implies all compact subsets of E are also in B. It is clear from the definitions that H, D, and the class F, consisting of all finite subsets of $\mathbb{T}$, are each hereditary families of compact subsets of $\mathbb{T}$. If B is any hereditary family of compact sets in $\mathbb{T}$ and E is any compact subset of $\mathbb{T}$, let the *B-derivate* of E, $d_B(E) = d_B^{(1)}(E)$, consist of those points x in E such that, for every open interval I containing x, the closure of $E \cap I$ does not belong to the family B.

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For $n>1$, let the nth B-*derivate* of E be defined inductively by $d_B^{(n)}(E) = d_B(d_B^{(n-1)}(E))$; to obtain future economy of expression, we adopt the convention $d_B^{(0)}(E)=E$. If there exists a positive integer n such that $d_B^{(n)}(E)$ is empty, then we say that E has *finite B-rank*; in this case, the least such integer n is called the *B-rank* of E. For the family F of finite sets, observe that $d_F(E)$ denotes the set of limit points of E, and that E has finite F-rank if and only if the classical Cantor–Bendixson rank of E is finite. For Cantor–Bendixson derivates, we use the classical notation E' for $d_F(E)$, and $E^{(n)}$ for $d_F^{(n)}(E)$. For a connection between the Cantor–Bendixson rank and Dirichlet sets, see [5].

We shall use the following B-rank result of S. Kahane [6].

PROPOSITION 1. *Let $n\in\mathbb{N}$, let E be a compact subset of $\mathbb{T}$, and let B be a hereditary family of compact subsets of $\mathbb{T}$. If E is the union of n sets from B, then the B-rank of E is at most n.*

Given x in $\mathbb{T}$, let $x=\sum_{k=1}^{\infty}x_k2^{-k},\ x_k\in\{0,1\}$, denote its binary expansion, and write $x=0.x_1x_2x_3\ldots$; this expression for x is unique if the terminating expansion is chosen whenever possible. Let $S_{-1}=\{0\}$ and, for each $n\in\mathbb{N}$, let S_n signify the set of all $x=0.x_1x_2x_3\ldots$ in $\mathbb{T}$ such that $\sum_{k=1}^{\infty}x_k=n+1$ and $x_k=0$ if $1\le k\le n$. Define $S=\bigcup_{n=-1}^{\infty}S_n$. Note that a point of $\mathbb{T}$ belongs to S if and only if the number of ones in the binary expansion of x does not exceed the number of its leading zeros by more than one. Clearly, S consists of rational points and hence is countable; it is not hard to see that S is closed (and hence compact) and has infinite Cantor–Bendixson rank ([4], or see Lemma 3 below).

THEOREM 1. *The set S has infinite Dirichlet rank.*

COROLLARY. *The set S cannot be expressed as the union of a finite number of Dirichlet sets.*

P r o o f. Proposition 1 implies that if S were a union of n Dirichlet sets, then the Dirichlet rank of S would not exceed n.

The proof of Theorem 1 will be based on the following three lemmas.

LEMMA 1. *If $y\in[0,1)\cap\mathbb{Q}$ and $N\in\mathbb{N}$, then*

$$\{y\}\cup\{y+2^{-m}:m\in\mathbb{N},\ m\ge N\}$$

is not a Dirichlet set.

P r o o f. Without loss of generality, we may assume that $N\ge 2$. It suffices to show that the set $J_{M,N}$ consisting of all nonnegative integers k such that

$$k\{y+2^{-m}:m\in\mathbb{N},m\ge N\}\subseteq[0,2^{-M}]\cup[1-2^{-M},1]$$

is finite for sufficiently large positive integers M.

If $y = 0$, let M be any integer not less than 2. If $y \neq 0$, then denote by δ the smallest nonzero element of the finite subgroup

$$G = \{jy : j \in \mathbb{Z}\}$$

of $\mathbb{T}$. Choose $M \in \mathbb{N}$ such that $2^{-M} < \delta$.

We first show that

$$(1) \qquad ky \equiv 0 \quad \text{for all } k \in J_{M,N}.$$

If $y = 0$ then (1) is clear, so suppose $y \neq 0$. Fix $k \in J_{M,N}$ and let $p \in \mathbb{N} \cap [0, \delta^{-1} - 1]$ be such that $ky \equiv p\delta$. Since $k2^{-n} \to 0^+$ as $n \to \infty$, it follows that

$$(2) \qquad k(y + 2^{-n}) \to p\delta^+ \quad \text{as } n \to \infty.$$

Because $2^{-M} < \delta$, the only element of G contained in $[0, 2^{-M}] \cup [1 - 2^{-M}, 1]$ is 0. But (2) and the facts that $p \in \mathbb{N} \cap [0, \delta^{-1} - 1]$ and $k \in J_{M,N}$ imply that $p = 0$, thus establishing (1).

Next, we show that for each $k \in J_{M,N}$,

$$(3) \qquad k(y + 2^{-n}) \in [0, 2^{-M}] \quad \text{for all } n \geq N.$$

To see this, fix $k \in J_{M,N}$. Since $ky \equiv 0$ and $0 < k2^{-n} < 2^{-M}$ for all n sufficiently large, it follows that there exists an integer $N_1 = N_1(k) \geq N$ such that

$$(4) \qquad k(y + 2^{-n}) \in [0, 2^{-M}] \quad \text{for all } n \geq N_1.$$

If (3) does not hold, then (4) implies that there exists a largest integer $\nu \geq N$ such that

$$(5) \qquad k(y + 2^{-\nu}) \in [1 - 2^{-M}, 1];$$

hence $k \in J_{M,N}$ implies

$$(6) \qquad k(y + 2^{-(\nu+1)}) \in [0, 2^{-M}].$$

But from (1) and (5), it follows that

$$(7) \qquad k2^{-\nu} = z + r \quad \text{where } z \in \mathbb{Z} \text{ and } r \in [1 - 2^{-M}, 1),$$

and (1) and (6) imply

$$(8) \qquad k2^{-(\nu+1)} = y + s \quad \text{where } y \in \mathbb{Z} \text{ and } s \in (0, 2^{-M}].$$

Dividing (7) by 2 yields

$$(9) \qquad k2^{-(\nu+1)} = (z + r)/2 \quad \text{where } r/2 \in [2^{-1} - 2^{-M-1}, 2^{-1}).$$

If z is even, then (8) and (9) imply $s \equiv r/2$, clearly a contradiction since $M \geq 2$ implies that $(0, 2^{-M}] \cap [2^{-1} - 2^{-M-1}, 2^{-1})$ is empty. If z is odd, then (8) and (9) yield $s \equiv (1 + r)/2$, again a contradiction since $(0, 2^{-M}] \cap [1 - 2^{-M-1}, 1)$ is empty. Therefore (3) is established.

Finally, we show that $J_{M,N}$ is finite. To this end, fix $k \in J_{M,N}$. By (1) and (3), we have

$$k2^{-N} = z + r \quad \text{where } z \in \mathbb{Z} \text{ and } r \in [0, 2^{-M}]. \tag{10}$$

We shall show that

$$z2^{-j} \in \mathbb{Z} \quad \text{for all } j \in \mathbb{N}, \tag{11}$$

so that $z = 0$. This will conclude the proof because (10) then implies $k = 2^N r \le 2^{N-M}$.

Note that (10) implies that (11) holds for $j = 0$. Suppose that (11) holds for some integer $j \ge 0$, but that $z2^{-(j+1)}$ is not an integer. Then

$$\begin{aligned} k2^{-(N+j+1)} &= (z+r)2^{-(j+1)} \\ &\equiv 2^{-1} + r2^{-(j+1)} \in [2^{-1}, 2^{-1} + 2^{-(M+j+1)}], \end{aligned}$$

in contradiction to (1) and (3). Therefore (11) holds by induction, and the proof of Lemma 1 is complete.

LEMMA 2. *Let $x = 0.x_1x_2x_3\ldots \in S \setminus \{0\}$, with x_{J+1} and x_{J+K} denoting the first and last nonzero binary digits of x, respectively. If $y \in S \setminus \{x\}$ and $|y - x| < 2^{-2(J+K+1)}$ then $y > x$ and $y_j = x_j$ for all $1 \le j \le J+K$.*

Proof. Let $y = 0.y_1y_2\ldots y_{J+L}$ denote the binary expansion of y. Suppose $x_j = y_j$ for all $j < j_0$ and $x_{j_0} \ne y_{j_0}$.

CASE 1: $x_{j_0} > y_{j_0}$. Note that this is precisely the case when $x > y$. If $y_{j_0+1} = 0$ then

$$2^{-2(J+K+1)} > |x-y| \ge 2^{-j_0} - \sum_{j=j_0+2}^{J+L} y_j 2^{-j} > 2^{-(j_0+1)}.$$

Consequently, $j_0 + 1 > 2(J + K + 1)$, and hence $x_j = 1$ for some $j = j_0 > J + K$, a contradiction. If $y_{j_0+1} = 1$ then, since $y \in S$ and y has at most j_0 leading zeros in its binary expansion, it follows that $\sum_{j=1}^{\infty} y_j \le j_0 + 1$. Arguing as when $y_{j_0+1} = 0$, we have

$$2^{-2(J+K+1)} > 2^{-j_0} - \sum_{j=j_0+1}^{J+L} y_j 2^{-j} \ge 2^{-j_0} - \sum_{j=j_0+1}^{2j_0+1} 2^{-j} = 2^{-(2j_0+1)}.$$

Thus, $2j_0 + 1 > 2(J + K + 1)$ and hence $j_0 > J + K$, a contradiction just as before. Therefore the case $x_{j_0} > y_{j_0}$ cannot occur.

CASE 2: $x_{j_0} < y_{j_0}$. Note that this is precisely the case when $y > x$. We have

$$2^{-2(J+K+1)} > |y-x| \ge 2^{-j_0} - \sum_{j=j_0+1}^{J+K} x_j 2^{-j}.$$

Since $x \in S$ and x has J leading zeros in its binary expansion, it follows that $\sum_{j=1}^{\infty} x_j \le J+1$. Therefore

$$2^{-j_0} - \sum_{j=j_0+1}^{J+K} x_j 2^{-j} \ge 2^{-j_0} - \sum_{j=j_0+1}^{j_0+J+1} 2^{-j} = 2^{-(j_0+J+1)}.$$

Combining the last pair of displayed inequalities gives $j_0 + J + 1 > 2(J + K + 1)$, and hence $j_0 > J + K$. This completes the proof of Lemma 2.

Definition. Let x be a nonzero element of $\mathbb{T}$ with binary expansion $x = 0.x_1x_2x_3\ldots$ (Recall that if x has two binary expansions, we agree to consider only the terminating expansion.) Suppose that $x_j = 0$ if $j \le J$ and $x_{J+1} = 1$. Define the *deficiency* of x by

$$\operatorname{def}(x) = 1 + J - \sum_{j=1}^{\infty} x_j.$$

Furthermore, define $\operatorname{def}(0) = \infty$.

The following properties of the deficiency are clear:

(a) $\operatorname{def}(x) > -\infty$ if and only if x is a binary rational number;
(b) $\operatorname{def}(x) \ge 0$ if and only if $x \in S$.

Lemma 3. *Let $n \in \mathbb{N}$ and $x \in S$. Then $x \in S^{(n)}$ if and only if $\operatorname{def}(x) \ge n$.*

Proof. The proof is by induction. The case $n=0$ is property (b) above. Suppose the result holds for $n \ge 0$. If $x \in S^{(n+1)}$, then there exists a sequence $\{y^{(m)}\}_{m=1}^{\infty}$ from $S^{(n)} \setminus \{x\}$ such that $y^{(m)} \to x$ as $m \to \infty$. By the induction hypothesis, $\operatorname{def}(y^{(m)}) \ge n$ for all $m \ge 1$. Lemma 2 implies that $\operatorname{def}(x) > \operatorname{def}(y^{(m)})$ for m sufficiently large. Hence $\operatorname{def}(x) \ge n+1$. Conversely, suppose $\operatorname{def}(x) \ge n+1$. For sufficiently large m, say $m \ge N$, we have

$$\operatorname{def}(x + 2^{-m}) = \operatorname{def}(x) - 1 \ge n.$$

The induction hypothesis implies that the sequence $\{x + 2^{-m}\}_{m=N}^{\infty}$ is contained in $S^{(n)} \setminus \{x\}$, and hence $x \in S^{(n+1)}$.

Proof of Theorem 1. By Lemma 3, we have $0 \in S^{(n)}$ for all $n \in \mathbb{N}$. Therefore it suffices to show that for each $n \in \mathbb{N}$, we have $S^{(n)} \subseteq d_D^{(n)}(S)$; for this we use induction. For $n = 0$ the inclusion is clear. Suppose the inclusion $S^{(n)} \subseteq d_D^{(n)}(S)$ holds for $n \ge 0$. Then

$$\begin{aligned}
d_D^{(n+1)}(S) &= d_D(d_D^{(n)}(S)) \\
&= \{x \in d_D^{(n)}(S)\colon \text{if } I \text{ is an open interval containing } x, \\
&\qquad\qquad \text{then } \overline{I \cap d_D^{(n)}(S)} \text{ is not a Dirichlet set}\} \\
&\supseteq \{x \in S^{(n)} \quad \colon \text{if } I \text{ is an open interval containing } x, \\
&\qquad\qquad \text{then } \overline{I \cap S^{(n)}} \text{ is not a Dirichlet set}\} \\
&= d_D(S^{(n)}).
\end{aligned}$$

To finish the proof, it therefore is enough to show that $S^{(n+1)} \subseteq d_D(S^{(n)})$. Let $x \in S^{(n+1)}$; by Lemma 3, we have $\operatorname{def}(x) \geq n+1$. Lemma 2 then implies that for sufficiently large m, say $m \geq N$, we have $\operatorname{def}(x+2^{-m}) = \operatorname{def}(x)-1 \geq n$. Thus $\{x + 2^{-m}\}_{m=N}^{\infty}$ is contained in $S^{(n)}$ by Lemma 3. If I is any open interval containing x, Lemma 1 then implies that $\overline{I \cap \{x + 2^{-m}\}_{m=N}^{\infty}} \subseteq \overline{I \cap S^{(n)}}$ is not a Dirichlet set. Hence $S^{(n+1)} \subseteq d_D(S^{(n)})$, and the proof of Theorem 1 is complete.

The question as to whether the set S is expressible as the union of finitely many H-sets cannot be answered so easily, as demonstrated by the next two results. A simple compactness argument yields the first assertion.

PROPOSITION 2. *Let $E \subseteq \mathbb{T}$ be compact and let B be a hereditary family of compact subsets of $\mathbb{T}$. If the B-rank of E is* 1 *then E can be expressed as the union of finitely many B-sets.*

THEOREM 2. *The H-rank of the set S is* 2.

The following lemma will be used to establish Theorem 2.

LEMMA 4. *For every $J \in \mathbb{N}$, $S \cap [2^{-J-1}, 1 - 2^{-J-1}]$ is an H-set.*

Proof. If $y \in S \cap [2^{-J-1}, 1-2^{-J-1}]$, then y has at most J leading zeros in its binary expansion, and consequently has at most $J+1$ ones. Thus, for all $j \in \mathbb{N}$, we have $2^j y \equiv x$ where

$$0 \leq x \leq \sum_{k=1}^{J+1} 2^{-k} = 1 - 2^{-(J+1)}.$$

Therefore $2^j(S \cap [2^{-J-1}, 1 - 2^{-J-1}])$ misses the interval $(1 - 2^{-J-1}, 1)$ for all $j \in \mathbb{N}$.

Proof of Theorem 2. It suffices to show that $d_H(S) = \{0\}$. Suppose that $y \in S \setminus \{0\}$, and choose $J \in \mathbb{N}$ such that $2^{-J-1} < y < 1 - 2^{-J-1}$. Then $I = (2^{-J-1}, 1 - 2^{-J-1})$ is an open interval containing y, and Lemma 4 implies that $\overline{S \cap I}$ is an H-set. Thus $d_H(S) \subseteq \{0\}$.

To show the reverse inclusion, suppose by way of contradiction that $0 \notin d_H(S)$. Then there is an open interval I containing 0 such that $\overline{Si \cap I}$ is an

H-set.Choose $J \in \mathbb{N}$ such that $\mathbb{T}$ is the union of I and

$$I_J = [2^{-J-1}, 1 - 2^{-J-1}].$$

Another application of Lemma 4 shows that $S = \overline{(S \cap I)} \cup (S \cap I_J)$ is the union of two H-sets, contradicting the Theorem of [4]. Thus $d_H(S) = \{0\}$.

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