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A RELATIVELY FREE TOPOLOGICAL GROUP THAT IS NOT VARIETAL FREE

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TO SID MORRIS ON HIS 50TH BIRTHDAY

Answering a 1982 question of Sidney A. Morris, we construct a topological group G and a subspace X such that (i) G is algebraically free over X, (ii) G is relatively free over X, that is, every continuous mapping from X to G extends to a unique continuous endomorphism of G, and (iii) G is not a varietal free topological group on X in any variety of topological groups.

0. Introduction. An abstract group G is called *relatively free* over a subset X if every map $X \to G$ extends to a unique endomorphism $G \to G$. It is well known [16, Th. 14.5] that a group G is relatively free over a subset $X \subseteq G$ if and only if G is free over X in a suitable variety of groups. (Equivalently: in the variety generated by G.) Does this result have an analogue for topological groups? The following has been open for fifteen years.

Problem (S. A. Morris, 1982, P 1254 in [14]). *If G is topologically relatively free with free generating space X and G is algebraically relatively free with free generating set X, is G necessarily $F(X, \mathcal{V}(G))$?*

Here G is *topologically relatively free* over a subspace X if every continuous map $X \to G$ lifts to a unique continuous endomorphism of G. A *variety of topological groups* in the sense of Morris [12–14] is a class of topological groups closed with respect to forming direct products of arbitrary subfamilies equipped with Tikhonov topology, proceeding to topological subgroups, and topological quotient groups. The symbol $\mathcal{V}(G)$ stands for the smallest variety containing a given topological group G, while $F(X, \mathcal{V})$ denotes the free topological group in a variety $\mathcal{V}$, that is, $F(X, \mathcal{V}) \in \mathcal{V}$ is a topological

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group with a distinguished subspace X, and every continuous mapping f from X to an arbitrary $G \in \mathcal{V}$ lifts to a unique continuous homomorphism $F(X, \mathcal{V}) \to G$.

The aim of the present note is to answer the problem in the negative through supplying a counter-example (§4), in which a topological group G is both relatively and algebraically free over a subspace X, but not topologically free in the variety generated by G.

All topological spaces are assumed to be Tikhonov, and all topological groups Hausdorff. Those concepts and elementary results from general topology and topological group theory appearing in this article, unless furnished with exact references, can be found, respectively, in [5] and [6].

1. Universal arrows and free topological groups. In order to give all the brands of free topological groups featuring in this article a unified treatment, we invoke a category-theoretic notion of a universal arrow, belonging to Samuel [19, 9]. If $S : D \to C$ is a covariant functor and c an object of C, a *universal arrow from c to S* is a pair $\langle r, u\rangle$ consisting of an object r of D and an arrow $u : c \to Sr$ of C, such that to every pair $\langle d, f\rangle$ with d an object of D and $f : c \to Sd$ an arrow of C, there is a *unique* arrow $f' : r \to d$ of D with $Sf' \circ u = f$.

This powerful concept describes a very wide range of mathematical constructions. However, we are only interested in a very particular case where S is the *forgetful* functor from a full subcategory, Ω, of the category $\mathcal{T}op\mathcal{G}r$ of all topological groups and continuous homomorphisms to the category $\mathcal{T}ikh$ of all Tikhonov spaces and continuous mappings. The following is a slight generalization of a concept from [12, 3].

1.1. Definition. Let Ω be a class of topological groups, and let X be a topological space. We will refer to a pair (G, i), where $G \in \Omega$ and $i : X \to G$ is continuous, as a *free topological group on X in the class Ω* whenever (G, i) is a universal arrow from X to the forgetful functor $\Omega \to \mathcal{T}ikh$, where Ω is viewed as a full subcategory of $\mathcal{T}op\mathcal{G}r$. We will denote G by $F(X, \Omega)$ and suppress i in our notation provided that $i : X \hookrightarrow G$ is a homeomorphic embedding.

1.2. Examples. 1. If Ω is the class of all topological groups, then $F(X, \Omega) \equiv F(X)$ is the Markov free topological group on X [10, 11].

2. If Ω is the class of all topological abelian groups, then $F(X, \Omega) \equiv A(X)$ is the Markov free abelian topological group [15].

3. If $\Omega = \mathcal{V}$ is a variety of topological groups, then $F(X, \mathcal{V})$ is the varietal free topological group on X in $\mathcal{V}$ [13, 14]. Among the best known examples of varietal free topological groups are free profinite groups, free precompact groups, and free nilpotent (or solvable) groups of a given class k.

4. Free compact groups [7], strictly speaking, are *not* free varietal groups, because compact groups do not form a variety (not every topological subgroup of a compact group is such). The situation can be corrected if we generalize the concept of a variety by making varieties closed with respect to a transition to *closed* topological subgroups only. However, here we are not interested in this generalization.

5. Relatively free topological groups G are exactly topological groups of the form $G = F(X, \{G\})$.

For a survey of some constructions of the above type, as well as similar constructions in functional analysis and Lie theory, see [18].

Comfort and van Mill have shown in [4] that the mere existence of free topological groups cannot be guaranteed in a most general situation where Ω is just any full subcategory of $\mathcal{T}op\mathcal{G}r$. (A similar phenomenon was already observed by Morris [12] who worked with a somewhat more restricted concept.) Still, there is no reason why one should limit oneself to considering free topological groups in the varieties only, and indeed the present investigation demonstrates a potential usefulness of those topological groups free in classes other than varieties.

We will need the following easily proved statement [3].

1.3. Assertion and definition. Every class, Ω, of topological groups is contained in the unique smallest variety of topological groups, $\mathcal{V}(\Omega)$. One says that $\mathcal{V}(\Omega)$ is *generated* by Ω. If Ω consists of a single topological group G, then $\mathcal{V}(\Omega)$ is denoted by $\mathcal{V}(G)$ and called a *singly generated* variety.

It is also helpful to keep in mind the following simple fact.

1.4. Assertion. *For a topological group G and a subspace X the following are equivalent*:

(i) *G is free on X in some variety of topological groups*;
(ii) *G is free on X in the variety of topological groups generated by G.*

Proof. While $\Leftarrow$ is trivial, assume (i) and let $\mathcal{V}$ be a variety of topological groups with $G = F(X, \mathcal{V})$. In particular, $G \in \mathcal{V}$. Let $f : X \to H \in \mathcal{V}(G)$ be a continuous mapping. Taking into account that $\mathcal{V}(G) \subseteq \mathcal{V}$ and $G = F(X, \mathcal{V})$, one concludes that f lifts to a unique continuous homomorphism $\bar{f} : G \to H$. Since obviously $G \in \mathcal{V}(G)$, we conclude that (ii) holds. ∎

2. Two background results on zero-dimensional groups. We say that a space X is *zero-dimensional* if X has a base consisting of sets that are both open and closed. (Using notation of dimension theory, $\operatorname{ind} X = 0$.) While it is well known and easily proved that the Tikhonov product of any family of zero-dimensional spaces, as well as any subspace of a zero-dimensional space, are again such, zero-dimensionality is destroyed by quotient mappings. Indeed, every Tikhonov space is the image of a zero-dimensional space (with a plethora of additional properties) under a continuous open surjection. (See [8] or [2], Coroll. in Sect. 7.) For topological groups the following important result was obtained by Arkhangel'skiĭ in 1981.

2.1. THEOREM (Arkhangel'skiĭ [1, 2]). *Every topological group H is a quotient group of a zero-dimensional topological group G. More precisely, given a topological group H, there exist a topological space Z and a continuous open onto homomorphism $\bar{f} : F(Z) \to H$ such that $F(Z)$ is zero-dimensional.* ■

Let Θ stand for the class of all zero-dimensional topological groups. The following is immediate.

2.2. COROLLARY. *The variety $\mathcal{V}(\Theta)$ generated by Θ is the class of all topological groups.* ■

The finest point of Arkhangel'skiĭ's proof of Theorem 2.1 is choosing a zero-dimensional space Z admitting a quotient map onto X and such that the free topological group $F(Z)$ is zero-dimensional. Indeed, it was shown by the second author (D.B.S.) [20] that the free topological group on a zero-dimensional space is not necessarily such. Two of the main ingredients of this subtle result are the so-called Dowker space [5] and a topological description of the subspace of the free topological group formed by all words of reduced length ≤ 2 obtained by the first author (V.G.P.) [17].

2.3. THEOREM (Shakhmatov [20]). *There exists a zero-dimensional space Y such that the free topological group $F(Y)$ is not zero-dimensional.* ■

An alternative proof of a somewhat more general result than 2.3 can be found in Theorem 4.5 and Corollaries 4.6 and 4.7 of [21].

3. Free zero-dimensional topological groups. Here is probably the most natural example of a topological group free in a class that fails to form a variety in a most spectacular fashion. (Cf. Corollary 2.2.) The following is well known in the topological folklore, though it has hardly ever been published by anyone.

3.1. THEOREM. *For every zero-dimensional Tikhonov space X there exists a (unique up to isomorphism) free zero-dimensional topological group $G = F(X, \Theta)$. The mapping $i : X \to G$ is a homeomorphic embedding, and as an abstract group, G is freely generated by X.*

Proof. For the sake of completeness we prove the result. Let $\mathfrak{T}^*$ be the topology of X. Denote by $\mathcal{F}$ the family of all zero-dimensional (not necessarily Hausdorff) group topologies, $\mathfrak{T}$, on the abstract free group $F(X)$ having the property $\mathfrak{T}|_X \subseteq \mathfrak{T}^*$, where $\mathfrak{T}|_X$ is the subspace topology induced on X

by $\mathfrak{T}$. Let $\mathfrak{T}_z$ be the supremum of all the topologies from $\mathcal{F}$, i.e. the topology generated by $\bigcup \mathcal{F}$ as a base. It is clear that $\mathfrak{T}_z$ is a zero-dimensional group topology on $F(X)$ and $\mathfrak{T}_z|_X \subseteq \mathfrak{T}$. Being zero-dimensional, X admits a homeomorphic embedding, $j : X \to \{-1,1\}^\tau$, into a Cantor cube of the same weight, τ, as X. If we think of $\{-1,1\}$ as a multiplicative topological subgroup of $\mathbb{R}$, then $\{-1,1\}^\tau$ becomes a zero-dimensional compact abelian topological group. Denote by $\widetilde{j} : F(X) \to \{-1,1\}^\tau$ the group homomorphism extending j. Since $\{-1,1\}^\tau$ is a zero-dimensional topological group, $\mathfrak{T}' = \{\widetilde{j}^{-1}(U) : U \text{ is open in } \{-1,1\}^\tau\}$ is a zero-dimensional group topology on $F(X)$ with $\mathfrak{T}'|_X = \mathfrak{T}^*$. Thus $\mathfrak{T}' \in \mathcal{F}$ and finally $\mathfrak{T}_z|_X = \mathfrak{T}^*$.

We claim that $\mathfrak{T}_z$ is Hausdorff. Let $g \in F(X)$ be arbitrary with $g \neq e$. Let $g = x_1^{\varepsilon_1} x_2^{\varepsilon_2} \dots x_n^{\varepsilon_n}$ be an irreducible word decomposition, where $n \geq 1$, $\varepsilon_i = \pm 1$, $x_{i+1} \neq x_i$ for all i, and $Y = \{x_1, \dots, x_n\} \subseteq X$. Since X is zero-dimensional and Hausdorff, we may choose open-and-closed subsets $U_1, \dots, U_n \subseteq X$ so that $x_i \in U_i$ and $U_i = U_j$ if $x_i = x_j$ and $U_i \cap U_j = \emptyset$ whenever $x_i \neq x_j$. The map $f : X \to F(\{x_1, \dots, x_n\})$ determined by $f(U_i) = \{x_i\}$, $i = 1, \dots, n$, and $f(X \setminus \bigcup_{i=1}^n U_i) \subseteq \{e\}$ is continuous if the free group $F(Y)$ is equipped with the discrete topology. Since clearly $\operatorname{ind}(F(Y)) = 0$, the group homeomorphism $\widetilde{f} : (F(X), \mathfrak{T}_z) \to F(Y)$ extending f is continuous. The image $f(g) = f(x_1)^{\varepsilon_1} f(x_2)^{\varepsilon_2} \dots f(x_n)^{\varepsilon_n}$ is different from the identity in $F(Y)$, since $n \geq 1$, $\varepsilon_i = \pm 1$, and $f(x_{i+1}) \neq f(x_i)$ for all i. As a consequence, $\widetilde{f}^{-1}(e)$ is an open neighbourhood of the identity not containing g, and the result follows. ∎

3.2. Remark. A similar result holds for free abelian zero-dimensional topological groups.

We finish this section with two rather obvious remarks. Say that a topological subgroup, H, of a topological group G is a *topological group retract* of G if there exists a continuous group endomorphism $r : G \to G$ (a *topological group retraction of* G *onto* H) with $r(G) = H$ and $r^2 = \mathrm{Id}_H$. (The last condition can be replaced with $r|_H = \mathrm{Id}_H$.)

3.3. Lemma. *Let $r : G \to G$ be a topological group retraction from a topological group G onto its subgroup H. Then $r : G \to H$ is an open group homomorphism*; *in particular*, *H is a topological factor group of G.*

Proof. Since a topological group retraction $r : G \to H$ is in particular a retraction of topological spaces, it is a quotient map. Now recall that a continuous homomorphism of topological groups is open if and only if it is quotient. ∎

3.4. Lemma. *Let $X = Y \oplus Z$ be the disjoint sum of two zero-dimensional spaces. Then the topological subgroup of the free zero-dimensional group*

$F(Y,\Theta)$, *generated by* Y, *is* (i) *canonically isomorphic to the free zero-dimensional group* $F(Y,\Theta)$, *and* (ii) *a topological group retract of* $F(Y,\Theta)$.

Furthermore, let $\varphi : X \to F(X,\Theta)$ *be the mapping determined by* $\varphi|_Z = \{e\}$ *and* $\varphi|_Y = \mathrm{Id}_Y$, *and let* $\overline{\varphi} : F(X,\Theta) \to F(X,\Theta)$ *be a unique group homomorphism extending* φ. *Then* φ *is a topological group retraction of* $F(X,\Theta)$ *onto* $F(Y,\Theta)$ *witnessing* (ii).

Proof. The mapping $\varphi : X \to F(X,\Theta)$ is continuous (since Y is open-and-closed in X) and therefore $\overline{\varphi} : F(X,\Theta) \to F(X,\Theta)$ is continuous as well. Using the algebraic freeness of the group $F(X,\Theta)$ over its subset X, it is easy to conclude that the image of $\overline{\varphi}$ coincides with the (algebraically free) subgroup $F(Y,\Theta)$. Since the restriction of $\overline{\varphi}$ to $F(Y,\Theta)$ is the identity map, one concludes that $\overline{\varphi}$ is a topological retraction of $F(X,\Theta)$ onto $F(Y,\Theta)$, proving (ii). To establish (i), for every continuous map $\psi : X \to F(X,\Theta)$ let $\overline{\psi} : F(X,\Theta) \to F(X,\Theta)$ denote a unique continuous group homomorphism extending ψ. Now observe that every continuous map $f : Y \to G \in \Theta$ extends in a unique fashion to a continuous homomorphism $\widetilde{f} : \overline{\varphi}(F(X,\Theta)) \to G$, where $\widetilde{f} = \overline{f \circ \varphi(F(X,\Theta))|_X}|_{\overline{\varphi}(F(X,\Theta))}$. ∎

4. Construction. Let a zero-dimensional space Y be as in Theorem 2.3 above. Using Arkhangel'skiĭ's Theorem 2.1, choose a Tikhonov space Z such that the free topological group $F(Y)$ is a topological factor-group of the free topological group $F(Z)$, and in addition the latter group is zero-dimensional.

The space $X = Y \oplus Z$ is zero-dimensional. Let $G = F(X,\Theta)$ be the free zero-dimensional topological group on X (Th. 3.1).

4.1. Claim. *The group* G *is algebraically free over* X.

Proof. Follows from Theorem 3.1. ∎

4.2. Claim. *The topological group* G *is topologically relatively free over* X.

Proof. Since $G \in \Theta$, any continuous mapping $f : X \to G$ lifts to a unique continuous homomorphism $\bar{f} : G \equiv F(X,\Theta) \to G$. ∎

4.3. Claim. *The free topological group* $F(Z)$ *is canonically isomorphic with* $F(Z,\Theta)$.

Proof. The desired canonical isomorphism, ι, is a unique extension to $F(Z)$ of the identity map Id_Z, the inverse to ι being the unique continuous homomorphism $F(Z,\Theta) \to F(Z)$ extending the same identity map Id_Z and existing because $F(Z) \in \Theta$. ∎

4.4. Claim. *The free topological group* $F(Y)$ *is contained in the variety of topological groups*, $\mathcal{V} = \mathcal{V}(G)$, *generated by* G.

Proof. According to Lemmas 3.4 and 3.3, $F(Z,\Theta)$ is a topological factor group of $G = F(X,\Theta)$ and therefore $F(Z,\Theta) \in \mathcal{V}(G)$. Using Claim 4.3, we conclude that $F(Y)$, being a topological factor group of the free topological group $F(Z) = F(Z, \Theta)$, is in $\mathcal{V}(G)$, too. ∎

4.5. Claim. *The topology of the free topological group $F(Y)$ is strictly finer than the topology of the free zero-dimensional topological group $F(Y, \Theta)$.*

Proof. According to a well-known property of free topological groups [2], the topology of $F(Y)$ is the finest group topology inducing the initial topology on Y. Therefore, it contains the topology of $F(Y, \Theta)$. Assuming the two coincide, one would conclude that $F(Y)$ is zero-dimensional, a contradiction with our choice of Y. ∎

4.6. Claim. *The topological group G is not free over X in the variety of topological groups, $\mathcal{V}(G)$, generated by G.*

Proof. Let $f : X = Y \oplus Z \to F(Y)$ be the mapping determined by $f|_Z = \{e\}$ and $f|_Y = \mathrm{Id}_Y$, and let $\bar{f} : F(X) \to F(Y)$ be a unique group homomorphism extending f. Note that f is continuous because Y is both open and closed in X. Assume G is free on X in the variety $\mathcal{V} = \mathcal{V}(G)$. Since $F(Y) \in \mathcal{V}$ by Claim 4.4, it then follows that $\bar{f} : F(X, \Theta) \to F(Y)$ must be continuous. According to Lemma 3.4, $\bar{f} = i \circ \overline{\varphi}$, where $i : F(Y, \Theta) \to F(Y)$ is an algebraic isomorphism. Since $\overline{\varphi} : F(X, \Theta) \to F(Y, \Theta)$ is a topological group retraction (Lemma 3.4), it is an open map (Lemma 3.3). From continuity of $\bar{f} = i \circ \overline{\varphi}$ it now follows that i must be continuous as well. But according to Claim 4.5, i is discontinuous, which is a contradiction. ∎

Combining Claims 4.1, 4.2, 4.6 and Assertion 1.4, we deduce the principal result of this article.

4.7. Main Theorem. *There exists a topological group G that is relatively free over a topological subspace X and algebraically free over X, yet not free over X in the variety of topological groups generated by G (therefore in any variety of topological groups).* ∎

5. Open question. It would be interesting to construct a topological group G that is topologically relatively free and algebraically free over a subspace X, but at the same time is not the free topological group in the variety generated by G over *any* subspace $Y \subseteq G$. Such an example would have further strengthened the main result of the present article.

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