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STRONG ASYMPTOTIC STABILITY FOR n-DIMENSIONAL THERMOELASTICITY SYSTEMS

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We use a new approach to prove the strong asymptotic stability for n-dimensional thermoelasticity systems. Unlike the earlier works, our method can be applied in the case of feedbacks with no growth assumption at the origin, and when LaSalle's invariance principle cannot be applied due to the lack of compactness.

1. Introduction. Consider the nonlinear damped thermoelasticity system

$$\text{(P)}\qquad \begin{cases} u'' - \Delta u + \gamma\nabla\theta + g(u') = 0 & \text{in } \Omega\times(0,\infty),\\ \theta' - k\Delta\theta + \gamma \operatorname{div} u' = 0 & \text{in } \Omega\times(0,\infty),\\ u = 0, \quad \theta = 0, & \text{on } \Gamma\times(0,\infty),\\ u(0) = u_0, \quad u'(0) = u_1, \quad \theta(0) = \theta_0 & \text{in } \Omega, \end{cases}$$

where Ω is an open set of finite measure in $\mathbb{R}^n$, having a boundary Γ of class C^2, k and γ are two positive constants, and $g : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function.

We recall that for $u = (u_1, \ldots, u_n) \in \mathcal{D}'(\Omega)^n$ and $f \in \mathcal{D}'(\Omega)$ we have

$$\Delta u = (\Delta u_1, \ldots, \Delta u_n), \qquad \operatorname{div} u = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i}, \qquad \nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

For $n \le 3$, problem (P) has its roots in the mathematical description of a thermoelastic system (see for example [6, pp. 54–67] and [7]); u and θ denote the displacement and the temperature, respectively.

The following assumptions on the nonlinear function $g = (g_1, \ldots, g_n)$ are made:

$$\text{(H1)}\qquad g_i \text{ is } C^1, \text{ strictly increasing and } g_i(0) = 0;$$

there exists $q \ge 2$ satisfying $(n-2)q \le 2n$ and two positive constants c_1, c_2 such that

$$\text{(H2)}\qquad c_1|x| \le |g_i(x)| \le c_2|x|^{q-1} \quad \text{for all } |x| \ge 1, \ x \in \mathbb{R}.$$

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The main goal of this paper is to prove that under the above hypotheses, the unique global weak solution $(u(t), \theta(t))$ to (P) decays to zero strongly when t goes to infinity.

The problem of existence, regularity and asymptotic behavior of solutions to thermoelasticity systems has attracted a lot of attention in recent years (see for example [5, 8, 9, 10, 12] and the references therein, to name just a few). Quite recently, Ouazza [11] has proved that if $g(x) := (\alpha(x_1), \ldots, \alpha(x_n))$ where $\alpha : \mathbb{R} \to \mathbb{R}$ is such that

$$\alpha(x) = \begin{cases} |x|^{p-1}x & \text{if } |x| \le 1, \\ \alpha(x) = x & \text{if } |x| > 1, \end{cases}$$

then the total energy $E(t) := \frac{1}{2}\|(u, u', \theta)\|^2_{H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2(\Omega)}$ has an exponential decay rate if $p = 1$ and a polynomial decay rate if $p > 1$. Ouazza's work was marked by the following features:

(a) the domain Ω is bounded;

(b) the dissipative term g is of a preassigned polynomial growth at the origin.

These assumptions are critically invoked in the proofs in the following ways:

(a) the boundedness of Ω allows the use of some compact imbedding theorems (and LaSalle's invariance principle can then be used to prove the strong asymptotic stability);

(b) the polynomial growth at the origin of the dissipative term g allows the construction of a standard Lyapunov function, or the use of some specific integral inequalities, which are then used to yield the desired decay rates.

Our goal in this paper is to dispense entirely with the above assumptions (a)–(b). Indeed, in our formulation Ω is not necessarily bounded, and no growth assumption at the origin is imposed on g. This results in major difficulties, which require the development of a new approach in successfully solving the problem of strong asymptotic stability. This approach, introduced by the author in [1], has already been used in the study of the strong asymptotic stability of some nonlinear wave equations and some plate models [1, 2], and for isotropic elasticity systems in [3].

The paper is organized as follows. In Section 2, we state the main result, and in Section 3 we give its proof.

2. Statement of the main theorem. The problem (P) is well posed and dissipative. Indeed, one can write it in the first order form

$$\begin{cases} U' + \mathcal{A}U + \mathcal{B}U = 0, \\ U(0) = U_0, \end{cases}$$

where $U = (u, u', \theta)$, $U_0 = (u_0, u_1, \theta_0)$ and the operators $\mathcal{A}$ and $\mathcal{B}$ are given by

$$\begin{aligned}\mathcal{A}(u, u', \theta) &= (-u', -\Delta u + \gamma\nabla\theta, -k\Delta\theta + \gamma \operatorname{div} u),\\ \mathcal{B}(u, u', \theta) &= (0, g(u'), 0),\\ D(\mathcal{A}) &= (H^2(\Omega) \cap H_0^1(\Omega))^n \times H_0^1(\Omega)^n \times (H^2(\Omega) \cap H_0^1(\Omega)),\\ D(\mathcal{B}) &= H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2(\Omega).\end{aligned}$$

We can easily verify (for details see Ouazza [11], Muñoz Rivera [10], Ball [4]) that there exists a global weak solution (u, θ) such that

$$u \in C(\mathbb{R}_+, H_0^1(\Omega)^n) \cap C^1(\mathbb{R}_+, L^2(\Omega)^n), \qquad \theta \in C(\mathbb{R}_+, L^2(\Omega))$$

for all given initial data $(u_0, u_1, \theta_0) \in H_0^1(\Omega)^n \times L^2(\Omega)^n \times L^2(\Omega)$.

Moreover, if $(u_0, u_1, \theta_0) \in D(\mathcal{A})$ then we have the following regularity property:

$$u \in C(\mathbb{R}_+, (H^2(\Omega) \cap H_0^1(\Omega))^n) \cap C^1(\mathbb{R}_+, H_0^1(\Omega)^n) \cap C^2(\mathbb{R}_+, L^2(\Omega)^n),$$
$$\theta \in C(\mathbb{R}_+, H^2(\Omega) \cap H_0^1(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega));$$

we say in this case that (u, θ) is a *strong solution*.

We define the energy of the solutions by the formula

$$E(t) := \frac{1}{2}\int_\Omega (|u'(t)|^2 + |\nabla u(t)|^2 + \theta(t)^2)\, dx.$$

If (u, θ) is a strong solution, then by a simple computation we have

$$E(S) - E(T) = \int_S^T \int_\Omega (k|\nabla\theta|^2 + u' \cdot g(u'))\, dx \qquad \text{for all } 0 \le S < T < \infty.$$

This identity remains valid for all mild solutions by an easy density argument. By (H1) we deduce that $E(t)$ is nonincreasing. Our main result is the following

Main Theorem. *We have $E(t) \to 0$ as $t \to \infty$ for every weak solution of* (P).

3. Proof of the main theorem. For the proof we need the following two lemmas (we use the summation convention for repeated indices):

Lemma 1. *We have*

$$\int_0^t \int_\Omega u_i g_i(u_i')\, dx\, ds = o(t), \qquad t \to \infty.$$

Lemma 2. *We have*

$$\int_0^t\int_\Omega u_i' u_i'\,dx\,ds = o(t), \qquad t\to\infty.$$

Proof of Lemma 1. By (H1), we see that $|g_i(x)|\le c|x|$ for $|x|\le 1$ (here and in the sequel c denotes various positive constants which may be different at different occurrences). Then

$$\begin{aligned}\int_{|u_i'|\le 1}|u_i g_i(u_i')|\,dx &\le c\int_{|u_i'|\le 1}(u_i' g_i(u_i'))^{1/2}|u_i|\,dx\\ &\le c\Big(\int_\Omega u_i' g_i(u_i')\,dx\Big)^{1/2}\|u\|_{L^2(\Omega)^n}.\end{aligned}$$

Similarly, by (H2) we have

$$\int_{|u_i'|>1}|u_i g_i(u_i')|\,dx \le c\Big(\int_\Omega u_i' g_i(u_i')\,dx\Big)^{1/q'}\|u\|_{L^q(\Omega)^n}$$

where $q'=q/(q-1)$ is the Hölder conjugate of q.

Then from Hölder's inequality we obtain

$$\begin{aligned}\int_0^t\int_\Omega u_i g_i(u_i')\,dx\,ds &\le c\Big(\int_0^t\int_\Omega u_i' g_i(u_i')\,dx\,ds\Big)^{1/2}\sqrt{t}\sup_{[0,t]}\|u(s)\|_{L^2(\Omega)^n}\\ &\quad + ct^{1/q}\Big(\int_0^t\int_\Omega u_i' g_i(u_i')\,dx\,ds\Big)^{1/q'}\sup_{[0,t]}\|u(s)\|_{L^q(\Omega)^n}.\end{aligned}$$

Using the Hölder, Sobolev and Poincaré inequalities we have

$$\|u(s)\|_{L^2(\Omega)^n}\le c\|u(s)\|_{L^q(\Omega)^n}\le cE(s)^{1/2}\le cE(0)^{1/2} \qquad \text{for all } s\ge 0.$$

From these estimates, it follows that

$$\int_0^t\int_\Omega u_i g_i(u_i')\,dx\,ds \le ct^{1/2}+ct^{1/q}=o(t), \qquad t\to\infty.$$

Proof of Lemma 2. Let ε be an arbitrarily small real and set

$$M_i(\varepsilon) := \sup\{x/g_i(x) : |x|\ge\sqrt{\varepsilon/|\Omega|}\} \quad \text{and} \quad M(\varepsilon) := \max_{i=1,\dots,n} M_i(\varepsilon).$$

By hypotheses (H1)–(H2), we have $M(\varepsilon)<\infty$.

Clearly,

$$\int_{|u_i'|<\sqrt{\varepsilon/|\Omega|}} u_i' u_i'\,dx \le n\varepsilon.$$

On the other hand,

$$\int_{|u_i'|\ge\sqrt{\varepsilon/|\Omega|}} u_i'u_i'\,dx = \int_{|u_i'|\ge\sqrt{\varepsilon/|\Omega|}} \frac{u_i'}{g_i(u_i')}u_i'\,g_i(u_i')\,dx$$
$$\le M(\varepsilon)\int_\Omega u_i'g_i(u_i')\,dx.$$

As

$$\int_{|u_i'|\ge\sqrt{\varepsilon/|\Omega|}} u_i'u_i'\,dx \le \sqrt{2E(0)}\Big(\int_{|u_i'|\ge\sqrt{\varepsilon/|\Omega|}} u_i'u_i'\,dx\Big)^{1/2},$$

we deduce that

$$\int_\Omega u_i'u_i'\,dx \le n\varepsilon + \sqrt{2E(0)M(\varepsilon)}\Big(\int_\Omega u_i'g_i(u_i')\,dx\Big)^{1/2}$$

and then by the Hölder inequality

$$\int_0^t\int_\Omega u_i'u_i'\,dx\,ds \le c\varepsilon t + c\sqrt{2E(0)M(\varepsilon)}\sqrt{t}\Big(\int_0^t\int_\Omega u_i'g_i(u_i')\,dx\,ds\Big)^{1/2}$$
$$\le c\varepsilon t + cE(0)\sqrt{2M(\varepsilon)}\sqrt{t} = o(t), \quad t\to\infty.$$

Proof of the main theorem. We multiply the first equation in (P) with u and integrate over $(0,t)\times\Omega$ to obtain

$$0 = \int_0^t\int_\Omega u\cdot(u'' - \Delta u + g(u') + \gamma\nabla\theta)\,dx\,ds$$
$$= \Big[\int_\Omega u\cdot u'\,dx\Big]_0^t + \int_0^t\int_\Omega(|\nabla u|^2 - |u'|^2 + u\cdot g(u') + \gamma u\cdot\nabla\theta)\,dx\,ds.$$

Hence

$$\Big[\int_\Omega u\cdot u'\Big]_0^t = \int_0^t\int_\Omega(2|u'|^2 - u\cdot g(u')) - 2\int_0^t E(s)\,ds + \int_0^t\int_\Omega(\theta^2 - \gamma u\cdot\nabla\theta).$$

Let $\varepsilon>0$ be a small real number to be chosen later. We have

$$\int_\Omega(\theta^2 - \gamma u\cdot\nabla\theta)\,dx \le \int_\Omega(c|\nabla\theta|^2 + \varepsilon|u|^2 + c(\varepsilon)|\nabla\theta|^2)\,dx$$
$$\le c(\varepsilon)\int_\Omega|\nabla\theta|^2\,dx + c\varepsilon\int_\Omega|\nabla u|^2\,dx$$
$$\le -c(\varepsilon)E' + c\varepsilon E.$$

After integration over $(0,t)$, we obtain

$$\int_0^t\int_\Omega (\theta^2 - \gamma u\cdot\nabla\theta)\,dx\,ds \le -c(\varepsilon)\int_0^t E'(s)\,ds + c\varepsilon\int_0^t E(s)\,ds$$
$$\le c(\varepsilon)E(0) + c\varepsilon\int_0^t E(s)\,ds.$$

We deduce that

$$\Big[\int_\Omega u\cdot u'\Big]_0^t \le \int_0^t\int_\Omega (2|u'|^2 - u\cdot g(u')) - (2-c\varepsilon)\int_0^t E(s)\,ds + c(\varepsilon)E(0).$$

By choosing $\varepsilon = 1/c$, we conclude that

$$\Big[\int_\Omega u\cdot u'\,dx\Big]_0^t \le \int_0^t\int_\Omega (2|u'|^2 - u\cdot g(u'))\,dx\,ds - \int_0^t E(s)\,ds + cE(0). \tag{3.1}$$

Assume that, contrary to our claim, $l := \lim_{t\to\infty} E(t) > 0$. Then putting $\Phi(t) = \int_\Omega u\cdot u'\,dx$ we have from (3.1),

$$\Phi(t) - \Phi(0) \le -lt + o(t) + cE(0), \qquad t\to\infty;$$

we have used the lemmas in the last step.

It follows that $\Phi(t)\to-\infty$ as $t\to\infty$. But this is impossible because

$$\Big|\int_\Omega u\cdot u'\,dx\Big| \le \frac{1}{2}\int_\Omega (|u|^2 + |u'|^2)\,dx \le c\int_\Omega (|\nabla u|^2 + |u'|^2)\,dx \le cE(0).$$

We conclude that $\lim_{t\to\infty} E(t) = 0$.

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