FURTHER PROPERTIES OF AN EXTREMAL SET OF UNIQUENESS

BY

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Let $T$ denote the group $[0, 1)$ with addition modulo one. In [4] we presented an elementary construction of a countable, compact subset $S$ of $T$ which could not be expressed as the union of two $H$-sets, and conjectured that $S$ is not expressible as the union of finitely many $H$-sets. Here we use a descriptive set theory result of S. Kahane [6] to help show that $S$ cannot be expressed as the union of finitely many Dirichlet sets. For the connection of this problem with that of characterizing sets of uniqueness for trigonometric series on $T$, see [7] and [4].

Let $\mathbb{Z}$ denote the integers and $\mathbb{N}$ the nonnegative integers. If $x$ and $y$ are real numbers then by $x \equiv y$ we shall mean $x - y \in \mathbb{Z}$, and in this case we identify $x$ and $y$ with a single point in $T$. A subset $E$ of $T$ is a set of uniqueness if the only trigonometric series $\sum_{n=-\infty}^{\infty} c(n)e^{2\pi inx}$ on $T$ which converges to zero for all $x$ outside $E$ is the zero series: $c(n) = 0$ for all $n$. A compact subset $E$ of $T$ is an $H$-set if there exists a nonempty open interval $I$ in $T$ such that

$$N(E; I) = \{n \in \mathbb{Z} : nx \notin I \text{ for all } x \in E\}$$

is infinite: $E$ is a Dirichlet set if $N(E; (\varepsilon, 1 - \varepsilon))$ is infinite for all $\varepsilon > 0$. The families of all $H$-sets and Dirichlet sets in $T$ will be denoted by $H$ and $D$, respectively. Every finite subset of $T$ is a Dirichlet set [3], every Dirichlet set is clearly an $H$-set, and every $H$-set is a set of uniqueness [8]. Indeed, any countable union of (compact) $H$-sets is a set of uniqueness [1].

A family $B$ of compact subsets of $T$ is hereditary if $E \in B$ implies all compact subsets of $E$ are also in $B$. It is clear from the definitions that $H, D$, and the class $F$, consisting of all finite subsets of $T$, are each hereditary families of compact subsets of $T$. If $B$ is any hereditary family of compact sets in $T$ and $E$ is any compact subset of $T$, let the $B$-derivate of $E$, $d_B(E) = d_B^{(1)}(E)$, consist of those points $x$ in $E$ such that, for every open interval $I$ containing $x$, the closure of $E \cap I$ does not belong to the family $B$.
For \( n > 1 \), let the \( n \)th \( B \)-derivate of \( E \) be defined inductively by \( d_B^{(n)}(E) = d_B(d_B^{(n-1)}(E)) \); to obtain future economy of expression, we adopt the convention \( d_B^{(0)}(E) = E \). If there exists a positive integer \( n \) such that \( d_B^{(n)}(E) \) is empty, then we say that \( E \) has finite \( B \)-rank; in this case, the least such integer \( n \) is called the \( B \)-rank of \( E \). For the family \( F \) of finite sets, observe that \( d_F(E) \) denotes the set of limit points of \( E \), and that \( E \) has finite \( F \)-rank if and only if the classical Cantor–Bendixson rank of \( E \) is finite. For Cantor–Bendixson derivates, we use the classical notation \( E' \) for \( d_F(E) \), and \( E^{(n)} \) for \( d_F^{(n)}(E) \). For a connection between the Cantor–Bendixson rank and Dirichlet sets, see [5].

We shall use the following \( B \)-rank result of S. Kahane [6].

**Proposition 1.** Let \( n \in \mathbb{N} \), let \( E \) be a compact subset of \( T \), and let \( B \) be a hereditary family of compact subsets of \( T \). If \( E \) is the union of \( n \) sets from \( B \), then the \( B \)-rank of \( E \) is at most \( n \).

Given \( x \) in \( T \), let \( x = \sum_{k=1}^{\infty} x_k 2^{-k} \), \( x_k \in \{0,1\} \), denote its binary expansion, and write \( x = 0.x_1x_2x_3\ldots \); this expression for \( x \) is unique if the terminating expansion is chosen whenever possible. Let \( S_{-1} = \{0\} \) and, for each \( n \in \mathbb{N} \), let \( S_n \) signify the set of all \( x = 0.x_1x_2x_3\ldots \) in \( T \) such that \( \sum_{k=1}^{\infty} x_k = n + 1 \) and \( x_k = 0 \) if \( 1 \leq k \leq n \). Define \( S = \bigcup_{n=1}^{\infty} S_n \). Note that a point of \( T \) belongs to \( S \) if and only if the number of ones in the binary expansion of \( x \) does not exceed the number of its leading zeros by more than one. Clearly, \( S \) consists of rational points and hence is countable; it is not hard to see that \( S \) is closed (and hence compact) and has infinite Cantor–Bendixson rank ([4], or see Lemma 3 below).

**Theorem 1.** The set \( S \) has infinite Dirichlet rank.

**Corollary.** The set \( S \) cannot be expressed as the union of a finite number of Dirichlet sets.

**Proof.** Proposition 1 implies that if \( S \) were a union of \( n \) Dirichlet sets, then the Dirichlet rank of \( S \) would not exceed \( n \).

The proof of Theorem 1 will be based on the following three lemmas.

**Lemma 1.** If \( y \in [0, 1) \cap \mathbb{Q} \) and \( N \in \mathbb{N} \), then
\[
\{y\} \cup \{y + 2^{-m} : m \in \mathbb{N}, m \geq N\}
\]
is not a Dirichlet set.

**Proof.** Without loss of generality, we may assume that \( N \geq 2 \). It suffices to show that the set \( J_{M,N} \) consisting of all nonnegative integers \( k \) such that
\[
k\{y + 2^{-m} : m \in \mathbb{N}, m \geq N\} \subseteq [0, 2^{-M}] \cup [1 - 2^{-M}, 1]
\]
is finite for sufficiently large positive integers \( M \).
If $y = 0$, let $M$ be any integer not less than 2. If $y \neq 0$, then denote by $\delta$ the smallest nonzero element of the finite subgroup

$$G = \{ jy : j \in \mathbb{Z} \}$$

of $T$. Choose $M \in \mathbb{N}$ such that $2^{-M} < \delta$.

We first show that

$$ky \equiv 0 \quad \text{for all } k \in J_{M,N}. \tag{1}$$

If $y = 0$ then (1) is clear, so suppose $y \neq 0$. Fix $k \in J_{M,N}$ and let $p \in \mathbb{N} \cap [0, \delta^{-1} - 1]$ be such that $ky \equiv p\delta$. Since $k2^{-n} \to 0^+$ as $n \to \infty$, it follows that

$$k(y + 2^{-n}) \to p\delta^+ \quad \text{as } n \to \infty. \tag{2}$$

Because $2^{-M} < \delta$, the only element of $G$ contained in $[0, 2^{-M}] \cup [1 - 2^{-M}, 1]$ is 0. But (2) and the facts that $p \in \mathbb{N} \cap [0, \delta^{-1} - 1]$ and $k \in J_{M,N}$ imply that $p = 0$, thus establishing (1).

Next, we show that for each $k \in J_{M,N}$,

$$k(y + 2^{-n}) \in [0, 2^{-M}] \quad \text{for all } n \geq N. \tag{3}$$

To see this, fix $k \in J_{M,N}$. Since $ky \equiv 0$ and $0 < k2^{-n} < 2^{-M}$ for all $n$ sufficiently large, it follows that there exists an integer $N_1 = N_1(k) \geq N$ such that

$$k(y + 2^{-n}) \in [0, 2^{-M}] \quad \text{for all } n \geq N_1. \tag{4}$$

If (3) does not hold, then (4) implies that there exists a largest integer $\nu \geq N$ such that

$$k(y + 2^{-\nu}) \in [1 - 2^{-M}, 1]; \tag{5}$$

hence $k \in J_{M,N}$ implies

$$k(y + 2^{-(\nu+1)}) \in [0, 2^{-M}]. \tag{6}$$

But from (1) and (5), it follows that

$$k2^{-\nu} = z + r \quad \text{where } z \in \mathbb{Z} \text{ and } r \in [1 - 2^{-M}, 1), \tag{7}$$

and (1) and (6) imply

$$k2^{-(\nu+1)} = y + s \quad \text{where } y \in \mathbb{Z} \text{ and } s \in (0, 2^{-M}]. \tag{8}$$

Dividing (7) by 2 yields

$$k2^{-(\nu+1)} = (z + r)/2 \quad \text{where } r/2 \in [2^{-1} - 2^{-M-1}, 2^{-1}). \tag{9}$$

If $z$ is even, then (8) and (9) imply $s \equiv r/2$, clearly a contradiction since $M \geq 2$ implies that $(0, 2^{-M}] \cap [2^{-1} - 2^{-M-1}, 2^{-1})$ is empty. If $z$ is odd, then (8) and (9) yield $s \equiv (1 + r)/2$, again a contradiction since $(0, 2^{-M}] \cap [1 - 2^{-M-1}, 1)$ is empty. Therefore (3) is established.
Finally, we show that \( J_{M,N} \) is finite. To this end, fix \( k \in J_{M,N} \). By (1) and (3), we have

\[ k2^{-N} = z + r \quad \text{where} \quad z \in \mathbb{Z} \text{ and } r \in [0, 2^{-M}]. \]

We shall show that

\[ z2^{-j} \leq z \quad \text{for all} \quad j \in \mathbb{N}, \]

so that \( z = 0 \). This will conclude the proof because (10) then implies \( k = 2^N r \leq 2^{N-M} \).

Note that (10) implies that (11) holds for \( j = 0 \). Suppose that (11) holds for some integer \( j \geq 0 \), but that \( z2^{-(j+1)} \) is not an integer. Then

\[
\begin{align*}
 k2^{-(N+j+1)} &= (z + r)2^{-(j+1)} \\
 &= 2^{-1} + r2^{-(j+1)} \in [2^{-1}, 2^{-1} + 2^{-(M+j+1)}],
\end{align*}
\]

in contradiction to (1) and (3). Therefore (11) holds by induction, and the proof of Lemma 1 is complete.

**Lemma 2.** Let \( x = 0.x_1x_2x_3\ldots \in S \setminus \{0\} \), with \( x_{j+1} \) and \( x_{j+K} \) denoting the first and last nonzero binary digits of \( x \), respectively. If \( y \in S \setminus \{x\} \) and \( |y - x| < 2^{-2(J+K+1)} \) then \( y > x \) and \( y_j = x_j \) for all \( 1 \leq j \leq J+K \).

**Proof.** Let \( y = 0.y_1y_2\ldots y_{J+L} \) denote the binary expansion of \( y \). Suppose \( x_j = y_j \) for all \( j < j_0 \) and \( x_{j_0} \neq y_{j_0} \).

**Case 1:** \( x_{j_0} > y_{j_0} \). Note that this is precisely the case when \( x > y \). If \( y_{j_0+1} = 0 \) then

\[
2^{-2(J+K+1)} > |x - y| \geq 2^{-j_0} - \sum_{j=j_0+2}^{J+L} y_j 2^{-j} > 2^{-(j_0+1)}.\]

Consequently, \( j_0 + 1 > 2(J + K + 1) \), and hence \( x_j = 1 \) for some \( j = j_0 > J + K \), a contradiction. If \( y_{j_0+1} = 1 \) then, since \( y \in S \) and \( y \) has at most \( j_0 \) leading zeros in its binary expansion, it follows that \( \sum_{j=j_0+1}^{\infty} y_j \leq j_0 + 1 \). Arguing as when \( y_{j_0+1} = 0 \), we have

\[
2^{-2(J+K+1)} > 2^{-j_0} - \sum_{j=j_0+1}^{J+L} y_j 2^{-j} \geq 2^{-j_0} - \sum_{j=j_0+1}^{2j_0+1} 2^{-j} = 2^{-(2j_0+1)}.\]

Thus, \( 2j_0 + 1 > 2(J + K + 1) \) and hence \( j_0 > J + K \), a contradiction just as before. Therefore the case \( x_{j_0} > y_{j_0} \) cannot occur.

**Case 2:** \( x_{j_0} < y_{j_0} \). Note that this is precisely the case when \( y > x \). We have
2^{-2(J+K+1)} > |y - x| \geq 2^{-j_0} - \sum_{j=j_0+1}^{J+K} x_j 2^{-j}.

Since \(x \in S\) and \(x\) has \(J\) leading zeros in its binary expansion, it follows that \(\sum_{j=1}^{\infty} x_j \leq J + 1\). Therefore

\[
2^{-j_0} - \sum_{j=j_0+1}^{J+K} x_j 2^{-j} \geq 2^{-j_0} - \sum_{j=j_0+1}^{j_0+J+1} 2^{-j} = 2^{-(j_0+J+1)}.
\]

Combining the last pair of displayed inequalities gives \(j_0 + J + 1 > 2(J + K + 1)\), and hence \(j_0 > J + K\). This completes the proof of Lemma 2.

**Definition.** Let \(x\) be a nonzero element of \(\mathbb{T}\) with binary expansion \(x = 0.x_1x_2x_3\ldots\). (Recall that if \(x\) has two binary expansions, we agree to consider only the terminating expansion.) Suppose that \(x_j = 0\) if \(j \leq J\) and \(x_{J+1} = 1\). Define the deficiency of \(x\) by

\[
def(x) = 1 + \sum_{j=1}^{\infty} x_j.
\]

Furthermore, define \(def(0) = \infty\).

The following properties of the deficiency are clear:

(a) \(\text{def}(x) > -\infty\) if and only if \(x\) is a binary rational number;
(b) \(\text{def}(x) \geq 0\) if and only if \(x \in S\).

**Lemma 3.** Let \(n \in \mathbb{N}\) and \(x \in S\). Then \(x \in S^{(n)}\) if and only if \(\text{def}(x) \geq n\).

**Proof.** The proof is by induction. The case \(n=0\) is property (b) above. Suppose the result holds for \(n \geq 0\). If \(x \in S^{(n+1)}\), then there exists a sequence \(\{y^{(m)}\}_{m=1}^{\infty}\) from \(S^{(n)} \setminus \{x\}\) such that \(y^{(m)} \to x\) as \(m \to \infty\). By the induction hypothesis, \(\text{def}(y^{(m)}) \geq n\) for all \(m \geq 1\). Lemma 2 implies that \(\text{def}(x) > \text{def}(y^{(m)})\) for \(m\) sufficiently large. Hence \(\text{def}(x) \geq n + 1\). Conversely, suppose \(\text{def}(x) \geq n + 1\). For sufficiently large \(m\), say \(m \geq N\), we have

\[
def(x + 2^{-m}) = \text{def}(x) - 1 \geq n.
\]

The induction hypothesis implies that the sequence \(\{x + 2^{-m}\}_{m=N}^{\infty}\) is contained in \(S^{(n)} \setminus \{x\}\), and hence \(x \in S^{(n+1)}\).

**Proof of Theorem 1.** By Lemma 3, we have \(0 \in S^{(n)}\) for all \(n \in \mathbb{N}\). Therefore it suffices to show that for each \(n \in \mathbb{N}\), we have \(S^{(n)} \subseteq d^{(n)}(S)\); for this we use induction. For \(n = 0\) the inclusion is clear. Suppose the inclusion \(S^{(n)} \subseteq d^{(n)}(S)\) holds for \(n \geq 0\). Then
\[ d_D^{(n+1)}(S) = d_D(d_D^{(n)}(S)) = \begin{cases} \{x \in d_D^{(n)}(S) : \text{if } I \text{ is an open interval containing } x, \\ \quad \text{then } I \cap d_D^{(n)}(S) \text{ is not a Dirichlet set} \} \\ \supseteq \{x \in S^{(n)} : \text{if } I \text{ is an open interval containing } x, \\ \quad \text{then } I \cap S^{(n)} \text{ is not a Dirichlet set} \} \end{cases} = d_D(S^{(n)}). \]

To finish the proof, it therefore is enough to show that \( S^{(n+1)} \subseteq d_D(S^{(n)}). \) Let \( x \in S^{(n+1)}; \) by Lemma 3, we have \( \text{def}(x) \geq n+1. \) Lemma 2 then implies that for sufficiently large \( m, \) say \( m \geq N, \) we have \( \text{def}(x+2^{-m}) = \text{def}(x)-1 \geq n. \) Thus \( \{x + 2^{-m}\}_{m=N}^{\infty} \) is contained in \( S^{(n)} \) by Lemma 3. If \( I \) is any open interval containing \( x, \) Lemma 1 then implies that \( I \cap \{x + 2^{-m}\}_{m=N}^{\infty} \subseteq I \cap S^{(n)} \) is not a Dirichlet set. Hence \( S^{(n+1)} \subseteq d_D(S^{(n)}), \) and the proof of Theorem 1 is complete.

The question as to whether the set \( S \) is expressible as the union of finitely many \( H \)-sets cannot be answered so easily, as demonstrated by the next two results. A simple compactness argument yields the first assertion.

**Proposition 2.** Let \( E \subseteq \mathbb{T} \) be compact and let \( B \) be a hereditary family of compact subsets of \( \mathbb{T}. \) If the \( B \)-rank of \( E \) is \( 1 \) then \( E \) can be expressed as the union of finitely many \( B \)-sets.

**Theorem 2.** The \( H \)-rank of the set \( S \) is \( 2. \)

The following lemma will be used to establish Theorem 2.

**Lemma 4.** For every \( J \in \mathbb{N}, \) \( S \cap [2^{-J-1}, 1-2^{-J-1}] \) is an \( H \)-set.

**Proof.** If \( y \in S \cap [2^{-J-1}, 1-2^{-J-1}], \) then \( y \) has at most \( J \) leading zeros in its binary expansion, and consequently has at most \( J+1 \) ones. Thus, for all \( j \in \mathbb{N}, \) we have \( 2^j y = x \) where
\[
0 \leq x \leq \sum_{k=1}^{J+1} 2^{-k} = 1 - 2^{-(J+1)}. \]

Therefore \( 2^j (S \cap [2^{-J-1}, 1-2^{-J-1}]) \) misses the interval \((1-2^{-J-1}, 1)\) for all \( j \in \mathbb{N}. \)

**Proof of Theorem 2.** It suffices to show that \( d_H(S) = \{0\}. \) Suppose that \( y \in S \setminus \{0\}, \) and choose \( J \in \mathbb{N} \) such that \( 2^{-J-1} < y < 1 - 2^{-J-1}. \) Then \( I = (2^{-J-1}, 1-2^{-J-1}) \) is an open interval containing \( y, \) and Lemma 4 implies that \( S \cap I \) is an \( H \)-set. Thus \( d_H(S) \subseteq \{0\}. \)

To show the reverse inclusion, suppose by way of contradiction that \( 0 \not\in d_H(S). \) Then there is an open interval \( I \) containing \( 0 \) such that \( S \cap I \) is an
Choose \( J \in \mathbb{N} \) such that \( T \) is the union of \( I \) and
\[
I_J = [2^{-J-1}, 1 - 2^{-J-1}].
\]
Another application of Lemma 4 shows that \( S = (S \cap I) \cup (S \cap I_J) \) is the union of two \( H \)-sets, contradicting the Theorem of [4]. Thus \( d_H(S) = \{0\} \).

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