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THE GROTHENDIECK GROUP OF GL(F) × GL(G)*-EQUIVARIANT MODULES OVER THE COORDINATE RING OF DETERMINANTAL VARIETIES*

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Introduction. Let K be a field and F,G two vector spaces over K of dimensions m,n respectively. Consider the affine space $X=\mathrm{Hom}_K(F,G)$ of linear maps from F to G. We identify X with the space $F^*\otimes G$. The coordinate ring A of X is naturally identified with the symmetric algebra $A=\mathrm{Sym}(F\otimes G^*)$. Under this identification, for fixed bases $\{f_1,\ldots,f_m\}$, $\{g_1,\ldots,g_n\}$ of F,G respectively, the tensor $f_i\otimes g_j^*$ corresponds to the (j,i)th entry function $t_{i,j}$ on X.

For each r with $0\le r\le \min(m,n)$ we denote by X_r the determinantal variety of maps of rank $\le r$:

$$X_r=\{\phi:F\to G\mid \operatorname{rank}\phi\le r\}. \tag{1}$$

We denote by A_r the coordinate ring of X_r.

The objective of this paper is the investigation of natural modules with support in X_r. By a natural module we mean the graded A_r-module with a $\mathrm{GL}(F)\times\mathrm{GL}(G)$ action compatible with the module structure.

Several families of such modules were constructed and investigated in [Ar], [B-E], [L]. However, there was no attempt to understand the structure of modules of that type.

In this paper we investigate the category $\mathcal{C}_r(F,G)$ of graded A_r-modules with the rational $\mathrm{GL}(F)\times\mathrm{GL}(G)$ action compatible with the module structure, and equivariant degree 0 maps. We denote by $K_0'(A_r)$ the Grothendieck group of the category $\mathcal{C}_r(F,G)$.

The main result is a complete description of $K_0'(A_r)$. We provide three families of modules, each of which gives the generators of $K_0'(A_r)$, with no relations. The three families come from three natural desingularizations of the determinantal variety X_r as the push downs of certain vector bundles on these desingularizations.

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Similar phenomena seem to occur in more general situations—for example for orbit closures in multiplicity free actions. This will be the subject of a separate paper.

In order to describe the results of the paper in more detail, let us define the three families of modules.

The variety X_r has two desingularizations Z_1 and Z_2 (comp. [L]):

$$Z_1 = \{(\phi, R) \in X \times \mathrm{Grass}(m-r, F) \mid \phi|_R = 0\}, \tag{2}$$

$$Z_2 = \{(\phi, \overline{R}) \in X \times \mathrm{Grass}(r, G) \mid \mathrm{Im}(\phi) \subset \overline{R}\}. \tag{3}$$

For $i = 1, 2$ we denote by q_i the projection of Z_i onto X_r, and by p_i the projection of Z_i onto the corresponding Grassmannian.

We also consider the fibre product $Z = Z_1 \times_{X_r} Z_2$ which can be described as

$$Z = \{(\phi, R, \overline{R}) \in X \times \mathrm{Grass}(m-r, F) \times \mathrm{Grass}(r, G) \mid \phi|_R = 0,\ \mathrm{Im}(\phi) \subset \overline{R}\}. \tag{4}$$

We denote by q the projection of Z onto X_r, by p the projection of Z onto the product of two Grassmannians, and by u_i the projection of Z onto Z_i.

Throughout the paper we denote by $0 \to \mathcal{R} \to F \to \mathcal{Q} \to 0$ the tautological sequence on $\mathrm{Grass}(m-r, F)$, and by $0 \to \overline{\mathcal{R}} \to G \to \overline{\mathcal{Q}} \to 0$ the tautological sequence on $\mathrm{Grass}(r, G)$.

We construct three families of sheaves over Z_1, Z_2 and Z.

Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an integral weight for $\mathrm{GL}(F)$. We set $\alpha' = (\alpha_1, \ldots, \alpha_r)$ and $\alpha'' = (\alpha_{r+1}, \ldots, \alpha_m)$. Let $\beta = (\beta_1, \ldots, \beta_n)$ be the integral weight for $\mathrm{GL}(G^*)$. We define $\beta' = (\beta_1, \ldots, \beta_r)$ and $\beta'' = (\beta_{r+1}, \ldots, \beta_n)$.

Let $\alpha = (\alpha', \alpha'')$. We assume both α', α'' to be dominant. Let β be a dominant weight. For each such pair (α, β) we define a sheaf

$$\mathcal{M}(\alpha, \beta) = p_1^*(S_{\alpha'}\mathcal{Q} \otimes S_{\alpha''}\mathcal{R}) \otimes S_\beta G^* \otimes \mathcal{O}_{Z_1} \tag{5}$$

on Z_1.

Let α be a dominant weight and $\beta = (\beta', \beta'')$ an arbitrary weight (we assume β', β'' to be dominant). For each such pair (α, β) we define a sheaf

$$\mathcal{N}(\alpha, \beta) = S_\alpha F \otimes p_2^*(S_{\beta'}\overline{\mathcal{R}}^* \otimes S_{\beta''}\overline{\mathcal{Q}}^*) \otimes \mathcal{O}_{Z_2} \tag{6}$$

on Z_2.

Finally, let α, β be arbitrary weights (assume that $\alpha', \alpha'', \beta', \beta''$ are dominant). On the variety Z we consider the sheaves

$$\mathcal{P}(\alpha, \beta) = p_1^*(S_{\alpha'}\mathcal{Q} \otimes S_{\alpha''}\mathcal{R}) \otimes p_2^*(S_{\beta'}\overline{\mathcal{R}}^* \otimes S_{\beta''}\overline{\mathcal{Q}}^*) \otimes \mathcal{O}_Z. \tag{7}$$

For arbitrary α, β we define

$$M(\alpha, \beta) = H^0(Z_1, \mathcal{M}(\alpha, \beta)), \tag{10}$$

$$N(\alpha,\beta) = H^0(Z_2,\mathcal{N}(\alpha,\beta)), \tag{11}$$

$$P(\alpha,\beta) = H^0(Z,\mathcal{P}(\alpha,\beta)). \tag{12}$$

Finally, for $q \in \mathbb{Z}$ and a graded module $M = \bigoplus_n M_n$ we denote by $M(q)$ the module M with gradation shifted by q, i.e. $M(q)_n = M_{q+n}$.

We will prove the following results:

THEOREM 1. *The group $K_0'(A_r)$ is generated by the classes of the modules of each of the families $M(\alpha,\beta)(q)$, $N(\alpha,\beta)(q)$, $P(\alpha,\beta)(q)$ where α,β are both dominant weights and $q \in \mathbb{Z}$.*

Theorem 1 allows us, in fact, to calculate the group $K_0'(A_r)$ explicitly.

We define, for any module M from $\mathcal{C}_r(F,G)$, $M = \bigoplus_{i\ge d} M_i$, its *graded character*:

$$\begin{aligned} \operatorname{char}(M) &:= \operatorname{char}(M,q) \\ &= \sum_i \operatorname{char}(M_i)q^i \in R(\mathrm{GL}(F)\times \mathrm{GL}(G))[[q]][q^{-1}]. \end{aligned} \tag{8}$$

This defines a homomorphism of abelian groups

$$\operatorname{char}\colon K_0'(A_r) \to R(\mathrm{GL}(F)\times \mathrm{GL}(G))[[q]][q^{-1}]. \tag{9}$$

THEOREM 2. (a) *The group $K_0'(A_r)$ is isomorphic to the additive subgroup of the ring $R(\mathrm{GL}(F)\times \mathrm{GL}(G))[[q]][q^{-1}]$ generated by the shifted characters of the modules $M(\alpha,\beta)$ (resp. of $N(\alpha,\beta), P(\alpha,\beta)$).*

(b) *$K_0'(A_r)$ is isomorphic to the additive group of the ring $R(\mathrm{GL}(F)\times \mathrm{GL}(G))[q,q^{-1}]$.*

In the remainder of the paper we work out the transition formulas between the generators given by each of the three families.

We also describe the degeneration sequence which is an acyclic complex of graded A_r-modules whose terms have composition series with factors $M(\alpha,\beta)$ and whose only homology is isomorphic to the coordinate ring of A_{r-1}.

Finally, we strengthen Theorem 1 to the assertion that every module M from $\mathcal{C}_r(F,G)$ has a canonical equivariant filtration whose factors have resolutions with terms being direct sums of modules of any of the three families.

The paper is organized as follows. A necessary step in the proof of Theorem 1 is the investigation of the cohomology of the sheaves $\mathcal{M}(\alpha,\beta)$, $\mathcal{N}(\alpha,\beta)$, $\mathcal{P}(\alpha,\beta)$ for the weights α, β not necessarily dominant. This is accomplished in Section 1. In Section 2 we prove Theorems 1 and 2. In Section 3 we write down expressions for the classes of the modules $M(\alpha,\beta)$ and $N(\alpha,\beta)$ in terms of the classes of the modules $P(\alpha,\beta)$, and vice versa.

In Section 4 we construct the degeneration sequence expressing the class in $K_0'(A_r)$ of the coordinate ring A_{r-1} of the smaller determinantal variety.

In fact, we show that the expression is a consequence of the exact sequence resolving A_{r-1} as an A_r-module by modules which filter into $M(\alpha,\beta)$'s. Finally, in Section 5 we prove the results on finite resolutions.

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1. The sheaves $\mathcal{M}(\alpha,\beta)$, $\mathcal{N}(\alpha,\beta)$, $\mathcal{P}(\alpha,\beta)$ and their cohomology. We start by stating the results of this section.

PROPOSITION 1. *Let α and β be two dominant weights.*

(a) $H^i(Z_1,\mathcal{M}(\alpha,\beta)) = 0$ *for* $i > 0$.
(b) $H^i(Z_2,\mathcal{N}(\alpha,\beta)) = 0$ *for* $i > 0$.
(c) $H^i(Z,\mathcal{P}(\alpha,\beta)) = 0$ *for* $i > 0$.

In the remaining results we assume, unless stated otherwise, that the characteristic of K equals 0. In order to state our results about cohomology we need a definition.

For arbitrary α we define the number $l(\alpha)$ as follows. Let ϱ be half the sum of positive roots of $\mathrm{GL}(F)$. Consider the weight $\alpha+\varrho = (u_1,\ldots,u_m)$. Then, by reverse induction on s (from $s = r$ to $s = 1$), define

$$\delta_s = \min\{t \mid t \ge \delta_{s+1},\tag{11}$$
$$t+m-s \notin \{\delta_{s+1}+m-s-1,\ldots,\delta_r+m-r,\alpha_{r+1}+m-r-1,\ldots,\alpha_m\}\}.$$

By construction the weight $(\delta_1,\ldots,\delta_r,\alpha_{r+1},\ldots,\alpha_m)+\varrho$ is not orthogonal to any root. By Bott's theorem there exists a unique l such that $H^l(\mathrm{Grass}(m-r,F), S_\delta\mathcal{Q}\otimes S''_\alpha\mathcal{R}) \ne 0$. We define $l(\alpha) := l$.

The first result about cohomology of the sheaves $\mathcal{M}(\alpha,\beta)$ specifies the largest i for which $H^i(Z_1,\mathcal{M}(\alpha,\beta))$ does not vanish.

PROPOSITION 2. (a) $H^i(Z_1,\mathcal{M}(\alpha,\beta)) = 0$ *for* $i > l(\alpha)$.
(b) *The cohomology module* $H^{l(\alpha)}(Z_1,\mathcal{M}(\alpha,\beta))$ *is nonzero.*

We show that some of the cohomology groups $H^i(Z_1,\mathcal{M}(\alpha,\beta))$ for $0 < i < l(\alpha)$ might be zero. We also describe the support of all the groups $H^i(Z_1,\mathcal{M}(\alpha,\beta))$.

In order to state these results we need to recall some basic definitions. We call a permutation σ of m an *r-grassmannian permutation* if $\sigma(1) > \sigma(2) > \ldots > \sigma(r)$ and $\sigma(r+1) > \ldots > \sigma(m)$. For each r-grassmannian permutation σ we denote by C_σ the Weyl chamber of all weights $(\gamma_1,\ldots,\gamma_m)$ such that the entries of the sequence $(\gamma_1+m-1,\gamma_2+m-2,\ldots,\gamma_{m-1}+1,\gamma_m)$ have no repetitions, and are in the same order as $(\sigma_1,\ldots,\sigma_m)$.

Then, for each r-grassmannian permutation σ of length i, we define the A_r-module $H^i(\mathrm{Grass}(m-r,F),\mathcal{M}(\alpha,\beta))_\sigma$ to be the part of the ith cohomology group coming from the weights from C_σ.

We prove the following result:

Proposition 3. *Let α be an arbitrary weight and let β be a dominant weight.*

(a) *The module $H^i(Z_1, \mathcal{M}(\alpha,\beta))_\sigma$ is nonzero if and only if there exists $\delta = (\delta_1,\dots,\delta_r)$ such that $\delta \geq \alpha'$ (termwise) and $(\delta,\alpha'') \in C_\sigma$.*

(b) *The support of the module $H^i(Z_1, \mathcal{M}(\alpha,\beta))_\sigma$ is the determinantal variety X_{s-1} for $s = \sigma(r+1)$.*

The results analogous to Propositions 2 and 3 are true for the sheaves $\mathcal{N}(\alpha,\beta)$. Let us just formulate these results.

Proposition 2$'$. (a) *$H^i(Z_2, \mathcal{N}(\alpha,\beta)) = 0$ for $i > l(\beta)$.*

(b) *The cohomology module $H^{l(\beta)}(Z_2, \mathcal{N}(\alpha,\beta))$ is nonzero.*

Proposition 3$'$. *Let α be a dominant weight and let β be an arbitrary weight.*

(a) *The module $H^i(Z_2, \mathcal{N}(\alpha,\beta))_\sigma$ is nonzero if and only if there exists $\delta = (\delta_1,\dots,\delta_r)$ such that $\delta \geq \beta'$ (termwise) and $(\delta,\beta'') \in C_\sigma$.*

(b) *The support of the module $H^i(Z_2, \mathcal{N}(\alpha,\beta))_\sigma$ is the determinantal variety X_{s-1} for $s = \sigma(r+1)$.*

The calculation of cohomology for our three families is based on Bott's theorem and a simple spectral sequence argument. Let us deal with the family $\mathcal{M}(\alpha,\beta)$. We observe that since p_1 is an affine map, $R^i(p_1)_*\mathcal{O}_{Z_1} = 0$ for $i > 0$. One can also see that $(p_1)_*\mathcal{O}_{Z_1} = \mathrm{Sym}(\mathcal{Q}\otimes G^*)$. Therefore by the Leray spectral sequence and the projection formula (assuming α arbitrary and β dominant) we have

$$(14)\qquad H^i(Z_1, \mathcal{M}(\alpha,\beta)) = H^i(\mathrm{Grass}(m-r,F), S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes \mathrm{Sym}(\mathcal{Q}\otimes G^*))\otimes S_\beta G^*.$$

Similarly (for α dominant and β arbitrary),

$$(15)\qquad H^i(Z_2, \mathcal{N}(\alpha,\beta)) = S_\alpha F\otimes H^i(\mathrm{Grass}(r,G), S_{\beta'}\overline{\mathcal{R}}^*\otimes S_{\beta''}\overline{\mathcal{Q}}^*\otimes \mathrm{Sym}(F\otimes \overline{\mathcal{R}}^*))$$

and (for both α and β arbitrary)

$$(16)\qquad H^i(Z, \mathcal{P}(\alpha,\beta)) = H^i(\mathrm{Grass}(m-r,F)\times \mathrm{Grass}(r,G), S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}^*\otimes S_{\beta'}\overline{\mathcal{R}}^*\otimes S_{\beta''}\overline{\mathcal{Q}}^*\otimes \mathrm{Sym}(\mathcal{Q}\otimes\overline{\mathcal{R}}^*)).$$

Proof of Proposition 1. The proposition follows from the formulas above by applying the Cauchy formulas for decomposition of symmetric powers ([MD], Ch. 1), the Littlewood–Richardson rule ([MD], Ch. 1) and Bott's theorem ([J]). The argument can be made characteristic free by using Kempf's vanishing theorem ([J]) and good filtrations ([D]), in particular Boffi's result [B] that a tensor product of Schur modules has a good filtration. ∎

Proof of Proposition 2. First of all, we can assume that $\beta = 0$ because by the projection formula tensoring with $S_\beta G^*$ commutes with taking cohomology. This means that by (14) we are reduced to calculating the cohomology

$$H^*(\mathrm{Grass}(m-r,F), S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes \mathrm{Sym}(\mathcal{Q}\otimes G^*)). \tag{17}$$

This can be rewritten as

$$\bigoplus_{\delta\in\alpha'\otimes\gamma} H^*(\mathrm{Grass}(m-r,F), S_\delta\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes S_\gamma G^*). \tag{18}$$

By the Littlewood–Richardson rule every weight δ occurring in $\alpha'\otimes\gamma$ is $\geq \alpha'$ termwise. Also, since $\dim \mathcal{Q} = r \leq \dim G^*$, all such δ will occur in the tensor product of α' with some γ.

Consider the weight $\alpha = (\alpha', \alpha'')$. Let δ_0 be the weight constructed in defining $l(\alpha)$. This is by definition the (termwise) minimal weight such that $S_{\delta_0}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}$ has nonzero cohomology. This cohomology occurs in degree $l(\alpha)$. Also it is clear by Bott's theorem that for weights $\delta \geq \delta_0$ termwise the cohomology of $S_\delta\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}$, if nonzero, has to occur in degrees $\leq l(\alpha)$. This proves parts (a) and (b) of the proposition. ∎

Examples. 1. Take $m = 6$, $r = 3$, $\alpha' = (2,1,1)$, $\alpha'' = (5,4,4)$. Then $\alpha + \varrho = (7,5,4,7,5,4)$. We get $\delta_3 = 3, \delta_2 = 4, \delta_1 = 4$. Therefore $(\delta, \alpha'') = (4,4,3,5,4,4)$ and $l(\alpha) = 1$. The proposition says that $H^i(\mathcal{M}(2,1,1;5,4,4))$ is nonzero for $i = 0,1$ and zero for $i \geq 2$.

2. Take $m = 3$, $r = 1$, $\alpha' = (1), \alpha'' = (4,4)$. Then $\alpha + \varrho = (3,5,4)$. The result is that $H^2(\mathcal{M}(\alpha,0) = S_{(3,3,3)}F$ and the only other nonzero cohomology group of $\mathcal{M}(\alpha,0)$ is $H^0(\mathcal{M}(\alpha,0))$. This shows that for $1 \leq i < l(\alpha)$ some of the cohomology groups of $\mathcal{M}(\alpha,0)$ might be zero.

Proof of Proposition 3. Again we can assume that $\beta = 0$. Choose an r-grassmannian permutation σ. We are interested in the support of the cohomology modules of

$$\mathcal{M}(\alpha,0)_\sigma = \bigoplus_{\gamma}\ \bigoplus_{\delta\in\alpha'\otimes\gamma,\,(\delta,\alpha'')\in C_\sigma} S_\delta\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes S_\gamma G^*. \tag{19}$$

Let $\sigma(r+1) = s$. Then we can increase $\delta_1,\ldots,\delta_{s-1}$ as we please to still get weights in C_σ. On the other hand, we can increase the indices $\delta_s,\ldots,\delta_r$ only in the limited way if we are to get a weight in C_σ. This shows that the support of our module is X_{s-1}. ∎

2. Proof of Theorems 1 and 2. We start with the proof of Theorem 1.

We prove the theorem for the modules $M(\alpha,\beta)$. The proof for $N(\alpha,\beta)$ is symmetric. The proof for $P(\alpha,\beta)$ will follow, since we will also show that the class of each $M(\alpha,\beta)$ in $K_0'(A_r)$ can be expressed through the classes of the $P(\alpha,\beta)$.

We prove the theorem in several steps.

Each equivariant sheaf $\mathcal{M}$ on Z_1 (corresponding to a graded $\mathrm{Sym}(\mathcal{Q} \otimes G^*)$-module) has Euler characteristic class

$$\chi(\mathcal{M}) = \sum_{i \geq 0} (-1)^i [H^i(Z_1, \mathcal{M})],$$

which can be treated as an element of $K_0'(A_r)$ because every cohomology group $H^i(Z_1, \mathcal{M})$ is clearly an object of $\mathcal{C}_r(F, G)$.

PROPOSITION 4. *The Euler characteristic classes* $\chi(\mathcal{M}(\alpha', \alpha'', \beta))$ *generate* $K_0'(A_r)$.

Proof. Consider an arbitrary graded A_r-module M with rational $\mathrm{GL}(F) \times \mathrm{GL}(G)$ action. Then the natural morphism

$$M \to (q_1)_* q_1^* M \tag{20}$$

has kernel and cokernel supported in X_{r-1}. It is therefore enough to show that:

1) The class in $K_0'(A_r)$ of any module supported in X_{r-1} is in the subgroup generated by the classes $\chi(\mathcal{M}(\alpha', \alpha'', \beta))$.

2) The class $\chi(q_1^* M)$ is in that subgroup (we use the fact that higher cohomology groups $H^i(q_1^* M)$ are supported in X_{r-1}).

To prove 1) it is enough to show that the Euler characteristic of each module of type $\mathcal{M}$ for X_{r-1} is in our subgroup of $K_0'(A_r)$. Consider the grassmannian $\mathrm{Grass}(m-r+1, F)$ with tautological sequence $0 \to \widehat{\mathcal{R}} \to F \to \widehat{\mathcal{Q}} \to 0$. Let us work over the partial flag variety $\mathrm{Flag}(m-r, m-r+1, F) := \mathrm{GL}(F)/P$ on which all bundles $\mathcal{R}, \mathcal{Q}, \widehat{\mathcal{R}}, \widehat{\mathcal{Q}}$ are defined. Modules of type $\mathcal{M}$ for X_{r-1} will be denoted by $\widehat{\mathcal{M}}$. We also give names to the natural projections

$$\mathrm{Grass}(m-r+1, F) \xleftarrow{v_2} \mathrm{Flag}(m-r, m-r+1, F) \xrightarrow{v_1} \mathrm{Grass}(m-r, F). \tag{21}$$

Since in this argument all constructions commute with tensoring by $S_\beta G^*$, we will drop it from our notation, dealing with the modules $\mathcal{M}(\alpha', \alpha'') := \mathcal{M}(\alpha', \alpha'', 0)$ and similarly for the modules $\widehat{\mathcal{M}}$.

We have, by definition,

$$\begin{aligned} &\widehat{\mathcal{M}}(\alpha_1', \ldots, \alpha_{r-1}', \alpha_1'', \ldots, \alpha_{m-r+1}'') \\ &\quad = v_{2*}(S_{(\alpha_1', \ldots, \alpha_{r-1}')} \widehat{\mathcal{Q}} \otimes S_{\alpha_1''}(\widehat{\mathcal{R}}/\mathcal{R}) \otimes S_{(\alpha_2'', \ldots, \alpha_{m-r+1}'')} \mathcal{R} \otimes \mathrm{Sym}(\widehat{\mathcal{Q}} \otimes G^*)). \end{aligned} \tag{22}$$

The higher direct images of the tensor product in brackets on the right hand side vanish.

This module has a Koszul type resolution on $\mathrm{Flag}(m-r, m-r+1, F)$ over $\mathrm{Sym}(\mathcal{Q}\otimes G^*)$ with terms

$$S_{(\alpha'_1,\ldots,\alpha'_{r-1})}\widehat{\mathcal{Q}}\otimes S_{\alpha''_1}(\widehat{\mathcal{R}}/\mathcal{R}) \otimes S_{(\alpha''_2,\ldots,\alpha''_{m-r+1})}\mathcal{R}\otimes {\bigwedge}^{\cdot}(\mathrm{Ker}(\mathcal{Q}\to\widehat{\mathcal{Q}})\otimes G^*)\otimes \mathrm{Sym}(\mathcal{Q}\otimes G^*), \tag{23}$$

which can be rewritten as

$$\bigoplus_t S_{(\alpha'_1,\ldots,\alpha'_{r-1})}\widehat{\mathcal{Q}}\otimes S_{\alpha''_1+t}(\widehat{\mathcal{R}}/\mathcal{R})\otimes S_{(\alpha''_2,\ldots,\alpha''_{m-r+1})}\mathcal{R}\otimes {\bigwedge}^t G^* \otimes \mathrm{Sym}(\mathcal{Q}\otimes G^*) \tag{24}$$

because $\mathrm{Ker}(\mathcal{Q}\to\widehat{\mathcal{Q}})$ is isomorphic to $\widehat{\mathcal{R}}/\mathcal{R}$ as can be seen from the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{R} & \to & F & \to & \mathcal{Q} & \to & 0\\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \widehat{\mathcal{R}} & \to & F & \to & \widehat{\mathcal{Q}} & \to & 0 \end{array}$$

Pushing down the terms of this resolution by v_{1*} we get (by the projection formula; $\mathcal{Q}\otimes G^*$ is induced from $\mathrm{Grass}(m-r, F)$) an expression for each term in the resolution as the Euler characteristic of a module of type $\mathcal{M}(\gamma', \gamma'', \delta)$, possibly with sign. We have thus expressed $\chi(\widehat{\mathcal{M}}(\alpha', \alpha''))$ as a linear combination in $K'_0(A_r)$ of the Euler characteristics $\chi(\mathcal{M}(\gamma', \gamma'', \delta))$. This proves statement 1).

Remark 5. *The same proof shows that if we start with* $\widehat{\mathcal{M}}(\alpha', \alpha'', \beta)$ *where* $\alpha = (\alpha', \alpha'')$ *is dominant, then its Euler characteristic class lies in the subgroup of* $K'_0(A_r)$ *generated by the Euler characteristics* $\chi(\mathcal{M}(\gamma', \gamma'', \delta))$ *with* $\gamma = (\gamma', \gamma'')$ *dominant.*

Proof. Pushing down the terms in the formula (24) on $\mathrm{Grass}(m-r, F)$ means we apply Bott's theorem to the sequence $(\alpha'_1, \ldots, \alpha'_{r-1}, \alpha''_1 + t)$. In the case when $\alpha = (\alpha', \alpha'')$ is dominant we can only get either 0 (meaning the corresponding Euler characteristic is 0) or weights which, together with $(\alpha''_2, \ldots, \alpha''_{m-r+1})$, form dominant weights. ∎

To prove statement 2) we notice that $q_1^* M$ is a sheaf of graded $\mathrm{Sym}(\mathcal{Q}\otimes G^*)$-modules. We take its finite free $\mathrm{GL}(\mathcal{Q})\times\mathrm{GL}(\mathcal{R})\times\mathrm{GL}(G)$-equivariant resolution. Its terms are up to filtration direct sums of modules of type $\mathcal{M}(\alpha', \alpha'', \beta)$ (or, if $\mathrm{char}\, K \neq 0$, they are equivalent to a combination of such terms in the Grothendieck group). Pushing this resolution down we see that $\chi(q_1^* M)$ is an alternating sum of the Euler characteristics $\chi(\mathcal{M}(\alpha', \alpha'', \beta))$. This completes the proof of Proposition 4. ∎

The second part of the proof of Theorem 1 is

Theorem 6. *The classes* $\chi(\mathcal{M}(\alpha', \alpha'', \beta))$ *such that* (α', α'') *is dominant generate* $K'_0(A_r)$.

P r o o f. We will assume throughout this proof that $\operatorname{char} K = 0$. The argument can be easily adjusted to the characteristic free case by using good filtrations (comp. [D]), but we leave it to the reader.

Consider an arbitrary sheaf $\mathcal{M}(\alpha', \alpha'', \beta)$. We use induction on $s := \alpha''_1 - \alpha'_r$. If $s \le 0$ then (α', α'') is dominant and there is nothing to prove. Suppose that for (γ', γ'') with smaller s the corresponding sheaves are in the subgroup of $K'_0(A_r)$ in question.

We will identify $\mathcal{M}(\alpha', \alpha'', \beta)$ with its direct image $p_{1*}\mathcal{M}(\alpha', \alpha'', \beta)$, i.e.

$$\mathcal{M}(\alpha', \alpha'', \beta) = S_{\alpha'}\mathcal{Q} \otimes S_{\alpha''}\mathcal{R} \otimes \operatorname{Sym}(\mathcal{Q} \otimes G^*). \tag{25}$$

Consider the subsheaf of $\mathcal{M}(\alpha', \alpha'', \beta)$ consisting of all sheaves $S_{\gamma'}\mathcal{Q} \otimes S_{\gamma''}\mathcal{R} \otimes S_\delta G^*$ such that $\gamma''_1 - \gamma'_r$ is smaller than s. It is clearly a $\operatorname{Sym}(\mathcal{Q} \otimes G^*)$-submodule of $\mathcal{M}(\alpha', \alpha'', \beta)$. We call it $\mathcal{M}_{<s}(\alpha', \alpha'', \beta)$. We also denote the factor $\mathcal{M}(\alpha', \alpha'', \beta)/\mathcal{M}_{<s}(\alpha', \alpha'', \beta)$ by $\mathcal{M}_s(\alpha', \alpha'', \beta)$. By definition we have an exact sequence of $\operatorname{Sym}(\mathcal{Q} \otimes G^*)$-modules

$$0 \to \mathcal{M}_{<s}(\alpha', \alpha'', \beta) \to \mathcal{M}(\alpha', \alpha'', \beta) \to \mathcal{M}_s(\alpha', \alpha'', \beta) \to 0.$$

The support of the module $\mathcal{M}_s(\alpha', \alpha'', \beta)$ (or rather of all its cohomology groups) is contained in X_{r-1}. Indeed, if we multiply the representation $S_{\gamma'}\mathcal{Q} \otimes S_{\gamma''}\mathcal{R} \otimes S_\delta G^*$ by $\bigwedge^r \mathcal{Q} \otimes \bigwedge^r G^*$ corresponding to $r \times r$ minors, we add one to each entry of γ' without changing γ'', so we decrease s. This means that the ideal generated by $r \times r$ minors annihilates all the cohomology groups of $\mathcal{M}_s(\alpha', \alpha'', \beta)$. Therefore, by induction on r and by Remark 5, the Euler characteristic $\chi(\mathcal{M}_s(\alpha', \alpha'', \beta))$ is contained in our subgroup of $K'_0(A_r)$.

Now, consider an $\mathrm{GL}(\mathcal{Q}) \times \mathrm{GL}(\mathcal{R}) \times \mathrm{GL}(G)$-equivariant resolution of $\mathcal{M}_{<s}(\alpha', \alpha'', \beta)$ by free $\operatorname{Sym}(\mathcal{Q} \otimes G^*)$-modules. Its terms are up to filtration direct sums of sheaves $\mathcal{M}(\gamma', \gamma'', \delta)$ and each term occurring in the resolution has smaller s than (α', α'') has. Indeed, the generators of the 0th syzygy come from $\mathcal{M}_{<s}(\alpha', \alpha'', \beta)$. The generators of the $(i+1)$-st syzygy are contained in $\mathcal{M}(\gamma', \gamma'', \delta) \otimes \operatorname{Sym}(\mathcal{Q} \otimes G^*)$ where $\mathcal{M}(\gamma', \gamma'', \delta)$ is a term in the ith syzygy, so by the Littlewood–Richardson rule the invariant s can only decrease. Therefore, by induction on s, the Euler characteristic $\chi(\mathcal{M}_{<s}(\alpha', \alpha'', \beta))$ lies in our subgroup of $K'_0(A_r)$. This concludes the proof of Proposition 6. ∎

Theorem 1 is now proven for the families $M(\alpha, \beta)$ and $N(\alpha, \beta)$. To conclude the proof for $P(\alpha, \beta)$ we need the following

PROPOSITION 7. *The classes of the modules $M(\alpha, \beta)$ (for α, β dominant) lie in the subgroup of $K'_0(A_r)$ generated by the classes of the modules $P(\gamma, \delta)$ (with γ, δ dominant).*

Before proving Proposition 7 we prove another useful statement.

PROPOSITION 8. *Let α, β be dominant. Then for every dominant γ the class in $K_0'(A_r)$ of $P(\alpha,\beta)\otimes S_\gamma G^*$ is an element of the subgroup generated by the classes of the modules $P(\delta,\varepsilon)$ (δ, ε dominant). Similarly, for every dominant γ the class in $K_0'(A_r)$ of $S_\gamma F\otimes P(\alpha,\beta)$ is an element of the subgroup generated by the classes of the modules $P(\delta,\varepsilon)$ (δ, ε dominant).*

Proof. Because of symmetry it is enough to prove the first statement. It is also enough to do the proof for $\operatorname{char} K = 0$ because any equality of characters involving Schur functors which is true in characteristic 0 is automatically true in the representation ring $R(\mathrm{GL}(F)\times\mathrm{GL}(G))$ in arbitrary characteristic.

Using the Jacobi–Trudi determinantal expression for $S_\gamma G^*$ as a combination of tensor products of exterior powers $\bigwedge^j G^*$ ([MD], Ch. 1, [A], [Z]), it is enough to show that the class in $K_0'(A_r)$ of $P(\alpha,\beta)\otimes\bigwedge^j G^*$ is an element of the subgroup generated by the classes of the modules $P(\delta,\varepsilon)$ (δ, ε dominant). Consider the sheaf $\mathcal{P}(\alpha,\beta)\otimes\bigwedge^j G^*$. It obviously has a filtration with the associated graded object

$$\bigoplus_{a+b=j}\mathcal{P}(\alpha,\beta)\otimes\bigwedge^a\overline{\mathcal{R}}^*\otimes\bigwedge^b\overline{\mathcal{Q}}^*. \tag{26}$$

Using Pieri's formula ([MD], Ch. 1) we notice that the above sheaf decomposes into a direct sum of sheaves of type $\mathcal{P}(\alpha,\xi)$ where the weight ξ satisfies $\xi_1\ge\ldots\ge\xi_r$, $\xi_{r+1}\ge\ldots\ge\xi_n$ and $\xi_r\ge\xi_{r+1}-1$. We notice that, by the formula (16) and by the Littlewood–Richardson rule the sheaves $\mathcal{P}(\alpha,\xi)$ have no higher cohomology. Therefore to conclude the proof it is enough to show that the modules $H^0(Z,\mathcal{P}(\alpha,\xi))$ are in the subgroup of $K_0'(A_r)$ generated by the modules $P(\delta,\varepsilon)$ (δ, ε dominant).

Consider the sheaf $\mathcal{P}(\alpha,\xi)$. If $\xi_r\ge\xi_{r+1}$ then the last statement is obvious since α and ξ are dominant. Therefore, assume $\xi_r=\xi_{r+1}-1$. Recall that by (16) and the Cauchy formula,

$$\mathcal{P}(\alpha,\xi)=S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes S_{\xi'}\overline{\mathcal{R}}^*\otimes S_{\xi''}\overline{\mathcal{Q}}^*\otimes\Big(\bigoplus_\gamma S_\gamma\mathcal{Q}\otimes S_\gamma\overline{\mathcal{R}}^*\Big). \tag{27}$$

Using the Littlewood–Richardson rule we see that the weights giving a nonzero contribution to $H^0(Z,\mathcal{P}(\alpha,\xi))$ are those corresponding to the highest weights $\psi=(\psi_1,\ldots,\psi_r)$ of irreducible representations in $S_{\xi'}\overline{\mathcal{R}}^*\otimes S_\gamma\overline{\mathcal{R}}^*$ for which $\psi_r>\xi_r$. The span of the coresponding representations obviously gives a graded $\mathrm{GL}(F)\times\mathrm{GL}(G^*)$-equivariant subsheaf $\mathcal{T}$ of $\mathcal{P}(\alpha,\xi)$. Identifying $\mathcal{T}$ with its direct image $p_*\mathcal{T}$ we can treat this sheaf as a sheaf of graded $\mathrm{Sym}(\mathcal{Q}\otimes\overline{\mathcal{R}}^*)$-modules.

CLAIM. *The sheaf $p_*\mathcal{T}$ has a finite graded equivariant resolution with terms which, up to filtration, are direct sums of terms*

$$S_{\alpha'}\mathcal{Q} \otimes S_{\alpha''}\mathcal{R} \otimes S_{\psi}\overline{\mathcal{R}}^* \otimes S_{\xi''}\overline{\mathcal{Q}}^* \otimes \mathrm{Sym}(\mathcal{Q} \otimes \overline{\mathcal{R}}^*)$$

with weights ψ satisfying $\psi_r > \xi_r$.

P r o o f. We can identify any $\mathrm{GL}(F) \times \mathrm{GL}(G^*)$-equivariant sheaf on $\mathrm{Grass}(m-r, F) \times \mathrm{Grass}(r, G)$ with a rational $P' \times P''$-module (where P', P'' denote the parabolic subgroups in $\mathrm{GL}(F)$, $\mathrm{GL}(G^*)$ respectively corresponding to the two grassmannians). Taking this point of view we see that resolving $p_*\mathcal{T}$ is just resolving the corresponding graded $\mathrm{Sym}(\mathcal{Q} \otimes \overline{\mathcal{R}}^*)$-module. The only thing we have to worry about is preserving the $P' \times P''$ action. Denote by L', L'' the Levi subgroups in P', P'' respectively and by U', U'' the unipotent radicals. For every $P' \times P''$-graded $\mathrm{Sym}(\mathcal{Q} \otimes \overline{\mathcal{R}}^*)$-module its graded components have canonical filtrations on whose subquotients the radicals U', U'' act trivially (we filter by the $\mathcal{Q}$-content and by the $\overline{\mathcal{R}}^*$-content). Moreover, multiplying by $\mathcal{Q} \otimes \overline{\mathcal{R}}^*$ is compatible with those filtrations. This means that every finitely generated $\mathrm{Sym}(\mathcal{Q} \otimes \overline{\mathcal{R}}^*)$-module with compatible $P' \times P''$ action has a finite filtration on whose factors $U' \times U''$ acts trivially.

For a statement in K_0' it is therefore enough to resolve $p_*\mathcal{T}$ as an $L' \times L''$-module. Then, by reductivity of $L' \times L''$, a finite, graded equivariant resolution of any $\mathrm{Sym}(\mathcal{Q} \otimes \overline{\mathcal{R}}^*)$-module exists. The statement about weights ψ follows from the fact that all weights in $p_*\mathcal{T}$ satisfy the inequality $\psi_r > \xi_r$, and from the Littlewood–Richardson rule. ∎

The claim implies that all the terms in the resolution have factors of type $\mathcal{P}(\delta, \varepsilon)$ for δ, ε dominant. Taking the sections of this resolution we deduce that $H^0(Z, \mathcal{P}(\alpha, \xi))$ has a resolution whose terms, up to filtration, are direct sums of modules $P(\delta, \varepsilon)$ (δ, ε dominant). This proves the proposition. ∎

Proof of Proposition 7. It follows from Proposition 8 that we can assume that $\beta = 0$, i.e. it is enough to express the class of $M(\alpha, 0)$ through the classes of $P(\gamma, \delta)$ with γ, δ dominant. On the other hand, it is clear that $M(\alpha, 0) = P(\alpha, 0)$ by direct calculation of the cohomology groups of the corresponding sheaves. ∎

Proof of Theorem 2. Part (a) of Theorem 2 is a consequence of the fact that the characters of the modules $M(\alpha, \beta)(-i)$ are linearly independent in $R(\mathrm{GL}(F) \times \mathrm{GL}(G))[[q]][q^{-1}]$.

To prove (b) we define the homomorphism of groups

$$\Psi : R(\mathrm{GL}(F) \times \mathrm{GL}(G))[q][q^{-1}] \to K_0'(A_r) \tag{28}$$

by sending $[S_\alpha F \otimes S_\beta G^*]q^i$ to $[M(\alpha, \beta)(-i)]$. By Theorem 1, Ψ is an epimorphism. It is also a monomorphism because the classes of the shifted modules $M(\alpha, \beta)(-i)$ are linearly independent in $K_0'(A_r)$. ∎

3. Transition formulas. In the preceding section we proved that the group $K_0'(A_r)$ has three sets of generators, the classes of the modules

$M(\alpha,\beta)(q)$, $N(\alpha,\beta)(q)$, $P(\alpha,\beta)(q)$, each indexed by pairs (α,β) of dominant weights and integers q. In this section we write down transition formulas allowing one to write an element of each basis as a linear combination of the elements of another basis. We are dealing only with the classes of $M(\alpha,\beta)$; the statements for $N(\alpha,\beta)$ are symmetric.

We prove all the formulas when $\operatorname{char} K = 0$, but they are true in arbitrary characteristic because, as already remarked, any equality involving Schur functors true in characteristic 0 is true in the representation ring $R(\mathrm{GL}(F)\times \mathrm{GL}(G))$ in arbitrary characteristic.

We start with the formula expressing the class of $P(\alpha,\beta)$ through the classes of modules $M(\alpha,\beta)$.

THEOREM 3. *Let $\alpha=(\alpha',\alpha'')$ and $\beta=(\beta',\beta'')$ be dominant. The class of the module $P(\alpha,\beta)$ in $K_0'(A_r)$ can be expressed through the classes of $M(\gamma,\delta)(q)$ (with $\gamma=(\gamma',\gamma'')$, $\delta=(\delta',\delta'')$ being dominant) as follows*:

$$(29)\qquad [P(\alpha,\beta)] = \sum_{\lambda}\sum_{\gamma'\in\widetilde{\lambda}\otimes\alpha'}\sum_{\eta''\in\beta''\otimes\lambda,\,\delta=\chi(\beta',\eta'')}(-1)^{|\lambda|+\varepsilon(\beta',\eta'')}M(\gamma',\alpha'';\delta)(-|\lambda|).$$

Here $\eta\in\mu\otimes\nu$ means that we take partitions from the tensor product on the right with proper multiplicities, and $\eta=\chi(\mu,\nu)$ means that we apply Bott's algorithm to the sequence (μ,ν), writing the appropriate summand with a proper sign $\varepsilon(\mu,\nu)$ The symbol $\widetilde{\lambda}$ stands for the partition conjugate to λ.

EXAMPLE. Take $m=n=4$, $r=2$, $\alpha'=(1,0),\alpha''=(0,0),\beta'=(1,1)$, $\beta''=(0,0)$. Then the class of $P(1,0,0,0;1,1,0,0)$ equals

$$[M(1,0;0,0;1,1;0,0)-M(2,0;0,0;1,1;1,0)-M(1,1;0,0;1,1;1,0) + M(3,0;0,0;1,1;1,1)+M(2,1;0,0;1,1;1,1)]$$

with the first summand corresponding to $\lambda=(0,0)$, two next ones to $\lambda=(1,0)$, and the last one to $\lambda=(1,1)$.

Proof of Theorem 3. The proof is based on the push down of the Koszul complex. Consider the sheaf $p_*\mathcal{P}(\alpha,\beta)$ as a sheaf on $\mathrm{Grass}(m-r,F)\times\mathrm{Grass}(r,G)$. It is a sheaf of graded $\mathrm{Sym}(\mathcal{Q}\otimes\overline{\mathcal{R}}^*)$-modules. Consider the Koszul complex $\bigwedge^{\cdot}(\mathcal{Q}\otimes\overline{\mathcal{Q}}^*)$ resolving $\mathrm{Sym}(\mathcal{Q}\otimes\overline{\mathcal{R}}^*)$ as a $\mathrm{Sym}(\mathcal{Q}\otimes G^*)$-module. Tensoring this complex with $S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes S_{\beta'}\overline{\mathcal{R}}^*\otimes S_{\beta''}\overline{\mathcal{Q}}^*$ we get a resolution of $p_*\mathcal{P}(\alpha,\beta)$ whose terms are sheaves whose push downs on $\mathrm{Grass}(m-r,F)$ are (up to sign) sheaves of type $\mathcal{M}(\gamma,\delta)$. Using the Cauchy formula for the decomposition of $\bigwedge^{\cdot}(\mathcal{Q}\otimes\overline{\mathcal{Q}}^*)$ and taking Euler characteristics we get the assertion of Theorem 3. ■

Let us give an algorithm for expressing the class of $M(\alpha,\beta)$ through the classes of $P(\gamma,\delta)$. We start with the case of $\beta=(1^j)$.

PROPOSITION 9. *Let $\beta=(1^j)$ and let $\alpha=(\alpha',\alpha'')$ be dominant. Then the module $M(\alpha,\beta)$ has a filtration with associated graded object*

$$\bigoplus_{(1^{n-r})\supset\delta''\supset\beta''}\ \bigoplus_{\gamma'\in\alpha'\otimes\delta/\beta} P(\gamma',\alpha'';\delta',\delta''). \tag{30}$$

Proof. The module $M(\alpha,\beta)$ can be identified with

$$M(\alpha,\beta)=H^0(\mathrm{Grass}(m-r,F)\times\mathrm{Grass}(r,G),\widehat{\mathcal{M}}(\alpha,\beta)) \tag{31}$$

where $\widehat{\mathcal{M}}(\alpha,\beta)$ is the sheaf over $\mathrm{Grass}(m,F)\times\mathrm{Grass}(r,G)$ defined by the formula

$$\widehat{\mathcal{M}}(\alpha,\beta)=S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes\bigwedge^j G^*\otimes\mathrm{Sym}(\mathcal{Q}\otimes\overline{\mathcal{R}}^*). \tag{32}$$

The sheaf $\mathcal{M}(\alpha,\beta)$ can be filtered in such a way that the associated graded object is

$$\bigoplus_{a+b=j} S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes\bigwedge^a\overline{\mathcal{R}}^*\otimes\bigwedge^b\overline{\mathcal{Q}}\otimes\mathrm{Sym}(\mathcal{Q}\otimes\overline{\mathcal{R}}^*). \tag{33}$$

No summand in (33) has higher cohomology, and the global sections of the summand corresponding to the pair (a,b) are easily identified with

$$\bigoplus_{\gamma'\in\alpha'\otimes 1^{r-a}} P(\gamma',\alpha'';(1^r),(1^b,0^{n-r-b})) \tag{34}$$

embedded by tensoring with $\bigwedge^{r-a}\mathcal{Q}\otimes\bigwedge^{r-a}\overline{\mathcal{R}}^*$. This implies the proposition, with the sheaf from (34) corresponding to the summand in (30) with $\delta''=(1^b)$. ∎

The proposition yields a formula for the class of an arbitrary $M(\alpha,\beta)$ by applying the Jacobi–Trudi formula expressing the functor $S_\beta G^*$ as a combination of tensor products of exterior powers (comp. [MD], Ch. 1), provided we give a formula for the class of $P(\alpha,\beta)\otimes\bigwedge^j G^*$ in $K_0'(A_r)$. The bundle $p^*\bigwedge^j G^*$ can be filtered, so the associated graded object is $\bigoplus_{a+b=j}\bigwedge^a\overline{\mathcal{R}}^*\otimes\bigwedge^b\overline{\mathcal{Q}}^*$. Using this filtration we see that

$$[P(\alpha,\beta)\otimes\bigwedge^j G^*]=\sum_{\xi'\in\beta'\otimes 1^a,\ \xi''\in\beta''\otimes 1^b}[P(\alpha,\xi',\xi'')].$$

All the sequences (ξ',ξ'') on the right hand side have the property $\xi'_r\ge\xi''_1-1$. If the inequality is strict, the corresponding weight is dominant. It therefore remains to express the class of the module $P(\alpha,\xi',\xi'')$ through the classes of dominant $P(\alpha,\beta)$ in the case $\xi'_r=\xi''_1-1$.

Before we state the result we need some notation.

For fixed α, ξ', ξ'' satisfying $\xi'_r = \xi''_1 - 1$ we define μ to be the partition $(\xi'_1 - \xi'_r, \ldots, \xi'_{r-1} - \xi'_r, 0)$. Then, for each $j \geq 1$, we define $\mu(j)$ to be the partition conjugate to

$$\widetilde{\mu(j)} = (r, \widetilde{\mu}_1 + 1, \ldots, \widetilde{\mu}_{j-1} + 1, \widetilde{\mu}_{j+1}, \ldots, \widetilde{\mu}_t)$$

and set $s_j = |\mu(j)/\mu|$. With this notation we have

PROPOSITION 10. *Let $\alpha = (\alpha', \alpha'')$ be dominant and let $\xi = (\xi', \xi'')$ be such that $\xi'_r = \xi''_1 - 1$. Then*

$$[P(\alpha, \xi', \xi'')] = \sum_{j\geq 1} (-1)^{j+1} \Big[\sum_{\gamma' \in \alpha' \otimes 1^{s_j},\, \gamma'' = \alpha''} \ \sum_{\delta'_s = r + \mu(j)_s, \delta'' = \xi''} P(\gamma, \delta)(-s_j) \Big].$$

Proof. Consider the sheaf $p_* \mathcal{P}(\alpha', \alpha'', \xi', \xi'')$ on $\mathrm{Grass}(m-r, F) \times \mathrm{Grass}(r, G)$. Let $\mathcal{N}$ be the subsheaf of $p_* \mathcal{P}(\alpha', \alpha'', \xi', \xi'')$ consisting of the subbundles with nonzero sections. We will exhibit an explicit resolution of $\mathcal{N}$.

We first treat a special case.

CLAIM. *Consider the sheaf of algebras $\mathcal{B} = \mathrm{Sym}(\mathcal{Q} \otimes \overline{\mathcal{R}}^*)$. Let $\mathcal{M}_\mu \subset S_\mu \overline{\mathcal{R}}^* \otimes \mathcal{B}$ be the subsheaf consisting of the subbundles $S_\phi \mathcal{Q} \otimes S_\psi \overline{\mathcal{R}}^*$ such that $\psi_r \geq 1$. Let $\mathcal{N}_\mu$ be the factor $S_\mu \overline{\mathcal{R}}^* \otimes \mathcal{B}/\mathcal{M}_\mu$. Then $\mathcal{N}_\mu$ has a finite resolution $\mathcal{G}_{\cdot,\mu}$ over $\mathcal{B} = \mathrm{Sym}(\mathcal{Q} \otimes \overline{\mathcal{R}}^*)$ with terms*

$$\mathcal{G}_{j,\mu} = \bigwedge^{s_j} \mathcal{Q} \otimes S_{\mu(j)} \overline{\mathcal{R}}^* \otimes \mathcal{B}(-s_j)$$

where the partition $\mu(j)$ and s_j are as above.

Proof. We will construct the resolution in question in each fibre. Let us therefore consider the corresponding affine situation, writing U instead of $\mathcal{Q}$, V^* instead of $\overline{\overline{R}}^*$ and $B = \mathrm{Sym}(U \otimes V^*)$ instead of $\mathcal{B}$. We assume that $\dim V^* = r$ but also assume $\dim U \geq r$. Then we have the obvious analogue N_μ of the sheaf $\mathcal{N}_\mu$.

Notice that N_μ consists of the representations $S_\phi U \otimes S_\psi V^*$ such that $\psi_r = 0$, i.e. of representations which are nonzero when substituting for V a space of dimension $r-1$. Such a module is the push down of a twisted module on the grassmannian $\mathrm{Grass}(1, V^*)$. If the tautological sequence is $0 \to \mathcal{S} \to V^* \to \mathcal{T} \to 0$ then the module whose push down is N_μ is $S_\mu \mathcal{T} \otimes \mathrm{Sym}(U \otimes \mathcal{T})$. The resolution is therefore given by the higher cohomology groups of $S_\mu \mathcal{T} \otimes \bigwedge^{\cdot}(U \otimes \mathcal{S})$ which by Bott's theorem give the terms provided in the claim. ■

Now we apply the claim to finish the proof of Proposition 10. Consider the sheaf $p_* \mathcal{P}(\alpha', \alpha'', \xi', \xi'')$ on $\mathrm{Grass}(m-r, F) \times \mathrm{Grass}(r, G)$ and its subsheaf $\mathcal{N}$ constructed above. Take $\mu = (\xi'_1 - \xi'_r, \ldots, \xi'_{r-1} - \xi'_r, 0)$. Then it is clear that $\mathcal{N}$ has a resolution with terms

$$\mathcal{G}_{j,\mu} \otimes S_{\alpha'}\mathcal{Q} \otimes S_{\alpha''}\mathcal{R} \otimes \bigwedge^r \overline{\mathcal{R}}^{*\otimes \xi'_r} \otimes S_{\xi''}\overline{\mathcal{Q}}^*$$

for $j \geq 1$. Taking sections of this resolution we get the assertion. ∎

Remark 11. One might hope that a stronger version analogous to Proposition 9 is true for arbitrary β. This is, however, not the case, as the following example shows. Take $m = n = 4$, $r = 3$. Consider the class $[M(0,0,0,0;2,2,0,0)]$. It equals

$$[P(0,0,0,0;2,2,0,0) + P(1,1,0,0;2,2,1,1)(-2) - P(1,1,1,0;2,2,2,1)(-3) + P(2,2,0,0;2,2,2,2)(-4)].$$

Since one of the coefficients is negative, this module does not have a filtration with factors of type $P(\gamma,\delta)$.

4. The degeneration sequence. According to Theorem 1 the class of any graded A_r-module with rational $\mathrm{GL}(F)\otimes\mathrm{GL}(G)$ action can be expressed as a linear combination of the classes of modules $M(\alpha,\beta)$. In this section we deal with a formula for the class of the coordinate ring A_{r-1} of the smaller determinantal variety.

Proposition 12. *The class of A_{r-1} in $K'_0(A_r)$ is given by the formula*

$$[A_{r-1}] = [A_r] - \sum_{i=0}^{n-r}[M((i+1,1^{r-1}),(1^{r+i},0^{n-r-i}))]. \tag{35}$$

Proof. Let $\mathcal{A}$ denote the graded algebra of sheaves $\mathcal{A} := \mathrm{Sym}(\mathcal{Q}\otimes G^*)$ over $\mathrm{Grass}(m-r,F)$. Consider the relative Eagon–Northcott complex over $\mathcal{A}$,

$$0 \to \mathcal{E}_{n-r} \to \mathcal{E}_{n-r-1} \to \ldots \to \mathcal{E}_1 \to \mathcal{E}_0 \to \mathcal{A}, \tag{36}$$

where

$$\mathcal{E}_i = D_i\mathcal{Q} \otimes \bigwedge^r \mathcal{Q} \otimes \bigwedge^n G^* \otimes \mathcal{A}(-i-r). \tag{37}$$

This is a sheaf resolution of a sheaf $\mathcal{B}$ of algebras. The sheaf $\mathcal{B}$ can be written, up to filtration, as

$$[\mathcal{B}] = \bigoplus_{\lambda=(\lambda_1,\ldots,\lambda_{r-1})} S_\lambda\mathcal{Q} \otimes S_\lambda G^*. \tag{38}$$

Therefore its higher cohomology vanishes by Kempf's vanishing theorem and the sections are equal to A_{r-1}. In characteristic 0 the proof is thus finished because we observe that $\mathcal{E}_i = \mathcal{M}((i+1,1^{r-1}),(1^{r+i},0^{n-r-i}))$. In positive characteristic the proof follows by the general principle stated several times above (we could also say that in arbitrary characteristic we have $\chi(\mathcal{E}_i) = M((i+1,1^{r-1}),(1^{r+i},0^{n-r-i}))$. ∎

Remark 13. The above result is even stronger in characteristic 0 than in positive characteristic. Taking sections of (36) we actually get the exact sequence

$$0 \to E_{n-r} \to E_{n-r-1} \to \ldots \to E_1 \to E_0 \to A_r \tag{39}$$

with

$$E_i = M((i+1, 1^{r-1}), (1^{r+i}, 0^{n-r-i}))(-r-i) \tag{40}$$

which is a resolution of A_{r-1}.

This sequence is not characteristic free. We can get its characteristic free analogue by replacing the Eagon–Northcott complex by a nonminimal free resolution of the determinantal ideal of $r \times r$ minors of an $r \times n$ matrix, each term of which has a good filtration by Schur modules. Such a complex, of length $n-r+1$, was constructed by David Buchsbaum in [B]. We need its relative version which is

$$0 \to \widehat{\mathcal{E}}_{n-r} \to \widehat{\mathcal{E}}_{n-r-1} \to \ldots \to \widehat{\mathcal{E}}_1 \to \widehat{\mathcal{E}}_0 \to \mathcal{A} \tag{41}$$

where

$$\widehat{\mathcal{E}}_i = \bigoplus_{j_1>0,\ldots,j_i>0,\, j_{i+1}=r} \bigwedge^{j_1} \mathcal{Q} \otimes \ldots \otimes \bigwedge^{j_{i+1}} \mathcal{Q} \otimes \bigwedge^{j_1+\ldots+j_{i+1}} G^* \otimes \mathcal{A}(-j_1 - \ldots - j_{i+1}). \tag{42}$$

The modules occurring in the terms of (41) do not have higher cohomology by Kempf's vanishing theorem. The tensor product of exterior powers has a good filtration by [Bo]. This means that by pushing down (41) we get a characteristic free resolution of A_{r-1} of length $n-r+1$ by modules which, up to filtration, are direct sums of modules of type $M(\alpha, \beta)$.

It would be interesting to investigate whether this construction could lead to some nice nonminimal resolutions of determinantal ideals.

5. Canonical filtrations of equivariant modules over determinantal rings. In this section we assume that the characteristic of K is 0. We strengthen our results and prove that every module M from $\mathcal{C}_r(F, G)$ has a canonical filtration whose factors have finite free resolutions by modules of type $M(\alpha, \beta)$ (resp. $N(\alpha, \beta)$, $P(\alpha, \beta)$).

Let M be a module from $\mathcal{C}_r(F, G)$. For each pair (α, β) of weights we denote by $M_{\alpha,\beta}$ the isotypic component of M corresponding to $S_\alpha F \otimes S_\beta G^*$.

Let γ'' be a partition with $m-r$ parts. We define

$$M_{l \geq \gamma''} = \bigoplus_{\alpha'' \geq \gamma''} M_{\alpha,\beta}. \tag{43}$$

This is obviously an equivariant A_r-submodule of M. The submodules $M_{l \geq \gamma''}$ define a filtration on M, called the *canonical left filtration* on M.

Similarly, for a partition δ'' with $n-r$ parts we define the submodule

$$M_{r\geq\delta''}=\bigoplus_{\beta''\geq\delta''}M_{\alpha,\beta}. \tag{44}$$

It is obviously an equivariant A_r-submodule of M. The submodules $M_{r\geq\delta''}$ define a filtration on M which is called the *canonical right filtration* on M. Similarly we define the canonical two-sided filtrations on M.

THEOREM 4. *The factors of the left* (*resp. right*, *two-sided*) *canonical filtration have finite resolutions by modules of type* $M(\alpha,\beta)$ (*resp.* $N(\alpha,\beta)$, $P(\alpha,\beta)$).

Proof. We give the proof only for the left filtrations and modules $M(\alpha,\beta)$. The proofs in the other two cases are analogous. The result follows at once from the following lemma.

LEMMA 14. *Fix a partition* γ'' *with* $m-r$ *parts. Let* M *be a module from* $\mathcal{C}_r(F,G)$ *such that for every nonzero isotypic component* $M_{\alpha,\beta}$ *we have* $\alpha''=\gamma''$. *Then* M *has a finite resolution with terms of type* $M(\alpha,\beta)$ (*with* $\alpha''=\gamma''$).

Proof. Consider the sheaf $\mathcal{G}$ of $\mathrm{Sym}(\mathcal{Q}\otimes G^*)$-modules modelled after M by writing $S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}$ instead of $S_\alpha F$. Such a sheaf exists because of the condition on M. It is of the form $\mathcal{H}\otimes S_{\alpha''}\mathcal{R}$ where $\mathcal{H}$ is a sheaf involving only representations of $\mathcal{Q}$. The sheaf $\mathcal{G}$ has a finite resolution by $\mathrm{Sym}(\mathcal{Q}\otimes G^*)$-modules with terms $\mathcal{M}(\gamma',\alpha'';\delta)$. Taking sections of this resolution we get the assertion of lemma.

This concludes the proof of Theorem 4. ∎

6. The depth of modules $M(\alpha,\beta)$ and $N(\alpha,\beta)$. In this section we give a formula for the depth of modules from the families $M(\alpha,\beta)$ and $N(\alpha,\beta)$. We deal only with the family $M(\alpha,\beta)$ because the results for the other family are symmetric. For most of the section we assume that the characteristic of the field K is zero. We conclude the section with an example showing that the depth of modules $M(\alpha,\beta)$ and even their being Cohen–Macaulay depends on the characteristic of the base field.

Let K be a field of characteristic 0.

Since $M(\alpha,\beta)=M(\alpha,0)\otimes S_\beta G^*$ the depth of $M(\alpha,\beta)$ does not depend on β, so we can assume that $\beta=0$. We will denote $M(\alpha,0)$ by $M(\alpha)$.

Let us start with $\alpha=(\alpha',\alpha'')$. In fact, we will not assume that α is dominant; we will just assume that the sheaf

$$p_{1*}\mathcal{M}(\alpha',\alpha'')=S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes\mathrm{Sym}(\mathcal{Q}\otimes G^*)$$

does not have higher cohomology.

PROPOSITION 15. *The sheaf $\mathcal{M}(\alpha',\alpha'')$ does not have higher cohomology if and only if $\alpha'_r \ge \alpha''_1 - t$ where t is such that $\alpha''_1 = \alpha''_2 = \ldots = \alpha''_t > \alpha''_{t+1}$.*

Proof. Let $\delta_1,\ldots,\delta_r$ be the numbers defined in the proof of Proposition 2. The condition above is clearly equivalent to the fact that $\delta_r \ge \alpha''_1$ and that means that $l(\alpha)=0$. ■

We assume therefore that the condition of Proposition 15 is satisfied and, in order to study the depth of $M(\alpha',\alpha'')$, we look at its free resolution which is given (because of the vanishing) by the push down of the Koszul complex

(45) $$S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes \bigwedge^{\cdot}(\mathcal{R}\otimes G^*)\otimes \mathcal{O}_{X\times \mathrm{Grass}(m-r,F)}.$$

We want to calculate the projective dimension of $M(\alpha',\alpha'')$ over $A=\mathrm{Sym}(F\otimes G^*)$ because, by the Auslander–Buchsbaum Theorem, $\operatorname{depth} M + \operatorname{pd}_A M = mn$ for every A-module M.

In order to find the length of the push down of the complex (45) we recall that its ith term $F_i(\alpha)$ is given by the formula

(46) $$F_i(\alpha) = \bigoplus_{j\ge 0} H^j(\mathrm{Grass}(m-r,F), S_{\alpha'}\mathcal{Q}\otimes S_{\alpha''}\mathcal{R}\otimes \bigwedge^{i+j}(\mathcal{R}\otimes G^*)\otimes A(-i-j)).$$

In order to study the top of the resolution, we start with the last term of the Koszul complex. The corresponding $\mathrm{GL}(F)$ weight is

$$t(\alpha) = (\alpha'_1,\ldots,\alpha'_r,\alpha''_1+n,\ldots,\alpha''_{m-r}+n).$$

Consider the weight

$$t(\alpha)+\varrho = (\alpha'_1+m,\alpha'_2+m-1,\ldots,\alpha'_r+m-r,\alpha''_1+n+m-r-1,\ldots,\alpha''_{m-r}+n+1).$$

Write

$$t(\alpha)+\varrho = (a_1,\ldots,a_r,b_1,\ldots,b_{m-r}).$$

For $j=1,\ldots,m-r$ define the sequences $t_j(\alpha)$ by induction on j, as follows:

$$t_j(\alpha)+\varrho = (a_1,\ldots,a_r,d_1,\ldots,d_j,b_{j+1},\ldots,b_{m-r})$$

where d_{j+1} is defined by

$$d_{j+1} = \max\{t \mid t\le b_{j+1},\ t<d_j,\ t\notin\{a_1,\ldots,a_r\}\}.$$

Notice that in this definition the condition $t<d_j$ could be skipped, because each b_j is essentially lowered to the first possible number that is not one of the a_i's or previous d_k's.

Let us also define the numbers

$$q_j = n-(b_j-d_j),\qquad p_j = q_j - \sharp\{i \mid a_i<d_j\}$$

for $j=1,\ldots,m-r$.

Lemma 16. *For every $j = 1, \ldots, m - r$ we have $p_j \geq n - r$.*

Proof. Let us imagine that we construct sequences $t_j(\alpha)$ by the following process. We look at b_j and start lowering it by 1 until we reach the number that is not equal to any a_i or d_k for $k < j$. Then every lowering by 1 accounts for some a_i satisfying $d_j + 1 \leq a_i \leq b_j$ or by some d_k (which comes from some previous $a_k > b_j$). This means that we have a set of $b_j - d_j$ a_k's which is disjoint from the set $\{i \mid a_i < d_j\}$. Therefore

$$p_j = n - (b_j - d_j) - \sharp\{i \mid a_i < d_j\} \geq n - r.$$

This concludes the proof of the lemma. ∎

Theorem 5. *Let K be a field of characteristic* 0. *Let (α', α'') satisfy the condition of Proposition* 15. *Then the projective dimension of $M(\alpha', \alpha'')$ over A equals $\sum_{j=1}^{m-r} p_j$.*

Proof. Decomposing the terms of the complex (45) using the Cauchy formula and the Littlewood–Richardson rule, we find that the terms of the resolution $F_{\cdot}(\alpha)$ have $\mathrm{GL}(F)$ weights which are of type (α', δ'') with δ'' containing α'' such that the difference of the corresponding terms does not exceed n. Also, all such weights do occur. For such weights we need to find the supremum of the numbers $|\delta''/\alpha''| - l(w)$ where w is a permutation ordering the weight $(\alpha', \delta'') + \varrho$. It is clear that the top number is obtained from the sequence $t_{m-r}(\alpha)$.

We need therefore to find in which $F_i(\alpha)$ the corresponding term occurs. The homogeneous degree is, however, $\sum_{j=1}^{m-r}(n - (b_j - d_j))$ and the length of w is $l(w) = \sum_{j=1}^{m-r} \sharp\{i \mid a_i < d_j\}$, and the statement of the theorem follows. ∎

Corollary 17. *Let (α', α'') satisfy the condition of Proposition* 15. *The depth of the module $M(\alpha', \alpha'')$ is equal to $mn - \sum_{j=1}^{r} p_j$. The module $M(\alpha', \alpha'')$ is a maximal Cohen–Macaulay module over A_r if and only if every p_j is equal to $n - r$.*

Let us look more closely at the weights giving maximal Cohen–Macaulay modules. Let us look at the process of getting sequences $t_j(\alpha)$ by lowering the numbers b_j to d_j. However, now at each stage we will modify the sequence $(a_1, \ldots, a_r)$ as follows. We define inductively the sets $\{a_1^{(j)}, \ldots, a_r^{(j)}\}$ by setting

$$\{a_1^{(0)}, \ldots, a_r^{(0)}\} = \{a_1, \ldots, a_r\},$$

$$\{a_1^{(j)}, \ldots, a_r^{(j)}\} = \begin{cases} \{a_1^{(j-1)}, \ldots, \widehat{a}_i^{(j-1)}, \ldots, a_r^{(j-1)}, d_j\} & \text{if } b_j = a_i^{(j-1)}; \\ \{a_1^{(j-1)}, \ldots, a_r^{(j-1)}\} & \text{otherwise.} \end{cases}$$

Now, $p_j = n - r$ for $j = 1, \dots, m - r$ if and only if we have $b_j \geq \max\{a_1^{(j-1)}, \dots, a_r^{(j-1)}\}$ for each $j = 1, \dots, m - r$. Indeed, each a_i either induces a number between b_j and $d_j + 1$ or is smaller than d_j so the overall number by which the projective dimension decreases at the jth stage is $n-r$. Let us state this result as

Corollary 18. *Let* (α', α'') *satisfy the condition of Proposition* 15. *Define the sets* $\{a_1^{(j)}, \dots, a_r^{(j)}\}$ *as above. Then* $M(\alpha', \alpha'')$ *is maximal Cohen–Macaulay if and only if for every* $j = 1, \dots, m - r$ *we have*

$$b_j \geq \max\{a_1^{(j-1)}, \dots, a_r^{(j-1)}\}.$$

We now show an example of a module of type $M(\alpha)$ with a characteristic free presentation such that each graded component of $M(\alpha)$ is characteristic free and $M(\alpha)$ is Cohen–Macaulay when $\operatorname{char} K \neq 2$ but fails to be Cohen–Macaulay when $\operatorname{char} K = 2$. The interesting feature of the example is that the projective dimension of $M(\alpha)$ over A equals 2 whenever $\operatorname{char} K \neq 2$.

Example. Take $r = 2$, $m = 3$, $n = 4$. Consider the module $M(2, 0, 0)$. It consists of the sections of the sheaf

$$\mathcal{M}(2, 0, 0) = S_2\mathcal{Q} \otimes \operatorname{Sym}(\mathcal{Q} \otimes G^*)$$

over $\operatorname{Grass}(1, F)$.

It follows from the Cauchy formula, Kempf's vanishing theorem and the fact that the tensor product of Schur functors has a good filtration that the higher cohomology groups of $\mathcal{M}(2, 0, 0)$ vanish and that each graded component of $M(2, 0, 0)$ has a good filtration as a $\mathrm{GL}(F) \times \mathrm{GL}(G^*)$-module, and therefore is characteristic free.

Let us analyze the free resolution of $M(2, 0, 0)$ over A, assuming that K has characteristic 0. The resolution consists of cohomology groups of sheaves occurring in

$$S_2\mathcal{Q} \otimes \textstyle\bigwedge^{\cdot}(\mathcal{R} \otimes G^*).$$

In this case $\dim \mathcal{R} = 1$, so for each i with $0 \leq i \leq 4$ we are dealing with the cohomology groups of $S_2\mathcal{Q} \otimes S_i\mathcal{R}$, tensored with $\bigwedge^i G^*$.

In characteristic 0 the terms of the resolution of $M(2, 0, 0)$ are clearly

$$0 \to S_{2,2,1}F \otimes \textstyle\bigwedge^3 G^* \otimes A(-3) \to S_{2,2,1}F \otimes \bigwedge^2 G^* \otimes A(-2) \to S_2F \otimes A.$$

In fact, this complex is exact whenever the characteristic of K is not 2. Otherwise, the bundle $S_2\mathcal{Q} \otimes S_4\mathcal{R}$ has nonzero cohomology, in fact

$$H^1(\operatorname{Grass}(1, F), S_2\mathcal{Q} \otimes S_4\mathcal{R}) = H^2(\operatorname{Grass}(1, F), S_2\mathcal{Q} \otimes S_4\mathcal{R}) = S_{2,2,2}F.$$

This is one of the first counterexamples to Bott's theorem in positive characteristic.

This makes the resolution of $M(\alpha)$ in characteristic 2 look like this:

$$0 \to S_{2,2,2}F \otimes \bigwedge^4 G^* \otimes A(-4)$$
$$\to S_{2,2,2}F \otimes \bigwedge^4 G^* \otimes A(-4) \oplus S_{2,2,1}F \otimes \bigwedge^3 G^* \otimes A(-3)$$
$$\to S_{2,2,1}F \otimes \bigwedge^2 G^* \otimes A(-2) \to S_2 F \otimes A.$$

The resolution is so small that it can be calculated explicitly by the program Macaulay.

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