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ON THE METRIC THEORY OF CONTINUED FRACTIONS

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For a positive integer n let $P(n)$ be the measure of the set of irrational numbers $x \in (0,1)$ such that the best approximation of x with denominator $\le n$ is a convergent of the continued fraction expansion (in the sequel c.f.e.) of x. We shall show

THEOREM.

$$P(n) = \frac{1}{2} + \frac{6}{\pi^2}(\log 2)^2 + O\left(\frac{1}{n}\right).$$

This answers a question proposed to A. Schinzel by M. Deleglise. The proof is based on several lemmas. We let

$$\frac{0}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \ldots < \frac{p_N}{q_N} = \frac{1}{1}$$

be the Farey sequence of order n.

LEMMA 1. *For each $i < N$ we have*

$$\frac{p_{i+1}}{q_{i+1}} - \frac{p_i}{q_i} = \frac{1}{q_i q_{i+1}} \tag{1}$$

and $q_i + q_{i+1} > n$.

Proof. See [4], Chapter 2, §1.

LEMMA 2. *For each pair of coprime positive integers a, b such that $a \le n$, $b \le n$ and $a + b > n$ there exists one and only one $i < N$ such that*

$$q_i = a, \quad q_{i+1} = b.$$

Proof. See [3], Lemma 1, or [1], Lemma 4.1.

LEMMA 3. *An irreducible fraction p/q, where $q > 1$, is a convergent of the c.f.e. of an irrational number x if and only if*

$$|p - xq| < |p' - xq'|$$

for all pairs $\langle p', q'\rangle \in \mathbb{Z}^2$, where $0 \le q' \le q$, $\langle p', q'\rangle \ne \langle 0,0\rangle, \langle p,q\rangle$.

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P r o o f. See [4], Chapter 2, Theorem 10.

LEMMA 4. *Let $x \in (0,1)$ be irrational and for $i = 0, 1, \dots, N-1$,*

$$m_i = \frac{p_i + p_{i+1}}{q_i + q_{i+1}}, \qquad c_i = \frac{1}{2}\left(\frac{p_i}{q_i} + \frac{p_{i+1}}{q_{i+1}}\right).$$

(i) *If $x \in (p_i/q_i, m_i)$, then p_i/q_i is a convergent of the c.f.e. of x.*
(ii) *If $x \in (m_i, c_i)$, then p_i/q_i is not a convergent of the c.f.e. of x.*
(iii) *If $x \in (c_i, m_i)$, then p_{i+1}/q_{i+1} is not a convergent of the c.f.e. of x.*
(iv) *If $x \in (m_i, p_{i+1}/q_{i+1})$, then p_{i+1}/q_{i+1} is a convergent of the c.f.e. of x.*

P r o o f. It suffices to prove (i) and (ii), the proof of (iii) and (iv) is analogous.

Assume that $x \in (p_i/q_i, m_i)$. Then

$$0 < q_i x - p_i < q_i m_i - p_i = \frac{1}{q_i + q_{i+1}}.$$

Hence, if $\langle p', q'\rangle \in \mathbb{Z}^2$, $0 \le q' \le q_i$, $\langle p', q'\rangle \ne \langle 0,0\rangle, \langle p_i, q_i\rangle$ we have either $q' = 0$, $p' \ne 0$ and

$$|p' - q'x| = |p'| \ge 1,$$

or $\langle p', q'\rangle = \langle p_{i+1}, q_{i+1}\rangle$ and

$$|p' - q'x| = |p_{i+1} - q_{i+1}x| > p_{i+1} - q_{i+1}m_i = \frac{1}{q_i + q_{i+1}},$$

or $q' > 0$, $\langle p', q'\rangle \ne \langle p_i, q_i\rangle, \langle p_{i+1}, q_{i+1}\rangle$ and

$$|p' - q'x| \ge q' \min\left\{\left|\frac{p'}{q'} - \frac{p_i}{q_i}\right|, \left|\frac{p'}{q'} - \frac{p_{i+1}}{q_{i+1}}\right|\right\} \ge \frac{1}{\max\{q_i, q_{i+1}\}}.$$

In any case $|p' - q'x| > |p_i - q_i x|$ and by Lemma 3, p_i/q_i is a convergent of the c.f.e. of x if $q_i > 1$. If $q_i = 1$ then $p_i/q_i = 0$ and obviously the same is true.

Assume now that $x \in (m_i, c_i)$. Then $m_i < c_i$, hence $q_i > q_{i+1}$ by (1). We have

$$0 < p_{i+1} - q_{i+1}x < p_{i+1} - q_{i+1}m_i = \frac{1}{q_i + q_{i+1}} < q_i x - p_i$$

and, by Lemma 3, p_i/q_i is not a convergent of the c.f.e. of x.

LEMMA 5. *We have*

$$P(n) = \frac{1}{2} + \sum_{i=0}^{N-1} \frac{1}{(q_i + q_{i+1})\max\{q_i, q_{i+1}\}}.$$

Proof. Denoting the Lebesgue measure by m we have

$$P(n)=\sum_{i=0}^{N-1} m(S_i),$$

where S_i is the set of irrational numbers in the interval $(p_i/q_i, p_{i+1}/q_{i+1})$ for which the best approximation with denominator $\le n$ is a convergent of the c.f.e. of x. Clearly, p_i/q_i is the best approximation in question for $x\in(p_i/q_i, c_i)$, and p_{i+1}/q_{i+1} for $x\in(c_i, p_{i+1}/q_{i+1})$. Hence, by Lemma 4,

$$S_i=\left(\left(\frac{p_i}{q_i},\min\{m_i,c_i\}\right)\cup\left(\max\{m_i,c_i\},\frac{p_{i+1}}{q_{i+1}}\right)\right)\setminus\mathbb{Q},$$

and

$$m(S_i)=\min\{m_i,c_i\}-\frac{p_i}{q_i}+\frac{p_{i+1}}{q_{i+1}}-\max\{m_i,c_i\}.$$

Using (1) we obtain

$$m(S_i)=\frac{1}{q_iq_{i+1}}-\frac{|q_i-q_{i+1}|}{q_iq_{i+1}(q_i+q_{i+1})}=\frac{1}{2q_iq_{i+1}}+\frac{1}{(q_i+q_{i+1})\max\{q_iq_{i+1}\}}.$$

Therefore, by (1),

$$\begin{aligned}P(n)&=\frac{1}{2}\sum_{i=0}^{N-1}\left(\frac{p_{i+1}}{q_{i+1}}-\frac{p_i}{q_i}\right)+\sum_{i=0}^{N-1}\frac{1}{(q_i+q_{i+1})\max\{q_i,q_{i+1}\}}\\&=\frac{1}{2}+\sum_{i=0}^{N-1}\frac{1}{(q_i+q_{i+1})\max\{q_i,q_{i+1}\}}.\end{aligned}$$

Lemma 6. *For $n>1$ we have*

$$P(n)=\frac{1}{2}+2\sum\nolimits^{*}\frac{1}{bc},$$

where the sum $\sum^*$ is taken over all pairs $\langle b,c\rangle\in\mathbb{N}^2$ such that $b\le n<c<2b$ and $(b,c)=1$.

Proof. By Lemmas 1 and 2 we have

$$\sum_{i=0}^{N-1}\frac{1}{(q_i+q_{i+1})\max\{q_i,q_{i+1}\}}=\sum_{\substack{a,b=1\\a+b>n\\(a,b)=1}}^{n}\frac{1}{(a+b)\max\{a,b\}}.$$

For $n>1$ the conditions $a+b>n$, $(a,b)=1$ imply $a\ne b$. Hence

$$\sum_{\substack{a,b=1\\a+b>n\\(a,b)=1}}^{n}\frac{1}{(a+b)\max\{a,b\}}=2\sum_{\substack{a,b=1\\a<b,\,a+b>n\\(a,b)=1}}^{n}\frac{1}{(a+b)b}.$$

The conditions $1 \le a < b$, $(a,b) = 1$ are equivalent to $b < a+b < 2b$, $(a+b,b) = 1$. Replacing $a+b$ by c and using Lemma 5 we obtain the lemma.

LEMMA 7. *For arbitrary positive numbers $A < B$ we have*

$$\sum_{B \ge i > A} \frac{1}{i} = \log \frac{B}{A} - \frac{1}{B}\psi(B) + \frac{1}{A}\psi(A) + O\left(\frac{1}{A^2}\right),$$

$$\sum_{B \ge i > A} \frac{\log i}{i} = \frac{1}{2}\log \frac{B}{A} \log BA - \frac{\log B}{B}\psi(B) + \frac{\log A}{A}\psi(A) + O\left(\frac{\log^+ A}{A^2}\right),$$

$$\sum_{B \ge i > A} \frac{1}{i^2} = \frac{1}{B} - \frac{1}{A} + O\left(\frac{1}{A^2}\right),$$

$$\sum_{B \ge i > A} \log i = B \log B - A \log A + O(B),$$

where $\psi(x) = \{x\} - 1/2$ and $\log^+ x = \max\{\log x, 1\}$.

Proof. See Walfisz [5], Chapter 1, §1, Hilfssatz 3.

LEMMA 8. *We have*

$$\sum_{d=1}^{n} \frac{\mu(d)}{d} = O(1), \qquad \sum_{d=1}^{n} \frac{\mu(d)}{d^2} = \frac{6}{\pi^2} + O\left(\frac{1}{n}\right).$$

Proof. See Landau [2], §152 and §153.

Proof of the Theorem. In the notation of Lemma 6 we have

$$\sum\nolimits^* \frac{1}{bc} = \sum_{b=1}^{n} \sum_{2b > c > n} \frac{1}{bc} \sum_{d \mid (b,c)} \mu(d) = \sum_{d=1}^{n} \frac{\mu(d)}{d^2} \sum_{k=1}^{[n/d]} \frac{1}{k} \sum_{2k > l > n/d} \frac{1}{l},$$

hence by Lemma 7,

$$\begin{aligned}\sum\nolimits^* \frac{1}{bc} &= \sum_{d=1}^{n} \frac{\mu(d)}{d^2} \sum_{n/d \ge k > n/(2d)} \frac{1}{k}\left(\log \frac{2kd}{n} - \frac{1}{4k} + \frac{d}{n}\psi\left(\frac{n}{d}\right) + O\left(\frac{d^2}{n^2}\right)\right) \\ &= \sum_{d=1}^{n} \frac{\mu(d)}{d^2}\left(\left(\log \frac{2d}{n} + \frac{d}{n}\psi\left(\frac{n}{d}\right) + O\left(\frac{d^2}{n^2}\right)\right) \sum_{n/d \ge k > n/(2d)} \frac{1}{k} \right. \\ &\quad \left. + \sum_{n/d \ge k > n/(2d)} \frac{\log k}{k} - \sum_{n/d \ge k > n/(2d)} \frac{1}{4k^2}\right).\end{aligned}$$

Now,

$$\left(\log\frac{2d}{n}+\frac{d}{n}\psi\left(\frac{n}{d}\right)+O\left(\frac{d^2}{n^2}\right)\right)\sum_{n/d\ge k>n/(2d)}\frac{1}{k}$$
$$=\left(\log\frac{2d}{n}+\frac{d}{n}\psi\left(\frac{n}{d}\right)+O\left(\frac{d^2}{n^2}\right)\right)$$
$$\times\left(\log 2-\frac{d}{n}\psi\left(\frac{n}{d}\right)+\frac{2d}{n}\psi\left(\frac{n}{2d}\right)+O\left(\frac{d^2}{n^2}\right)\right)$$
$$=\log 2\log\frac{2d}{n}-\frac{d}{n}\log\frac{2d}{n}\psi\left(\frac{n}{d}\right)$$
$$+\frac{2d}{n}\log\frac{2d}{n}\psi\left(\frac{n}{2d}\right)+\frac{d}{n}\log 2\,\psi\left(\frac{n}{d}\right)+O\left(\frac{d^2}{n^2}\log^+\frac{n}{d}\right),$$

while

$$\sum_{n/d\ge k>n/(2d)}\frac{\log k}{k}=\frac{1}{2}\log 2\log\frac{n^2}{2d^2}-\frac{d}{n}\log\frac{n}{d}\psi\left(\frac{n}{d}\right)$$
$$+\frac{2d}{n}\log\frac{n}{2d}\psi\left(\frac{n}{2d}\right)+O\left(\frac{d^2}{n^2}\log^+\frac{n}{d}\right),$$
$$\sum_{n/d\ge k>n/(2d)}\frac{1}{4k^2}=\frac{d}{4n}+O\left(\frac{d^2}{n^2}\right).$$

Hence, by Lemma 8,

$$\sum\nolimits^*\frac{1}{bc}=\sum_{d=1}^{n}\frac{\mu(d)}{d^2}\left(\frac{1}{2}(\log 2)^2-\frac{d}{4n}+O\left(\frac{d^2}{n^2}\log^+\frac{n}{d}\right)\right)$$
$$=\frac{1}{2}(\log 2)^2\frac{6}{\pi^2}+O\left(\frac{1}{n}\right)-\frac{1}{4n}\sum_{d=1}^{n}\frac{\mu(d)}{d}+O\left(\frac{1}{n^2}\sum_{d=1}^{n}\log^+\frac{n}{d}\right)$$
$$=\frac{3}{\pi^2}(\log 2)^2+O\left(\frac{1}{n}\right)+O\left(\frac{1}{n^2}\sum_{d=1}^{n}\log^+\frac{n}{d}\right).$$

We have, by Lemma 7,

$$\sum_{d=1}^{n}\log^+\frac{n}{d}=\sum_{d=1}^{n/e}\log\frac{n}{d}+\sum_{d>n/e}^{n}1=O(n),$$

hence

$$\sum\nolimits^*\frac{1}{bc}=\frac{3}{\pi^2}(\log 2)^2+O\left(\frac{1}{n}\right).$$

Together with Lemma 6 this gives the theorem.

REMARK. The main term in the asymptotic formula for $P(n)$, but not the error term, can be derived from Theorem 1.3 of [1] and from Lemma 5.

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