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ON A LIMIT POINT ASSOCIATED WITH THE abc-CONJECTURE

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Let $Q(n)$ denote the squarefree part of n so that $Q(n)=\prod_{p|n} p$. Throughout, we set a,b, and c to be positive relatively prime integers with $c = a + b$. Define

$$L_{a,b} = \frac{\log c}{\log Q(abc)}.$$

The *abc*-conjecture of Masser and Oesterlé asserts that the greatest limit point of the double sequence $\{L_{a,b}\}$ is 1. Recently, in joint work with Browkin, Greaves, Schinzel, and the first author [1], it was shown that the *abc*-conjecture is equivalent to the assertion that the precise set S of limit points of $\{L_{a,b}\}$ is the interval $[1/3, 1]$. Unconditionally, using certain polynomial identities and a theorem concerning squarefree values of binary forms, they showed that $[1/3, 15/16] \subseteq S$. Further polynomial identities of Greaves and Nitaj (private communication) imply that $[1/3, 36/37] \subseteq S$. By considering $a = 1$ and $b = 2^n$, it is easy to see that $\{L_{a,b}\}$ has a limit point ≥ 1 in the extended real line. The purpose of this note is to establish the following:

THEOREM. $S \cap [1, 3/2) \neq \emptyset$.

In other words, we prove that there is a limit point of $\{L_{a,b}\}$ somewhere in the interval $[1, 3/2)$.

Before proving the theorem, it is of some value to discuss simpler arguments for two weaker results. First, we observe that the existence of a finite limit point ≥ 1 can be established as follows. Fix a positive integer k, and let $n \geq 2$ be a squarefree number. Observe that

$$n \leq Q(n(n^k - 1)) \leq n^{k+1}.$$

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Taking $a = 1$ and $b = n^k - 1$, we deduce that

$$\frac{k}{k+1} \le L_{a,b} \le k.$$

As n varies, we obtain infinitely many such $L_{a,b}$. Suppose $S \cap (1, \infty) = \emptyset$. Then the existence of infinitely many values of $L_{a,b}$ in $[k/(k+1), k]$ implies that there must be infinitely many a and b for which

$$\frac{k}{k+1} \le L_{a,b} \le 1 + \frac{1}{k}.$$

As this must be true for each positive integer k, we conclude that $1 \in S$. In other words, it follows that $S \cap [1, \infty) \ne \emptyset$.

Next, we show that $S \cap [1, 2] \ne \emptyset$. Let n be a positive integer, and let t be the smallest integer $> 2^n$ for which

$$Q(t(t-1)) \le 2t \quad \text{and} \quad Q((t+1)t) \ge 2(t+1).$$

The above inequalities can be seen to be possible as the first inequality holds when $t = 2^n + 1$ and the second holds when $t + 1$ is squarefree. Observe that

$$\begin{aligned} 2(t+1) &\le Q((t+1)t) \le Q(t(t-1)(t+1)) \\ &\le Q(t(t-1))(t+1) \le 2t(t+1). \end{aligned}$$

We take $a = 1$ and $b = (t-1)(t+1)$ so that $c = a + b = t^2$. As a function of n (or t), we see from the above inequality that

$$1 + o(1) \le \frac{2 \log t}{\log(2t(t+1))} \le L_{a,b} \le \frac{2 \log t}{\log(2(t+1))} \le 2.$$

The conclusion that $S \cap [1, 2] \ne \emptyset$ follows.

Our above result that S contains a number in $[1, 2]$ can be viewed as following from the simple polynomial identity

$$1 + (x-1)(x+1) = x^2.$$

To establish our main result, we modify the above argument somewhat and, in particular, replace the use of the above polynomial identity with

$$x^2(x-9) + 27(x-1) = (x-3)^3. \tag{1}$$

It may be possible that other polynomial identities will lead to a further shortening of the interval in the statement of the theorem. In this regard, we will also make use of the fact that the polynomial $f(x) = x(x-1)(x-3)$ is such that $f(m)$ is squarefree for infinitely many positive integers m. This follows from simple sieve considerations. More is true which may be of value for future identities of the type given in (1). As first noted by Gouvêa and Mazur [2], work of Hooley [3] implies that if $f(x) \in \mathbb{Z}[x]$ with each irreducible factor of $f(x)$ having degree ≤ 3, then there are infinitely many positive integers t for which $f(t)/R$ is squarefree where

$$R = \prod_{p^e || D} p^{e-1} \quad \text{with} \quad D = \gcd(f(m) : m \in \mathbb{Z}).$$

An analogous result for binary forms of degree ≤ 6 can be found in [1].

Proof of Theorem. Fix $\varepsilon > 0$ sufficiently small. Let t be a large positive integer, say $t \geq t_0(\varepsilon)$, with $t(t-1)(t-3)$ squarefree (as noted above, such t exist). Observe that

$$Q(t(t-1)(t-3)) \geq t^{2+2\varepsilon}.$$

We choose a positive integer m as small as possible such that

$$Q((3^{m-1}t)(3^{m-1}t-1)(3^{m-1}t-3)) \geq (3^{m-1}t)^{2+2\varepsilon}$$

and

$$Q((3^m t)(3^m t-1)(3^m t-3)) \leq (3^m t)^{2+2\varepsilon}.$$

Such an m exists as the first inequality holds when $m = 1$ and the second inequality holds if m is sufficiently large. Combining these two inequalities, we deduce

$$(3^{m-1}t)^{2+2\varepsilon} \leq Q((3^m t)(3^m t-1)(3^m t-3)(3^m t-9)) \leq (3^m t)^{3+2\varepsilon}. \tag{2}$$

We use the equation (1) with $x = 3^m t$. With this substitution, each of the three terms appearing in (1) is divisible by 27. As we wish for a and b to be relatively prime, we set

$$a = (3^{m-1}t)^2(3^{m-1}t-3) \quad \text{and} \quad b = 3^m t - 1.$$

Here $c = a + b = (3^{m-1}t-1)^3$, and from (2) we obtain

$$\frac{1}{3}(3^{m-1}t)^{2+2\varepsilon} \leq Q(abc) \leq (3^m t)^{3+2\varepsilon}.$$

Recalling that t is large, it is easy to see that

$$\frac{3}{3+3\varepsilon} \leq L_{a,b} \leq \frac{3}{2+\varepsilon}. \tag{3}$$

We finish the proof by supposing that $S \cap (1, 3/2) = \emptyset$ and proving that $1 \in S$. Since (3) holds for infinitely many different pairs (a, b) (as there are infinitely many choices for t that give rise to such a pair), $S \cap (1, 3/2) = \emptyset$ implies that there are infinitely many (a, b) for which $L_{a,b} \in [3/(3+3\varepsilon), 1/(1-\varepsilon)]$. As this is true for each choice of $\varepsilon > 0$ sufficiently small, it follows that $1 \in S$, completing the proof. ∎

We end the paper by noting that the interval $[1, 3/2)$ in the theorem can be shifted to the left. More specifically, a slight modification of the argument above gives that

$$S \cap \left[\frac{3}{3+\varepsilon}, \frac{3}{2+\varepsilon}\right] \neq \emptyset$$

for every $\varepsilon \in (0,1)$. Thus, for example, there must be an $\alpha \in S$ satisfying

$$\frac{36}{37} < 0.98 \leq \alpha \leq \frac{147}{101} < 1.46$$

though currently no such α is known.

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