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TWO-PARAMETER MULTIPLIERS ON HARDY SPACES

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1. Introduction. In an earlier paper (see Simon [2]) we investigated some multiplier operators from the so-called dyadic Hardy space H^p to itself $(0 < p \le 1)$. By means of those multipliers and by suitable transformations from ℓ_2 to ℓ_2 we defined operators A on H^p which characterize the space H^p in the following sense: a function f belongs to H^p if and only if $Af \in L^p$. Moreover, $\|f\|_{H^p} \sim \|Af\|_p$, where "$\sim$" means that the ratio of the two sides lies between positive constants, independently of f. Among others, for the Sunouchi operator U we showed the equivalence $\|f\|_{H^p} \sim \|Uf\|_p$ $(1/2 < p \le 1,\ \int_0^1 f = 0)$.

In the present work we generalize those results to two-dimensional spaces. As a special case we get the (H^p, L^p)-boundedness of the two-dimensional Sunouchi operator if $0 < p \le 1$. This improves a theorem of Weisz [7]. Furthermore, the equivalence $\|f\|_{H^p} \sim \|Uf\|_p$ $(1/2 < p \le 1,\ \int_0^1 f = 0)$ is shown also in the two-dimensional case.

To prove these results we apply the atomic decomposition of L^p-bounded martingales. It is well known that the atomic characterization of the Hardy spaces H^p $(0 < p \le 1)$ plays an important role in the one-dimensional case. In the two-dimensional case the situation is much more complicated because the support of a two-dimensional atom can be an arbitrary open set, not only a dyadic rectangle. However, by a theorem of Weisz [7] in the definition of p-quasi-locality of operators it is enough to take p-atoms supported on dyadic rectangles. Furthermore, a p-quasi-local operator which is bounded from L^2 into L^2 is also bounded from H^p into L^p $(0 < p \le 1)$.

2. Notations. In this section some definitions and notations are introduced. We give a short summary of the basic concepts of Walsh–Fourier analysis and formulate some known results which play an important role in

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our further investigations. In this connection as well as for more details see the book by Schipp–Wade–Simon [1].

First of all recall the definition of the Walsh (–Paley) functions w_n $(n = 0, 1, \ldots)$. Let r be the function defined on $[0, 1)$ by

$$r(x) := \begin{cases} 1 & (0 \le x < 1/2), \\ -1 & (1/2 \le x < 1), \end{cases}$$

extended to the real line by periodicity of period 1. The *Rademacher functions* r_n $(n = 0, 1, \ldots)$ are given by $r_n(x) := r(2^n x)$ $(0 \le x < 1)$. The system of the functions r_n $(n = 0, 1, \ldots)$ is orthonormal (in the usual $L^2[0, 1)$ sense) but incomplete. The product system w_n $(n = 0, 1, \ldots)$ generated by r_n's is already a complete and orthonormal system of functions. That is, $w_n := \prod_{k=0}^{\infty} r_k^{n_k}$, where $n = \sum_{k=0}^{\infty} n_k 2^k$ $(n_k = 0, 1)$ is the binary expansion of the natural number $n = 0, 1, \ldots$

For $f \in L^1[0, 1)$ let $\widehat{f}(n) := \int_0^1 f w_n$ $(n = 0, 1, \ldots)$ be the nth *Walsh–Fourier coefficient* of the function f. The symbol $\widehat{f}$ will denote the sequence $(\widehat{f}(n),\ n = 0, 1, \ldots)$. The nth partial sum $S_n f$ and the nth $(C, 1)$-mean $\sigma_n f$ of the Walsh–Fourier series $\sum_{k=0}^{\infty} \widehat{f}(k) w_k$ are defined by

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) w_k, \qquad \sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f \qquad (n = 1, 2, \ldots).$$

It is clear that $\sigma_n f$ can be written directly in terms of the Walsh–Fourier coefficients of f as

$$\sigma_n f = \sum_{k=0}^{n-1} (1 - k/n) \widehat{f}(k) w_k \qquad (n = 1, 2, \ldots).$$

If $x, y \in [0, 1)$ are arbitrary and $x = \sum_{k=0}^{\infty} x_k 2^{-k-1},\ y = \sum_{k=0}^{\infty} y_k 2^{-k-1}$ are their dyadic expansions (i.e. $x_k, y_k = 0, 1$, where $\lim_k x_k \neq 1, \lim_k y_k \neq 1$), then let

$$x \dotplus y := \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{2^{k+1}}$$

be the so-called *dyadic sum* of x and y. Furthermore, the (*dyadic*) *convolution* of $f, g \in L^1[0, 1)$ is defined by

$$f * g(x) := \int_0^1 f(t) g(x \dotplus t)\, dt \qquad (x \in [0, 1)).$$

It follows immediately that for all $f \in L^1[0, 1)$ and $n = 1, 2, \ldots$,

$$S_n f = f * D_n, \qquad \sigma_n f = f * K_n,$$

where

$$D_n := \sum_{k=0}^{n-1} w_k, \qquad K_n := \frac{1}{n}\sum_{k=1}^{n} D_k \qquad (n = 1, 2, \ldots)$$

are the exact analogues of the well known (trigonometric) kernel functions of Dirichlet's and Fejér's type, respectively. The functions D_{2^n} $(n = 0, 1, \ldots)$ have a nice property which plays a central role in Walsh–Fourier analysis:

$$D_{2^n}(x) = \begin{cases} 2^n & (0 \le x < 2^{-n}) \\ 0 & (2^{-n} \le x < 1) \end{cases} \qquad (n = 0, 1, \ldots). \tag{1}$$

Moreover, the following statements will also be used:

$$\sum_{k=0}^{n-1} k w_k = n(D_n - K_n) \qquad (n = 1, 2, \ldots), \tag{2}$$

$$0 \le K_{2^s}(x) = \frac{1}{2}\Big(2^{-s} D_{2^s}(x) + \sum_{l=0}^{s} 2^{l-s} D_{2^s}(x \dotplus 2^{-l-1})\Big), \tag{3}$$

$$|K_l(x)| \le \sum_{t=0}^{s} 2^{t-s-1} \sum_{i=t}^{s} (D_{2^i}(x) + D_{2^i}(x \dotplus 2^{-t-1})) \qquad (2^s \le l < 2^{s+1}), \tag{4}$$

$$\sum_{k=2^s}^{\infty} \frac{w_k}{k} = \sum_{l=2^s+1}^{\infty} K_l\left(\frac{1}{l-1} - \frac{1}{l+1}\right) - \frac{K_{2^s}}{2^s+1} - \frac{D_{2^s}}{2^s} \qquad (s = 0, 1, \ldots;\ x \in [0, 1)). \tag{5}$$

The Kronecker product $w_{n,m}$ $(n, m = 0, 1, \ldots)$ of two Walsh systems is said to be the *two-dimensional Walsh system*. Thus

$$w_{n,m}(x, y) := w_n(x) w_m(y) \qquad (x, y \in [0, 1)).$$

For the two-dimensional Walsh–Fourier coefficients of a function $f \in L^1[0,1)^2$ the same notations will be used as in the one-dimensional case. That is, let

$$\widehat{f}(n, m) := \int_0^1\int_0^1 f(x, y) w_{n,m}(x, y)\, dx\, dy \qquad (n, m = 0, 1, \ldots)$$

and $\widehat{f} := (\widehat{f}(n, m);\ n, m = 0, 1, \ldots)$. Furthermore, let

$$S_{n,m} f := \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \widehat{f}(k, l) w_{k,l} \qquad (n, m = 1, 2, \ldots)$$

be the (n, m)th (rectangular) partial sum of the two-dimensional Walsh–

Fourier series $\sum_{k,l=0}^{\infty,\infty} \widehat{f}(k,l)w_{k,l}$ of $f \in L^1[0,1)^2$. It is easy to show that

$$S_{n,m}f(x,y) = \int_0^1\int_0^1 f(t,u)D_n(x \dot{+} t)D_m(y \dot{+} u)\,dt\,du \qquad (x,y \in [0,1)).$$

In the special case $n = 2^k,\ m = 2^l\ (k,l = 0,1,\dots)$ we have, by (1),

$$S_{2^k,2^l}f(x,y) = 2^{k+l}\int_{I(x,y)} f \qquad (x,y \in [0,1)),$$

where the dyadic rectangle $I(x,y)$ is defined to be the Cartesian product

$$I_{k,l}(x,y) := I_k(x) \times I_l(y).$$

Here $I_j(z)$ $(j = 0,1,\dots;\ z \in [0,1))$ stands for the (unique) dyadic interval

$$I_j(z) := [\nu 2^{-j}, (\nu+1)2^{-j}) \qquad (\nu = 0,\dots,2^j - 1)$$

containing z.

The one-dimensional operator

$$L^1[0,1) \ni f \mapsto \Big(\sum_{n=0}^{\infty} |S_{2^n}f - \sigma_{2^n}f|^2\Big)^{1/2} =: \widetilde{U}f$$

was defined and first investigated by Sunouchi [3], [4]. A simple calculation shows that

$$S_nf - \sigma_nf = \sum_{k=0}^{n-1} \frac{k}{n}\widehat{f}(k)w_k \qquad (n = 1,2,\dots),$$

which leads obviously to the definition of the so-called *two-dimensional Sunouchi operator* U:

$$Uf := \bigg[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\bigg(\sum_{k=0}^{2^n-1}\sum_{l=0}^{2^m-1}\frac{kl}{2^{n+m}}\widehat{f}(k,l)w_{k,l}\bigg)^2\bigg]^{1/2} \qquad (f \in L^1[0,1)^2)$$

(see also Weisz [7]). By the Parseval equality it is clear that U is a bounded operator from $L^2[0,1)^2$ to itself. This result was extended to the L^p-spaces $(1 < p < \infty)$ in the one-dimensional case by Sunouchi [4] and in the two-dimensional case by Weisz [6]. For further comments in this connection see Section 4.

3. Preliminaries. The Hardy spaces play a very important role in Walsh–Fourier analysis, especially in the two-dimensional case. (For details see Weisz [5].) To define them, let $\mathcal{F}_{n,m}$ $(n,m = 0,1,\dots)$ be the σ-algebra generated by the dyadic rectangles $I_{n,m}(x,y)$ $(x,y \in [0,1))$. Hence,

$$\mathcal{F}_{n,m} := \sigma(\{[k2^{-n},(k+1)2^{-n}) \times [l2^{-m},(l+1)2^{-m}) : \\ k=0,\dots,2^n-1;\ l=0,\dots,2^m-1\}),$$

where $\sigma(\mathcal{S})$ denotes the σ-algebra generated by an arbitrary set system $\mathcal{S}$. Then the conditional expectation operator relative to $\mathcal{F}_{n,m}$ is just $S_{2^n,2^m}$. A sequence $f=(f_{n,m};\ n,m=0,1,\dots)$ of integrable functions is said to be a *martingale* if

(i) $f_{n,m}$ is $\mathcal{F}_{n,m}$-measurable for all $n,m=0,1,\dots$ and
(ii) $S_{2^n,2^m}f_{k,l}=f_{n,m}$ for all $n,m,k,l=0,1,\dots$ such that $n\le k$ and $m\le l$.

In other words, for all $n,m=0,1,\dots$ the function $f_{n,m}$ is a two-dimensional Walsh polynomial of the form

$$f_{n,m}=\sum_{k=0}^{2^n-1}\sum_{l=0}^{2^m-1}\alpha_{k,l}w_{k,l}$$

(with suitable real coefficients $\alpha_{k,l}$ independent of n,m). For example, if $f\in L^1[0,1)^2$ then the sequence $(S_{2^n,2^m}f;\ n,m=0,1,\dots)$ is evidently a martingale (called the martingale *generated by* f). Of course, $f_1 := (f_{n,0}, n=0,1,\dots)$ and $f_2 := (f_{0,m}, m=0,1,\dots)$ are (one-dimensional) martingales with respect to the sequence of σ-algebras

$$\sigma(\{[j2^{-k},(j+1)2^{-k}) : j=0,\dots,2^k-1\}) \qquad (k=0,1,\dots).$$

The concept of Walsh–Fourier coefficients can be extended to martingales by setting $\widehat{f}(k,l):=\alpha_{k,l}$ $(k,l=0,1,\dots)$. That is, $\widehat{f}$ will denote the sequence of the Walsh–Fourier coefficients of the function or martingale f.

Let $\|g\|_p := (\int_0^1\int_0^1 |g(x,y)|^p\,dx\,dy)^{1/p}$ $(0<p<\infty)$ be the usual L^p-norm (or quasi-norm) of $g\in L^1[0,1)^2$. We say that a martingale $f=(f_{n,m};n,m=0,1,\dots)$ is L^p*-bounded* if

$$\|f\|_p := \sup_{n,m}\|f_{n,m}\|_p < \infty.$$

The set of L^p-bounded martingales will be denoted by L^p. Thus, if $F\in L^p[0,1)^2$ then it can be seen that the martingale f generated by F belongs to L^p and their L^p-norms are equivalent. This means that there exist positive constants c_p, C_p depending only on p such that $c_p\|f\|_p \le \|F\|_p \le C_p\|f\|_p$. (Also later the symbols c_p, C_p denote such constants, although not always the same at different occurrences.) If $p>1$ then L^p and $L^p[0,1)^2$ can be identified.

The *maximal function* f^* and the *quadratic variation* Qf of a martingale $f=(f_{n,m};\ n,m=0,1,\dots)$ are defined by

$$f^* := \sup_{n,m}|f_{n,m}|$$

and

$$Qf := \Big(\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}|f_{n,m}-f_{n-1,m}-f_{n,m-1}+f_{n-1,m-1}|^2\Big)^{1/2},$$

where $f_{-1,k} := f_{k,-1} := 0$ $(k=-1,0,1,\ldots)$. It can be shown that for each $0<p<\infty$ the norms (or quasi-norms) $\|f^*\|_p$ and $\|Qf\|_p$ are equivalent:

$$c_p\|f^*\|_p \le \|Qf\|_p \le C_p\|f^*\|_p.$$

We introduce the *martingale Hardy spaces* for $0<p<\infty$ as follows: denote by H^p the space of martingales f for which

$$\|f\|_{H^p} := \|f^*\|_p < \infty.$$

By the equivalence $\|f^*\|_p \sim \|Qf\|_p$ we get $\|f\|_{H^p} \sim \|Qf\|_p$. We remark that with the help of the well known Khinchin inequality it is possible to linearize the quadratic variation in the following sense:

$$\begin{aligned}(6)\qquad c_p\|Qf\|_p &\le \int_0^1\!\!\int_0^1 \Big\|\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} r_n(x)r_m(y)\\ &\qquad\times(f_{n,m}-f_{n-1,m}-f_{n,m-1}+f_{n-1,m-1})\Big\|_p\,dx\,dy\\ &\le C_p\|Qf\|_p \qquad (0<p\le 1)\end{aligned}$$

(for details see Simon [2]).

The atomic decomposition of martingales is a useful characterization in the theory of some Hardy spaces. Unfortunately, in two dimensions this characterization is much more complicated. Indeed, in the two-dimensional case the support of an atom is not a dyadic rectangle but an open set. However, a finer atomic decomposition can be given, that is, the atoms can be decomposed into elementary rectangle particles (see Weisz [7]). This makes it possible in some investigations to examine only atoms supported on dyadic rectangles. To this end, let $0<p\le 1$. A function $a\in L^2[0,1)^2$ is called a *rectangle p-atom* if either a is identically equal to 1 or there exists a dyadic rectangle I such that

$$(7)\qquad \begin{aligned}&\operatorname{supp} a\subset I, \qquad \|a\|_2\le |I|^{1/2-1/p},\\ &\int_0^1 a(x,t)\,dt=\int_0^1 a(u,y)\,du=0 \qquad (x,y\in[0,1)),\end{aligned}$$

where $|I|$ is the (two-dimensional) Lebesgue measure of I. We say that a is *supported on I*. Although the elements of H^p cannot be decomposed into rectangle p-atoms, in the investigations of the so-called p-quasi-local operators it is enough to take such atoms.

To define the quasi-locality let $\mathcal{M}$ be the set of all martingales defined above and T be a mapping from $\mathcal{M}$ to itself. Assume that T is sublinear and bounded from L^2 into L^2 (see also Simon [2]). Then T is called *p-quasi-local* if there exists $\delta > 0$ such that for every rectangle p-atom a supported on the dyadic rectangle I and for all $r = 0, 1, \ldots$ one has

$$\int_{[0,1)^2 \setminus I^r} |Ta|^p \le C_p 2^{-\delta r}. \tag{8}$$

Here I^r is the dyadic rectangle defined as follows: $I^r := I_1^r \times I_2^r$, where $I = I_1 \times I_2$ for some dyadic intervals I_1, I_2, and I_j^r is the (unique) dyadic interval for which $I_j \subset I_j^r$ and the ratio of the lengths of I_j^r and I_j is equal to 2^r $(j = 1, 2)$. Then a simple modification of a theorem of Weisz [7] says that for T to be bounded from H^p into L^p it is enough that T be p-quasi-local. Hence, in this case $\|Tf\|_p \le C_p \|f\|_{H^p}$ $(f \in H^p)$.

Let $x, y \in [0, 1)$ and

$$R_{x,y} f := \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} r_n(x) r_m(y) \times (f_{n,m} - f_{n-1,m} - f_{n,m-1} + f_{n-1,m-1}) \qquad (f \in H^p).$$

If $TR_{x,y} = R_{x,y} T$ for all $x, y \in [0, 1)$, then T is also bounded from H^p to itself. Indeed, by (6), for every $f \in H^p$ we get

$$\|Tf\|_{H^p} \le C_p \int_0^1 \int_0^1 \|T(R_{x,y} f)\|_p \, dx \, dy \le C_p \int_0^1 \int_0^1 \|R_{x,y} f\|_{H^p} \, dx \, dy \le C_p \|f\|_{H^p}.$$

Furthermore, if T is invertible and its inverse is bounded from H^p to H^p, then Tf can be estimated in H^p norm from below: $\|f\|_{H^p} = \|T^{-1}(Tf)\|_{H^p} \le C_p \|Tf\|_{H^p}$. Moreover, $\|Tf\|_{H^p}$ is equivalent to $\|f\|_{H^p}$ $(f \in H^p)$.

4. Results. In this work we investigate multiplier operators $T := T_\lambda$, i.e. a bounded sequence $\lambda = (\lambda_{k,l};\ k, l = 0, 1, \ldots)$ of real numbers is given and $\widehat{T_\lambda f} = \lambda \widehat{f}$ $(f \in \mathcal{M})$. The boundedness of λ and the well known Parseval equality imply that T_λ is obviously bounded from L^2 into L^2.

Let $0 < p \le 1$. If T_λ is p-quasi-local, then by our previous remarks $T_\lambda : H^p \to H^p$ is bounded. Moreover, in the case $\inf_{k,l} |\lambda_{k,l}| > 0$ the inverse T_λ^{-1} of T_λ is bounded from L^2 into L^2. Consequently, the p-quasi-locality of T_λ^{-1} is enough for $T_\lambda^{-1} : H^p \to H^p$ to be bounded. This leads to the equivalence $\|T_\lambda f\|_{H^p} \sim \|f\|_{H^p}$.

Let $T_\lambda f$ be written in the following form:

$$T_\lambda f = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \lambda_{j,k}\widehat{f}(j,k)w_{j,k} = \sum_{n=-1}^{\infty}\sum_{m=-1}^{\infty} \Lambda_{n,m}^{(\lambda)} * f,$$

where $\Lambda_{-1,-1}^{(\lambda)} * f := \lambda_{0,0}\widehat{f}(0,0)w_{0,0}$,

$$\Lambda_{-1,m}^{(\lambda)} * f := \sum_{k=2^m}^{2^{m+1}-1} \lambda_{0,k}\widehat{f}(0,k)w_{0,k},$$

$$\Lambda_{n,-1}^{(\lambda)} * f := \sum_{j=2^n}^{2^{n+1}-1} \lambda_{j,0}\widehat{f}(j,0)w_{j,0}$$

$$\Lambda_{n,m}^{(\lambda)} * f := \sum_{j=2^n}^{2^{n+1}-1}\sum_{k=2^m}^{2^{m+1}-1} \lambda_{j,k}\widehat{f}(j,k)w_{j,k} \qquad (n,m = 0,1,\ldots).$$

Consider the sequence df of functions defined by

$$df := (\Lambda_{n-1,m-1}^{(\lambda)} * f;\ n,m = 0,1,\ldots).$$

Then $Q(T_\lambda f)(x,y) = \|df(x,y)\|_{\ell_2}$ for all $x,y \in [0,1)$. If ℓ denotes the set of two-dimensional real sequences and $\delta : \ell \to \ell$ is a map satisfying the ℓ_2-boundedness condition $\|\delta(u)\|_{\ell_2} \le C_\delta \|u\|_{\ell_2}$ ($u \in \ell, C_\delta > 0$ is a constant depending only on δ), then define

$$\Delta f(x,y) := \delta(df(x,y)) \qquad (x,y \in [0,1)).$$

Since $\|\Delta f(x,y)\|_{\ell_2} \le C_\delta \|df(x,y)\|_{\ell_2} \le C_\delta Q(T_\lambda f)(x,y)$ $(x,y \in [0,1))$, the operator A defined by

$$Af(x,y) := \|\Delta f(x,y)\|_{\ell_2} \qquad (f \in H^p,\ x,y \in [0,1))$$

satisfies the estimate

$$\|Af\|_p \le C_p \|T_\lambda f\|_{H^p} \qquad (f \in H^p).$$

Furthermore, if δ is invertible and its inverse δ^{-1} is ℓ_2-bounded, then $df(x,y) = \delta^{-1}(\Delta f(x,y))$ $(x,y \in [0,1))$, i.e.

$$Q(T_\lambda f)(x,y) = \|df(x,y)\|_{\ell_2} \le C_{\delta^{-1}} \|\Delta f(x,y)\|_{\ell_2} \le C_{\delta^{-1}} Af \qquad (f \in H^p).$$

This implies the estimate

$$\|f\|_{H^p} \le C_p \|Af\|_p \qquad (f \in H^p),$$

that is, $\|f\|_{H^p} \sim \|Af\|_p$. For example, let $\bullet$ be the usual convolution in ℓ and, for a fixed sequence $b \in \ell_1$ consider

$$\delta(u) := u \bullet b \qquad (u \in \ell).$$

Then $\|\delta(u)\|_{\ell_2} \le \|b\|_{\ell_1} \|u\|_{\ell_2}$ $(u \in \ell)$, i.e. δ is ℓ_2-bounded.

By a special choice of b and λ we get the Sunouchi operator U as follows. Let b and λ be defined in the following way:

$$b_{n,m} := \frac{1}{2^{n+m+2}},$$

$$\lambda_{0,0} := 1, \quad \lambda_{i,j} := \frac{ij}{2^{n+m}}, \quad \lambda_{i,0} := i2^{-n}, \quad \lambda_{0,j} := j2^{-m},$$

where $2^n \leq i < 2^{n+1}$, $2^m \leq j < 2^{m+1}$ $(n,m = 0,1,\ldots)$. Hence, for $f \in H^p$, $i,l = 1,2,\ldots,$

$$\Lambda^{(\lambda)}_{-1,-1} * f = \widehat{f}(0,0)w_{0,0},$$

$$\Lambda^{(\lambda)}_{-1,l-1} * f = 2^{1-l} \sum_{j=2^{l-1}}^{2^l-1} \widehat{f}(0,j)jw_{0,j},$$

$$\Lambda^{(\lambda)}_{i-1,-1} * f = 2^{1-i} \sum_{k=2^{i-1}}^{2^i-1} \widehat{f}(k,0)kw_{k,0},$$

$$\Lambda^{(\lambda)}_{i-1,l-1} * f = 2^{-i-l-2} \sum_{k=2^{i-1}}^{2^i-1} \sum_{j=2^{l-1}}^{2^l-1} \widehat{f}(k,j)kjw_{k,j}$$

and the sequence $\Delta f = ((\Delta f)_{n,m};\ n,m = 0,1,\ldots)$ is the following:

$$\begin{aligned}
(\Delta f)_{n,m} &= \sum_{i=0}^{n} \sum_{l=0}^{m} 2^{-n-m+i+l-2} \Lambda^{(\lambda)}_{i-1,l-1} * f \\
&= 2^{-n-m-2}\widehat{f}(0,0) + 2^{-n-m-1} \sum_{l=1}^{m} \sum_{j=2^{l-1}}^{2^l-1} j\widehat{f}(0,j)w_{0,j} \\
&\quad + 2^{-n-m-1} \sum_{i=1}^{n} \sum_{k=2^{i-1}}^{2^i-1} k\widehat{f}(k,0)w_{k,0} \\
&\quad + 2^{-n-m} \sum_{i=1}^{n} \sum_{l=1}^{m} \sum_{k=2^{i-1}}^{2^i-1} \sum_{j=2^{l-1}}^{2^l-1} kj\widehat{f}(k,j)w_{k,j} \\
&= 2^{-n-m-2}\widehat{f}(0,0) + 2^{-n-m-1} \sum_{j=1}^{2^m-1} j\widehat{f}(0,j)w_{0,j} \\
&\quad + 2^{-n-m-1} \sum_{k=1}^{2^n-1} k\widehat{f}(k,0)w_{k,0} + 2^{-n-m} \sum_{k=1}^{2^n-1} \sum_{j=1}^{2^m-1} kj\widehat{f}(k,j)w_{k,j}.
\end{aligned}$$

It follows that

$$c(Uf - |\widehat{f}(0,0)| - \widetilde{U}f_1 - \widetilde{U}f_2) \leq Af \leq C(|\widehat{f}(0,0)| + \widetilde{U}f_1 + \widetilde{U}f_2 + Uf),$$

where $f_1 := (f_{0,m}, m = 0,1,\ldots)$, $f_2 := (f_{n,0}, n = 0,1,\ldots)$ and c, C are positive constants independent of f. Recall that $\|\widetilde{U} f_j\|_{H^p} \le C_p \|f\|_{H^p}$ $(j = 1,2)$ (see the one-dimensional case in Simon [2]). We will prove

THEOREM. *Let* λ *be defined as above and* $0 < p \le 1$. *Then* $T_\lambda : H^p \to H^p$ *is bounded. Moreover, if* $1/2 < p \le 1$, *then* $T_{1/\lambda} : H^p \to H^p$ *is bounded.*

On account of our previous remarks the first part of the Theorem implies

COROLLARY 1. *For all* $0 < p \le 1$ *there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$\|Uf\|_p \le C_p \|f\|_{H^p} \qquad (f \in H^p).$$

This improves a result of Weisz [7]. More specifically, he proved the same statement (by another argument) assuming $2/3 < p \le 1$.

A simple calculation shows that the mapping $\ell \ni u \mapsto b \bullet u \in \ell$ is a bijection and its inverse is $\ell \ni u \mapsto \widetilde{b} \bullet u \in \ell$ with the sequence $\widetilde{b}$ given by

$$\widetilde{b}_{n,m} := \begin{cases} 4 & (n = m = 0), \\ -2 & (n = 1, m = 0 \text{ or } n = 0, m = 1), \\ 1 & (n = m = 1), \\ 0 & (\text{for other } n, m = 0,1,\ldots). \end{cases}$$

This means that from the second part of the Theorem we get

COROLLARY 2. *If* $1/2 < p \le 1$, *then there exists a constant* $C_p > 0$ *depending only on* p *such that*

$$\|f\|_{H^p} \le C_p \| |\widehat{f}(0,0)| + \widetilde{U} f_1 + \widetilde{U} f_2 + Uf\|_p \qquad (f \in H^p).$$

Of course, for some martingales f the norms $\|f\|_{H^p}$ and $\|Uf\|_p$ are equivalent, that is, if $f_{0,m} = f_{n,0} = 0$ $(n, m = 0,1,\ldots)$, then

$$c_p \|f\|_{H^p} \le \|Uf\|_p \le C_p \|f\|_{H^p} \qquad (f \in H^p).$$

5. Proof of the Theorem. Let $0 < p \le 1$. We prove the boundedness of T_λ. It is enough to show that T_λ is p-quasi-local, i.e. (8) is true for all rectangle p-atoms a supported on I. Without loss of generality it can be assumed that

$$I = [0, 2^{-N}) \times [0, 2^{-M})$$

for some $N, M = 0,1,\ldots$ Let $r = 0,1,\ldots$ Then

$$\int_{[0,1)^2 \setminus I^r} |T_\lambda a|^p \le \sum_{i=1}^{4} \int_{A_i} |T_\lambda a|^p,$$

where

$$A_1 := [2^{-N+r}, 1) \times [0, 2^{-M}), \qquad A_2 := [2^{-N}, 1) \times [2^{-M+r}, 1),$$
$$A_3 := [0, 2^{-N}) \times [2^{-M+r}, 1), \qquad A_4 := [2^{-N+r}, 1) \times [2^{-M}, 1).$$

We will show that

$$\int_{A_i} |T_\lambda a|^p \le C_p 2^{-r\delta} \qquad (i = 1,2,3,4) \tag{9}$$

with a suitable positive δ independent of a and r. It is clear that the proof for $i = 3$ and 4 is the same as for $i = 1$ and 2, respectively. Consequently, we give details for $i = 1$ and $i = 2$ only.

First we examine the case $i = 1$. By the definition of the rectangle p-atom (see (7)) we have

$$\widehat{a}(n,m) = 0 \tag{10}$$

if $n < 2^N$ or $m < 2^M$. Therefore

$$T_\lambda a = \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} \sum_{k=2^i}^{2^{i+1}-1} \sum_{l=2^j}^{2^{j+1}-1} \frac{kl}{2^{i+j}} \widehat{a}(k,l) w_{k,l},$$

i.e.

$$\begin{aligned}\int_{A_1} |T_\lambda a|^p &= \int_{2^{-N+r}}^{1} \int_{0}^{2^{-M}} |T_\lambda a|^p \\ &\le \int_{2^{-N+r}}^{1} \int_{0}^{2^{-M}} \sum_{i=N}^{\infty} \left| \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} \sum_{j=M}^{\infty} \sum_{l=2^j}^{2^{j+1}-1} \frac{l}{2^j} \widehat{a}(k,l) w_{k,l} \right|^p \\ &= \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \int_{0}^{2^{-M}} \left| \int_0^{2^{-N}} \int_0^{2^{-M}} a(s,t) \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} w_k(x \dot{+} s) \right. \\ &\qquad \left. \times \sum_{j=M}^{\infty} \sum_{l=2^j}^{2^{j+1}-1} \frac{l}{2^j} w_l(y \dot{+} t) \, ds \, dt \right|^p dy \, dx.\end{aligned}$$

Using Hölder's inequality we conclude that

$$\begin{aligned}\int_{A_1} |T_\lambda a|^p &\le 2^{-M(1-p)} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \left(\int_0^{2^{-M}} \left| \int_0^{2^{-N}} \int_0^{2^{-M}} a(s,t) \right. \right. \\ &\qquad \left. \left. \times \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} w_k(x \dot{+} s) \sum_{j=M}^{\infty} \sum_{l=2^j}^{2^{j+1}-1} \frac{l}{2^j} w_l(y \dot{+} t) \, ds \, dt \right| dy \right)^p dx \\ &\le 2^{-M(1-p)} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \left(\int_0^{2^{-N}} \int_0^{2^{-M}} \left| \int_0^{2^{-M}} a(s,t) \right. \right. \\ &\qquad \left. \left. \times \sum_{j=M}^{\infty} \sum_{l=2^j}^{2^{j+1}-1} \frac{l}{2^j} w_l(y \dot{+} t) \, dt \right| dy \left| \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} w_k(x \dot{+} s) \right| ds \right)^p dx.\end{aligned}$$

It follows by Cauchy's inequality that

$$\int_{A_1} |T_\lambda a|^p \le 2^{-M(1-p)} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \Big(\int_0^{2^{-N}} 2^{-M/2} \Big[\int_0^1 \Big| \int_0^{2^{-M}} a(s,t)$$

$$\times \sum_{j=M}^{\infty} \sum_{l=2^j}^{2^{j+1}-1} \frac{l}{2^j} w_l(y \dot{+} t)\, dt \Big|^2 dy \Big]^{1/2} \Big| \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} w_k(x \dot{+} s) \Big| ds \Big)^p dx$$

$$\le 2^{-M(1-p/2)} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \Big(\int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2\, dt \Big]^{1/2}$$

$$\times \Big| \sum_{k=2^i}^{2^{i+1}-1} \frac{k}{2^i} w_k(x \dot{+} s) \Big| ds \Big)^p dx.$$

Now, applying the formulas (1)–(3) we obtain

$$\int_{A_1} |T_\lambda a|^p \le 2^{-M(1-p/2)} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} \Big(\int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2\, dt \Big]^{1/2}$$

$$\times \sum_{l=0}^{i} 2^{l-i-1} D_{2^i}(x \dot{+} s \dot{+} 2^{-l-1})\, ds \Big)^p dx$$

$$\le 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-(i+1)p}$$

$$\times \int_{2^{-N+r}}^{1} \Big(\sum_{l=0}^{N-r-1} 2^l \int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2\, dt \Big]^{1/2}$$

$$\times D_{2^i}(x \dot{+} s \dot{+} 2^{-l-1})\, ds \Big)^p dx$$

$$\le 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-(i+1)p}$$

$$\times \sum_{l=0}^{N-r-1} 2^{pl} \int_{2^{-N+r}}^{1} \Big(\int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2\, dt \Big]^{1/2}$$

$$\times D_{2^i}(x \dot{+} s \dot{+} 2^{-l-1}) ds \Big)^p dx$$

$$= 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-(i+1)p}$$

$$\times \sum_{l=0}^{N-r-1} 2^{pl} \int\limits_{2^{-l-1}}^{2^{-l-1}+2^{-N}} \Big(\int\limits_0^{2^{-N}} \Big[\int\limits_0^1 |a(s,t)|^2 \, dt \Big]^{1/2}$$

$$\times D_{2^i}(x \dot{+} s \dot{+} 2^{-l-1}) \, ds \Big)^p \, dx$$

$$\leq 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-(i+1)p}$$

$$\times \sum_{l=0}^{N-r-1} 2^{pl} \int\limits_{2^{-l-1}}^{2^{-l-1}+2^{-N}} \Big(\Big[\int\limits_0^{2^{-N}} \int\limits_0^1 |a(s,t)|^2 \, dt \, ds \Big]^{1/2}$$

$$\times \Big[\int\limits_0^{2^{-N}} D_{2^i}^2 (x \dot{+} s \dot{+} 2^{-l-1}) \, ds \Big]^{1/2} \Big)^p \, dx$$

$$= 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-(i+1)p} \|a\|_2^p \sum_{l=0}^{N-r-1} 2^{pl-N} 2^{ip/2}$$

$$\leq 2^{-M(1-p/2)} 2^{-(N+M)(p/2-1)} \sum_{i=N}^{\infty} 2^{-(i/2+1)p-N} \sum_{l=0}^{N-r-1} 2^{pl}$$

$$\leq C_p 2^{-Np/2} 2^{-Np/2} 2^{(N-r)p} = C_p 2^{-rp}.$$

Hence, (9) is true for $i = 1$ with $\delta := p$.

To show (9) for $i = 2$ we refer to (10) and to the definition (7) of the atoms, which gives

$$\int\limits_{A_2} |T_\lambda a|^p = \int\limits_{2^{-N}}^1 \int\limits_{2^{-M+r}}^1 |T_\lambda a|^p$$

$$\leq \sum_{i=N}^{\infty} \sum_{j=M}^{\infty} 2^{N+M} \int\limits_{2^{-N}}^1 \int\limits_{2^{-M+r}}^1 \Big(\int\limits_0^{2^{-N}} \int\limits_0^{2^{-M}} \Big| \sum_{k=2^i}^{2^{i+1}-1} \frac{k w_k (x \dot{+} s)}{2^i} \Big|$$

$$\times \Big| \sum_{l=2^j}^{2^{j+1}-1} \frac{l w_l (y \dot{+} t)}{2^j} \Big| \, ds \, dt \Big)^p \, dx \, dy$$

$$\leq \Big[\sum_{i=N}^{\infty} 2^N \int\limits_{2^{-N}}^1 \Big(\int\limits_0^{2^{-N}} \Big| \sum_{k=2^i}^{2^{i+1}-1} \frac{k w_k (x \dot{+} s)}{2^i} \Big| \, ds \Big)^p \, dx \Big]$$

$$\times \Big[\sum_{j=M}^{\infty} 2^M \int\limits_{2^{-M+r}}^1 \Big(\int\limits_0^{2^{-M}} \Big| \sum_{l=2^j}^{2^{j+1}-1} \frac{l w_l (y \dot{+} t)}{2^j} \Big| \, dt \Big)^p \, dy \Big] =: AB.$$

As in the proof for $i = 1$ we get

$$\begin{aligned} A &\le \sum_{i=N}^{\infty} 2^N \int_{2^{-N}}^{1} \Big(\int_{0}^{2^{-N}} K_{2^i}(x \dotplus s)\, ds\Big)^p dx \\ &\le \sum_{i=N}^{\infty} 2^{N-p(i+1)} \int_{2^{-N}}^{1} \Big(\int_{0}^{2^{-N}} \sum_{l=0}^{N-1} 2^{l-1} D_{2^i}(x \dotplus s \dotplus 2^{-l-1})\, ds\Big)^p dx \\ &\le \sum_{i=N}^{\infty} 2^{N-p(i+1)} \int_{2^{-N}}^{1} \sum_{l=0}^{N-1} 2^{p(l-1)} \Big(\int_{0}^{2^{-N}} D_{2^i}(x \dotplus s \dotplus 2^{-l-1})\, ds\Big)^p dx \\ &= \sum_{i=N}^{\infty} 2^{N-p(i+1)} \sum_{l=0}^{N-1} 2^{p(l-1)} \\ &\quad \times \int_{2^{-l-1}}^{2^{-l-1}+2^{-N}} \Big(\int_{0}^{2^{-N}} D_{2^i}(x \dotplus s \dotplus 2^{-l-1})\, ds\Big)^p dx \\ &= \sum_{i=N}^{\infty} 2^{N-p(i+1)} \sum_{l=0}^{N-1} 2^{p(l-1)-N} \le C_p. \end{aligned}$$

For B the proof is similar. The only difference is that we have to write the sum $\sum_{l=0}^{M-r-1}$ instead of $\sum_{l=0}^{N-1}$ (and, of course, M instead of N). Therefore

$$B \le \sum_{j=M}^{\infty} 2^{M-p(i+1)} \sum_{l=0}^{M-r-1} 2^{p(l-1)-M} \le C_p 2^{-rp}.$$

This completes the proof of (9) with $\delta := p$, that is, the Theorem is true for T_λ.

The proof for $T_{1/\lambda}$ is much more complicated. Assume $1/2 < p \le 1$. As above it is enough to prove (9) for $i = 1, 2$ and for $T_{1/\lambda}$ instead of T_λ. First we consider the case $i = 1$. Let a be a rectangular p-atom supported on $[0, 2^{-N}) \times [0, 2^{-M})$ for some $N, M = 0, 1, \dots$ and let $r = 0, 1, \dots$ Then—as in the proof for T_λ—we get

$$\begin{aligned} \int_{A_1} |T_{1/\lambda} a|^p \le 2^{-M(1-p/2)} \sum_{i=N}^{\infty} \int_{2^{-N+r}}^{1} &\Big(\int_{0}^{2^{-N}} \Big[\int_{0}^{1} |a(s,t)|^2\, dt \Big]^{1/2} \\ &\times \Big| \sum_{k=2^i}^{2^{i+1}-1} \frac{2^i}{k} w_k(x \dotplus s) \Big| ds \Big)^p dx, \end{aligned}$$

from which by the formulas (1) and (5) it follows that

$$\int\limits_{A_1} |T_{1/\lambda} a|^p \le 2^{-M(1-p/2)}$$

$$\times \sum_{i=N}^{\infty} \int\limits_{2^{-N+r}}^{1} \Big(\int\limits_0^{2^{-N}} \Big[\int\limits_0^1 |a(s,t)|^2 \, dt \Big]^{1/2} K_{2^i}(x \dot+ s) \, ds \Big)^p \, dx$$

$$+ 2^{-M(1-p/2)} \sum_{i=N}^{\infty} \int\limits_{2^{-N+r}}^{1} \Big(\int\limits_0^{2^{-N}} \Big[\int\limits_0^1 |a(s,t)|^2 \, dt \Big]^{1/2}$$

$$\times K_{2^{i+1}}(x \dot+ s) \, ds \Big)^p \, dx$$

$$+ 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{ip} \int\limits_{2^{-N+r}}^{1} \Bigg(\int\limits_0^{2^{-N}} \Big[\int\limits_0^1 |a(s,t)|^2 \, dt \Big]^{1/2}$$

$$\times \Bigg| \sum_{l=2^i+1}^{2^{i+1}-1} K_l(x \dot+ s) \Big(\frac{1}{l-1} - \frac{1}{l+1} \Big) \Bigg| \, ds \Bigg)^p \, dx$$

$$=: \Sigma^{(1)} + \Sigma^{(2)} + \Sigma^{(3)}.$$

Taking into account (1), (4) and the proof for T_λ we obtain

$$\Sigma^{(i)} \le C_p 2^{-rp} \qquad (i = 1, 2).$$

Furthermore, (1) and (4) imply that

$$\Sigma^{(3)} \le C_p 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{ip} \int\limits_{2^{-N+r}}^{1} \Big(\int\limits_0^{2^{-N}} \Big[\int\limits_0^1 |a(s,t)|^2 \, dt \Big]^{1/2}$$

$$\times 2^{-i} \sum_{\nu=0}^{i} 2^{\nu-i} \sum_{m=\nu}^{i} (D_{2^m}(x \dot+ s) + D_{2^m}(x \dot+ s \dot+ 2^{-\nu-1})) \, ds \Big)^p \, dx$$

$$\le C_p 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-ip} \int\limits_{2^{-N+r}}^{1} \Big(\int\limits_0^{2^{-N}} \Big[\int\limits_0^1 |a(s,t)|^2 \, dt \Big]^{1/2}$$

$$\times \sum_{m=0}^{i} \sum_{\nu=0}^{m} 2^{\nu} (D_{2^m}(x \dot+ s) + D_{2^m}(x \dot+ s \dot+ 2^{-\nu-1})) \, ds \Big)^p \, dx$$

$$\le C_p 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-ip} \int\limits_{2^{-N+r}}^{1} \Big(\int\limits_0^{2^{-N}} \Big[\int\limits_0^1 |a(s,t)|^2 \, dt \Big]^{1/2}$$

$$\times \Big(\sum_{m=0}^{N-r-1} 2^m D_{2^m}(x \dot+ s) + \sum_{m=0}^{N-r-1} \sum_{\nu=0}^{m} 2^{\nu} D_{2^m}(x \dot+ s \dot+ 2^{-\nu-1})$$

$$
\begin{aligned}
&+\sum_{m=N-r}^{i}\sum_{\nu=0}^{m}2^{\nu}D_{2^m}(x\dot{+}s\dot{+}2^{-\nu-1})\Big)\,ds\Big)^p\,dx\\
&\le C_p2^{-M(1-p/2)}\sum_{i=N}^{\infty}2^{-ip}\int\limits_{2^{-N+r}}^{1}\Big(\int\limits_{0}^{2^{-N}}\Big[\int\limits_0^1|a(s,t)|^2\,dt\Big]^{1/2}\\
&\quad\times\sum_{m=0}^{N-r-1}2^mD_{2^m}(x\dot{+}s)\,ds\Big)^p\,dx\\
&\quad+C_p2^{-M(1-p/2)}\sum_{i=N}^{\infty}2^{-ip}\int\limits_{2^{-N+r}}^{1}\Big(\int\limits_{0}^{2^{-N}}\Big[\int\limits_0^1|a(s,t)|^2\,dt\Big]^{1/2}\\
&\quad\times\sum_{m=0}^{N-r-1}\sum_{\nu=0}^{m}2^{\nu}D_{2^m}(x\dot{+}s\dot{+}2^{-\nu-1})\,ds\Big)^p\,dx\\
&\quad+C_p2^{-M(1-p/2)}\sum_{i=N}^{\infty}2^{-ip}\int\limits_{2^{-N+r}}^{1}\Big(\int\limits_{0}^{2^{-N}}\Big[\int\limits_0^1|a(s,t)|^2\,dt\Big]^{1/2}\\
&\quad\times\sum_{m=N-r}^{i}\sum_{\nu=0}^{m}2^{\nu}D_{2^m}(x\dot{+}s\dot{+}2^{-\nu-1})\,ds\Big)^p\,dx\\
&=:\Sigma^{(31)}+\Sigma^{(32)}+\Sigma^{(33)}.
\end{aligned}
$$

For $\Sigma^{(31)}$ it follows that

$$
\begin{aligned}
\Sigma^{(31)}&\le C_p2^{-M(1-p/2)}\sum_{i=N}^{\infty}2^{-ip}\sum_{m=0}^{N-r-1}2^{pm}\\
&\quad\times\int\limits_{2^{-N+r}}^{1}\Big(\int\limits_{0}^{2^{-N}}\Big[\int\limits_0^1|a(s,t)|^2\,dt\Big]^{1/2}D_{2^m}(x\dot{+}s)\,ds\Big)^p\,dx\\
&\le C_p2^{-M(1-p/2)}\|a\|_2^p\sum_{i=N}^{\infty}2^{-ip}\sum_{m=0}^{N-r-1}2^{pm}\\
&\quad\times\sum_{l=2^r}^{2^{N-m}-1}\int\limits_{l2^{-N}}^{(l+1)2^{-N}}\Big(\int\limits_0^{2^{-N}}D_{2^m}^2(x\dot{+}s)\,ds\Big)^{p/2}\,dx\\
&\le C_p2^{-M(1-p/2)}2^{-(N+M)(p/2-1)}2^{-pN}\\
&\quad\times\sum_{m=0}^{N-r-1}2^{pm}\sum_{l=2^r}^{2^{N-m}-1}2^{-N}(2^{2m-N})^{p/2}
\end{aligned}
$$

$$\begin{aligned}&\le C_p 2^{-Np/2} 2^N 2^{-pN} \sum_{m=0}^{N-r-1} 2^{2pm} 2^{-N} 2^{N-m} 2^{-Np/2}\\&= C_p 2^{-2pN} 2^N 2^{(2p-1)(N-r)}\\&= C_p 2^{-r(2p-1)}.\end{aligned}$$

Now, we estimate $\Sigma^{(32)}$ as follows:

$$\begin{aligned}\Sigma^{(32)} &\le C_p 2^{-M(1-p/2)-pN} \int_{2^{-N+r}}^{1} \Big(\int_{0}^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2\,dt \Big]^{1/2}\\&\quad\times \sum_{\nu=0}^{N-r-1} 2^{\nu} \sum_{m=\nu}^{N-r-1} D_{2^m}(x \dotplus s \dotplus 2^{-\nu-1})\,ds \Big)^p dx\\&\le C_p 2^{-M(1-p/2)-pN} \sum_{\nu=0}^{N-r-1} 2^{\nu p} \int_{2^{-N+r}}^{1} \Big(\int_0^{2^{-N}} \Big[\int_0^1 |a(s,t)|^2\,dt \Big]^{1/2}\\&\quad\times \sum_{m=\nu}^{N-r-1} D_{2^m}(x \dotplus s \dotplus 2^{-\nu-1})\,ds \Big)^p dx\\&\le C_p 2^{-M(1-p/2)-pN} \|a\|_2^p \sum_{\nu=0}^{N-r-1} 2^{\nu p}\\&\quad\times \int_{2^{-N+r}}^{1} \Big(\int_0^{2^{-N}} \Big(\sum_{m=\nu}^{N-r-1} D_{2^m}(x \dotplus s \dotplus 2^{-\nu-1}) \Big)^2 ds \Big)^{p/2} dx\\&= C_p 2^{-M(1-p/2)-pN} \|a\|_2^p \sum_{\nu=0}^{N-r-1} 2^{\nu p}\\&\quad\times \Big(\sum_{b=0}^{N-r-\nu-2} \int_{2^{b-N+r}}^{b-N+r+1} \Big(\int_0^{2^{-N}} D^2_{2^\nu}(x \dotplus s \dotplus 2^{-\nu-1}) \Big)\,ds \Big)^{p/2} dx\\&\quad+ \int_{2^{-\nu-1}}^{2^{-\nu}} \Big(\int_0^{2^{-N}} \Big(\sum_{m=\nu}^{N-r-1} D_{2^m}(x \dotplus s \dotplus 2^{-\nu-1}) \Big)^2 ds \Big)^{p/2} dx \Big)\\&=: \Sigma_1^{(32)} + \Sigma_2^{(32)},\end{aligned}$$

where

$$\begin{aligned}\Sigma_2^{(32)} &\le C_p 2^{-M(1-p/2)-pN} 2^{-(N+M)(p/2-1)}\\&\quad\times \sum_{\nu=0}^{N-r-1} 2^{\nu p} \sum_{b=0}^{N-r-\nu-2} 2^{b-N+r} (2^{2\nu-N})^{p/2}\end{aligned}$$

$$\le C_p 2^{-pN-N(p/2-1)} \sum_{\nu=0}^{N-r-1} 2^{2\nu p} 2^{-Np/2-N+r} 2^{N-r-\nu}$$
$$\le C_p 2^{-2pN+N} 2^{(2p-1)(N-r)}$$
$$= C_p 2^{-r(2p-1)}.$$

The analogous estimate for $\Sigma_2^{(32)}$ can be verified in the following way:

$$\Sigma_2^{(32)} \le C_p 2^{-M(1-p/2)-pN} \|a\|_2^p \sum_{\nu=0}^{N-r-1} 2^{\nu p}$$
$$\times \int_0^{2^{-\nu}} \Big(\int_0^{2^{-N}} \Big(\sum_{m=\nu}^{N-r-1} D_{2^m}(x \dotplus s) \Big)^2 ds \Big)^{p/2} dx$$
$$= C_p 2^{-M(1-p/2)-pN} \|a\|_2^p \sum_{\nu=0}^{N-r-1} 2^{\nu p}$$
$$\times \Big(\int_0^{2^{-N+r}} \Big(\int_0^{2^{-N}} \Big(\sum_{m=\nu}^{N-r-1} D_{2^m}(x \dotplus s) \Big)^2 ds \Big)^{p/2} dx$$
$$+ \sum_{d=1}^{N-r-\nu} \int_{2^{-\nu-d}}^{2^{-\nu-d+1}} \Big(\int_0^{2^{-N}} \Big(\sum_{m=\nu}^{\nu+d-1} D_{2^m}(x \dotplus s) \Big)^2 ds \Big)^{p/2} dx \Big)$$
$$\le C_p 2^{-M(1-p/2)-pN} \|a\|_2^p \sum_{\nu=0}^{N-r-1} 2^{\nu p}$$
$$\times \Big(2^{-N+r} 2^{-Np/2} 2^{(N-r)p} + \sum_{d=1}^{N-\nu-r} 2^{-\nu-d} 2^{-Np/2} 2^{p(\nu+d)} \Big)$$
$$= C_p 2^{-M(1-p/2)-2pN} \|a\|_2^p \sum_{\nu=0}^{N-r-1} 2^{(2p-1)\nu}$$
$$\times \Big(2^{(N-r)(p-1)} + \sum_{d=1}^{N-r-\nu} 2^{(p-1)d} \Big)$$
$$\le C_p 2^{-M(1-p/2)-2pN} 2^{-(N+M)(p/2-1)}$$
$$\times \sum_{\nu=0}^{N-r-1} (N-r-\nu) 2^{(2p-1)\nu}$$
$$\le C_p 2^{-2pN-N(p/2-1)} 2^{(2p-1)(N-r)}$$
$$\le C_p 2^{-r(2p-1)}.$$

To complete the proof for $i = 1$ we have to estimate $\Sigma^{(33)}$:

$$\Sigma^{(33)} \leq C_p 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-ip}$$
$$\times \sum_{m=N-r}^{i} \int_{2^{-N+r}}^{1} \Big(\int_{0}^{2^{-N}} \Big[\int_{0}^{1} |a(s,t)|^2 \, dt \Big]^{1/2}$$
$$\times \sum_{\nu=0}^{m} 2^{\nu} D_{2^m}(x \dotplus s \dotplus 2^{-\nu-1}) \, ds \Big)^p \, dx$$
$$= C_p 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-ip}$$
$$\times \sum_{m=N-r}^{i} \sum_{l=0}^{N-r-1} \int_{2^{-l-1}}^{2^{-l-1}+2^{-m}} \Big(\int_{0}^{2^{-N}} \Big[\int_{0}^{1} |a(s,t)|^2 \, dt \Big]^{1/2}$$
$$\times 2^l D_{2^m}(x \dotplus s \dotplus 2^{-\nu-1}) \, ds \Big)^p \, dx$$
$$\leq C_p 2^{-M(1-p/2)} \sum_{i=N}^{\infty} 2^{-ip} \sum_{m=N-r}^{i} \|a\|_2^p \sum_{l=0}^{N-r-1} 2^{pl}$$
$$\times \int_{2^{-l-1}}^{2^{-l-1}+2^{-m}} \Big(\int_{0}^{2^{-N}} D_{2^m}^2(x \dotplus s \dotplus 2^{-\nu-1}) \, ds \Big)^{p/2} \, dx$$
$$\leq C_p 2^{-M(1-p/2)} \|a\|_2^p \sum_{i=N}^{\infty} 2^{-ip}$$
$$\times \Big(\sum_{m=N-r}^{N} 2^{-m} \sum_{l=0}^{N-r-1} 2^{pl} (2^{2m-N})^{p/2}$$
$$+ \sum_{m=N+1}^{i} 2^{-m} \sum_{l=0}^{N-r-1} 2^{pl} 2^{mp/2} \Big)$$
$$\leq C_p 2^{-M(1-p/2)} \|a\|_2^p \sum_{i=N}^{\infty} 2^{-ip}$$
$$\times \Big(2^{-Np/2} \sum_{m=N-r}^{N} 2^{(p-1)m} 2^{p(N-r)}$$
$$+ 2^{p(N-r)} \sum_{m=N+1}^{i} 2^{p/2-1-m} \Big).$$

If $p < 1$, then

$$\begin{aligned}\Sigma^{(33)} &\le C_p 2^{-M(1-p/2)} \|a\|_2^p \sum_{i=N}^{\infty} 2^{-ip} \\ &\quad \times (2^{-Np/2} 2^{p(N-r)} 2^{(p-1)(N-r)} + 2^{p(N-r)} 2^{(p/2-1)N}) \\ &\le C_p 2^{-N(p/2-1)} 2^{p(N-r)-pN} (2^{-Np/2+pN-N-(p-1)r} + 2^{pN/2-N}) \\ &\le C_p 2^{-rp}.\end{aligned}$$

On the other hand, for $p = 1$ we obtain

$$\begin{aligned}\Sigma^{(33)} &\le C_1 2^{-M/2} \|a\|_2 \sum_{i=N}^{\infty} 2^{-i} r (2^{-N/2} 2^{N-r} + 2^{N-r} 2^{-N/2}) \\ &\le C_1 r 2^{-r} \le C_1 2^{-r/2}.\end{aligned}$$

Finally, we consider the case $i = 2$:

$$\begin{aligned}\int_{A_2} |T_{1/\lambda} a|^p &\le \Big[\sum_{i=N}^{\infty} 2^N \int_{2^{-N}}^{1} \Big(\int_0^{2^{-N}} \Big| \sum_{k=2^i}^{2^{i+1}-1} \frac{2^i}{k} w_k(x \dotplus s) \Big| \, ds \Big)^p dx \Big] \\ &\quad \times \Big[\sum_{j=M}^{\infty} 2^M \int_{2^{-M}}^{1} \Big(\int_0^{2^{-M}} \Big| \sum_{l=2^j}^{2^{j+1}-1} \frac{2^j}{l} w_l(y \dotplus t) \Big| \, dt \Big)^p dy \Big] =: RV,\end{aligned}$$

where $R \le C_p$ by the one-dimensional case (see Simon [2]). We will show a stronger estimate, that is, $V \le C_p 2^{-r\delta}$ with a suitable $\delta > 0$ independent of a, r and M. To this end, estimate V as follows (see the analogous situation above):

$$\begin{aligned}V &\le C_p \sum_{j=M}^{\infty} 2^{M+jp} \\ &\quad \times \int_{2^{-M+r}}^{1} \Big(\int_0^{2^{-M}} \Big| \sum_{l=2^j+1}^{2^{j+1}-1} K_l(y \dotplus t) \Big(\frac{1}{l-1} - \frac{1}{l+1} \Big) \Big| \, dt \Big)^p dy \\ &\quad + C_p \sum_{j=M}^{\infty} 2^M \int_{2^{-M+r}}^{1} \Big[\Big(\int_0^{2^{-M}} K_{2^j}(y \dotplus t) \, dt \Big)^p \\ &\quad + \Big(\int_0^{2^{-M}} K_{2^{j+1}}(y \dotplus t) \, dt \Big)^p \Big] dy \\ &=: V_1 + V_2.\end{aligned}$$

As in the proof for $i = 1$ (see the estimation of B) we get $V_2 \le C_p 2^{-rp}$. For V_1 we follow the method of the proof for the case $i = 1$ (see the estimation of $\Sigma^{(3)}$). Hence,

$$
\begin{aligned}
V_1 &\le C_p 2^M \sum_{j=M}^{\infty} 2^{-jp} \int_{2^{-M+r}}^{1} \Big(\int_0^{2^{-M}} \Big[\sum_{m=0}^{M-r-1} 2^m D_{2^m}(y \dotplus t) \\
&\quad + \sum_{m=0}^{M-r-1} \sum_{\nu=0}^{m} 2^\nu D_{2^m}(y \dotplus t \dotplus 2^{-\nu-1}) \\
&\quad + \sum_{m=M-r}^{j} \sum_{\nu=0}^{m} 2^\nu D_{2^m}(y \dotplus t \dotplus 2^{-\nu-1}) \Big] dt \Big)^p dy \\
&\le C_p 2^M \sum_{j=M}^{\infty} 2^{-jp} \int_{2^{-M+r}}^{1} \Big[\Big(\int_0^{2^{-M}} \sum_{m=0}^{M-r-1} 2^m D_{2^m}(y \dotplus t)\, dt \Big)^p \\
&\quad + \Big(\int_0^{2^{-M}} \sum_{m=0}^{M-r-1} \sum_{\nu=0}^{m} 2^\nu D_{2^m}(y \dotplus t \dotplus 2^{-\nu-1})\, dt \Big)^p \\
&\quad + \Big(\int_0^{2^{-M}} \sum_{m=M-r}^{j} \sum_{\nu=0}^{m} 2^\nu D_{2^m}(y \dotplus t \dotplus 2^{-\nu-1})\, dt \Big)^p \Big] dy \\
&=: V_1^{(1)} + V_1^{(2)} + V_1^{(3)}.
\end{aligned}
$$

Now, for $V_1^{(1)}$ we have (see the examination of $\Sigma^{(31)}$ in the proof for the case $i = 1$)

$$
\begin{aligned}
V_1^{(1)} &\le C_p 2^M \sum_{j=M}^{\infty} 2^{-jp} \sum_{m=0}^{M-r-1} 2^{pm} \\
&\quad \times \sum_{l=2^r}^{2^{M-m}-1} \int_{l2^{-M}}^{(l+1)2^{-M}} \Big(\int_0^{2^{-M}} D_{2^m}(y \dotplus t)\, dt \Big)^p dy \\
&\le C_p 2^{M-pM} \sum_{m=0}^{M-r-1} 2^{pm} \sum_{l=2^r}^{2^{M-m}-1} 2^{-M} (2^{m-M})^p \\
&\le C_p 2^{-2pM} \sum_{m=0}^{M-r-1} 2^{2pm} 2^{M-m} \\
&\le C_p 2^{-r(2p-1)}.
\end{aligned}
$$

Similarly to the estimation of $\Sigma^{(32)}$ in the case $i = 2$ we have

$$
V_1^{(2)} \le C_p 2^{M-pM} \sum_{\nu=0}^{M-r-1} 2^{\nu p}
$$

$$\times \Big[\sum_{b=0}^{M-r-\nu-2} \int_{2^{b-M+r}}^{2^{b-M+r+1}} \Big(\int_0^{2^{-M}} D_{2^\nu}(y \dot{+} t \dot{+} 2^{-\nu-1})\, dt \Big)^p dy$$

$$+ \int_{2^{-\nu-1}}^{2^{-\nu}} \Big(\int_0^{2^{-M}} \Big(\sum_{m=\nu}^{M-r-1} D_{2^m}(y \dot{+} t \dot{+} 2^{-\nu-1})\, dt \Big)^p dy \Big]$$

$$=: V_{11}^{(2)} + V_{12}^{(2)},$$

where

$$V_{11}^{(2)} \le C_p 2^{M-pM} \sum_{\nu=0}^{M-r-1} 2^{\nu p} \sum_{b=0}^{M-r-\nu-2} 2^{b-M+r}(2^{\nu-M})^p \le C_p 2^{-r(2p-1)}$$

and

$$V_{12}^{(2)} \le C_p 2^{M-pM} \sum_{\nu=0}^{M-r-1} 2^{\nu p} \Big[\int_0^{2^{-M+r}} \Big(\int_0^{2^{-M}} \Big(\sum_{m=\nu}^{M-r-1} D_{2^m}(y \dot{+} t)\, dt \Big)^p dy$$

$$+ \sum_{d=1}^{M-r-\nu} \int_{2^{-\nu-d}}^{2^{-\nu-d+1}} \Big(\int_0^{2^{-M}} \sum_{m=\nu}^{\nu+d-1} D_{2^m}(y \dot{+} t)\, dt \Big)^p dy \Big]$$

$$\le C_p 2^{M-pM} \sum_{\nu=0}^{M-r-1} 2^{\nu p}$$

$$\times \Big(2^{-M+r} 2^{-Mp} 2^{(M-r)p} + \sum_{d=1}^{M-r-\nu} 2^{-\nu-d} 2^{-Mp} 2^{p(\nu+d)} \Big)$$

$$\le C_p 2^{-r(2p-1)}.$$

To examine $V_1^{(3)}$ we apply again the argument from the case $i = 1$, i.e. similarly to the estimation of $\Sigma^{(33)}$ we get

$$V_1^{(3)} \le C_p 2^M \sum_{j=M}^{\infty} 2^{-jp}$$

$$\times \sum_{m=M-r}^{j} \int_{2^{-M+r}}^{1} \Big(\int_0^{2^{-M}} \sum_{\nu=0}^{m} 2^\nu D_{2^m}(y \dot{+} t \dot{+} 2^{-\nu-1})\, dt \Big)^p dy$$

$$\le C_p 2^M \sum_{j=M}^{\infty} 2^{-jp}$$

$$\times \sum_{m=M-r}^{j} \sum_{l=0}^{M-r-1} 2^{pl} \int_{2^{-l-1}}^{2^{-l-1}+2^{-m}} \Big(\int_0^{2^{-M}} 2^l D_{2^m}(y \dot{+} t \dot{+} 2^{-\nu-1})\, dt \Big)^p dy$$

$$\leq C_p 2^M \sum_{j=M}^{\infty} 2^{-jp}$$
$$\times \Big(\sum_{m=M-r}^{M} 2^{-m} \sum_{l=0}^{M-r-1} 2^{pl} (2^{m-M})^p + \sum_{m=M+1}^{j} 2^{-m} \sum_{l=0}^{M-r-1} 2^{pl} \Big)$$
$$\leq C_p 2^M \sum_{j=M}^{\infty} 2^{-jp} \Big(2^{-pM} \sum_{m=M-r}^{M} 2^{pm-m+p(M-r)} + 2^{-j+p(M-r)} \Big)$$
$$\leq C_p \begin{cases} 2^{-r(2p-1)} & (p<1), \\ 2^{-r/2} & (p=1). \end{cases}$$

This completes the proof of the theorem.

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