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ON 4-FIELDS AND 4-DISTRIBUTIONS IN 8-DIMENSIONAL VECTOR BUNDLES OVER 8-COMPLEXES

BY

MARTIN ČADEK AND JIŘÍ VANŽURA (BRNO)

Let ξ be an oriented 8-dimensional spin vector bundle over an 8-complex. In this paper we give necessary and sufficient conditions for ξ to have 4 linearly independent sections or to be a sum of two 4-dimensional spin vector bundles, in terms of characteristic classes and higher order cohomology operations. On closed connected spin smooth 8-manifolds these operations can be computed.

1. Introduction. While the existence of 3-fields and 3-distributions in vector bundles over manifolds has been treated by many authors (see for instance [AD], [CS], [D], [K1], [K2], [N2], [R2], [T4]) and more or less completely solved, the results on the existence of 4-fields and 4-distributions are rare and not so complete (see [AR], [N1], [N2], [R1], [K1]). Especially, the case of $4k$-dimensional vector bundles over $4k$-manifolds seems to be difficult to deal with.

In this paper we solve the problem for 8-dimensional oriented spin vector bundles over 8-manifolds. The method of the Postnikov tower enables us to reveal that there is a generating class (see [T2]) in this case and that the obstructions can be computed using secondary and tertiary cohomology operations. The computation of these operations over closed connected smooth spin 8-manifolds has been carried out in our previous paper [CV3] which serves as an important preliminary material for the present one.

Our main results are Theorem 3.1 and Corollary 3.2 on the existence of 4-dimensional spin vector bundles over 8-manifolds (in Section 3), Theorem 4.1 with Corollaries 4.2, 4.3 on the existence of 4-fields and Theorem 4.4 on the existence of 4-distributions (in Section 4). Section 2 has auxiliary

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character and summarizes facts needed for the statements and proofs of the main results.

2. Notation and auxiliary results. In this section we introduce notation and recall some facts about the singular cohomology of classifying spaces.

We will use $w_m(\xi)$ for the mth Stiefel–Whitney class of the vector bundle ξ, $p_m(\xi)$ for the mth Pontryagin class, and $e(\xi)$ for the Euler class. For a complex vector bundle ξ the symbol $c_m(\xi)$ denotes the mth Chern class. The w_m, p_m, e and c_m will stand for the characteristic classes of the universal vector bundles over the classifying spaces $\mathrm{BSO}(n)$ and $\mathrm{BU}(n)$, respectively. The pullbacks of the Stiefel–Whitney, Pontryagin and Euler classes in $H^*(\mathrm{BSpin}(n))$ will be denoted by the same letters.

The mappings $i_* : H^*(X,\mathbb{Z}_2) \to H^*(X,\mathbb{Z}_4)$ and $\varrho_m : H^*(X,\mathbb{Z}) \to H^*(X,\mathbb{Z}_m)$ are induced by the inclusion $\mathbb{Z}_2 \to \mathbb{Z}_4$ and the reduction mod m, respectively. We will also use the Steenrod operations $\mathrm{Sq}^i : H^n(X;\mathbb{Z}_2) \to H^{n+i}(X;\mathbb{Z}_2)$ and $P_3^i : H^n(X;\mathbb{Z}_3) \to H^{n+4i}(X;\mathbb{Z}_3)$.

We say that $x \in H^*(X;\mathbb{Z})$ is an *element of order* n ($n = 2,3,4,\ldots$) if and only if $x \neq 0$ and n is the least positive integer such that $nx = 0$ (if it exists).

The Eilenberg–MacLane space with the nth homotopy group G will be denoted by $K(G,n)$, and ι_n will stand for the fundamental class in $H^n(K(G,n);G)$. When writing the fundamental class, it will be always clear which group G we have in mind.

Now we summarize some facts about the groups $\mathrm{Spin}(3)$ and $\mathrm{Spin}(4)$ and the cohomologies of their classifying spaces. It is well known that $\mathrm{Spin}(3)$ is isomorphic with the group $\mathrm{Sp}(1)$ of unit quaternions. So, identifying $\mathrm{Spin}(3)\times\mathrm{Spin}(3)$ with $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$, we can define a homomorphism $\overline{\vartheta} : \mathrm{Spin}(3)\times\mathrm{Spin}(3) \to \mathrm{SO}(4)$ using the representation

$$(\alpha,\beta)\cdot v = \alpha v\overline{\beta},$$

where $\alpha,\beta \in \mathrm{Sp}(1)$, $v \in \mathbb{H} \cong \mathbb{R}^4$ and $^-$ stands for conjugation. Since the kernel of this homomorphism is $\{(1,1),(-1,-1)\} \cong \mathbb{Z}_2$, there is an isomorphism

$$\vartheta : \mathrm{Spin}(3)\times\mathrm{Spin}(3) \to \mathrm{Spin}(4).$$

It induces a homeomorphism on the level of classifying spaces which will be denoted by the same letter.

LEMMA 2.1. *The cohomology ring of* $\mathrm{BSpin}(3)$ *is*

$$H^*(\mathrm{BSpin}(3);\mathbb{Z}) \cong \mathbb{Z}[r],$$

where $p_1 = 4r$.

The cohomology ring of BSpin(4) *is*

$$H^*(\mathrm{BSpin}(4);\mathbb{Z}) \cong \mathbb{Z}[q,s],$$

where q *and* s *are defined with the aid of the first Pontryagin class and the Euler class by the relations*

$$p_1 = 2q, \qquad e = 2s - q.$$

Moreover,

$$\vartheta^*(q) = r\otimes 1 + 1\otimes r, \qquad \vartheta^*(s) = 1\otimes r.$$

Proof. The cohomology of BSpin(3) is well known. The existence of q and $s \in H^4(\mathrm{BSpin}(4);\mathbb{Z})$ follows from the relation

$$\varrho_4 p_1 = \mathfrak{P} w_2 + i_* w_4,$$

where $\mathfrak{P}$ is the Pontryagin square, which in $H^*(\mathrm{BSpin}(4);\mathbb{Z})$ reads as

$$\varrho_4(p_1 + 2e) = 0.$$

Since $\vartheta^* : H^*(\mathrm{BSpin}(4);\mathbb{Z}) \to H^*(\mathrm{BSpin}(3)\times\mathrm{BSpin}(3);\mathbb{Z}) \cong \mathbb{Z}[r\otimes 1, 1\otimes r]$, it is sufficient to prove the last part of our lemma.

Computing $\overline{\vartheta} : \mathrm{Sp}(1)\times\mathrm{Sp}(1) \to \mathrm{SO}(4)$ on the standard tori and using the classical results of Borel and Hirzebruch (see [BH]), we easily get

$$\overline{\vartheta}^*(p_1) = 2(r\otimes 1 + 1\otimes r), \qquad \overline{\vartheta}^* e = 1\otimes r - r\otimes 1.$$

Hence

$$\vartheta^* q = r\otimes 1 + 1\otimes r, \qquad \vartheta^* s = 1\otimes r.$$

Further, we recall the definition of two higher order cohomology operations introduced in [CV3].

Definition 2.2. Let Σ denote the secondary cohomology operation associated with the relation

$$\mathrm{Sq}^2 \circ \mathrm{Sq}^2 \varrho_2 = 0$$

on integral cohomology classes of dimension 4.

Let Φ be the tertiary cohomology operation associated with the relation

$$i_* \mathrm{Sq}^2 \circ \Sigma = 0$$

on integral cohomology classes of dimension 4 and uniquely determined by the properties

$$\Phi(r) = 0, \quad \Phi(2r) = -\varrho_4 r^2$$

for $r \in H^4(\mathrm{BSpin}(3);\mathbb{Z})$.

Let Ω be the secondary cohomology operation associated with the relation

$$i_* \mathrm{Sq}^2 \circ \mathrm{Sq}^2 = 0$$

in dimension 5.

Let X be a CW-complex. The operations Σ, Ω and Φ are defined on the sets $\mathrm{Def}(\Sigma, X) = \{x \in H^4(X;\mathbb{Z}) : \mathrm{Sq}^2 \varrho_2 x = 0\}$, $\mathrm{Def}(\Omega, X) = \{x \in H^5(X;\mathbb{Z}_2) : \mathrm{Sq}^2 x = 0\}$ and $\mathrm{Def}(\Phi, X) = \{x \in H^4(X;\mathbb{Z}) : \mathrm{Sq}^2 \varrho_2 x = 0,$ $0 \in \Sigma(x)\}$, respectively. The values of $\Sigma(x)$ form a subset of $H^7(X;\mathbb{Z}_2)$, while $\Omega(x)$ and $\Phi(x)$ are subsets of $H^8(X;\mathbb{Z}_4)$. The indeterminacies of Σ and Ω are $\mathrm{Indet}(\Sigma, X) = \mathrm{Sq}^2 H^5(X;\mathbb{Z}_2)$ and $\mathrm{Indet}(\Omega, X) = i_* \mathrm{Sq}^2 H^6(X;\mathbb{Z}_2)$, respectively. The indeterminacy of the remaining operation is $\mathrm{Indet}(\Phi, X) = \Omega\, \mathrm{Def}(\Omega, X)$.

For further properties of Σ, Ω and Φ we refer to [CV3]. In particular, the formula

$$\Phi(x+y) = \Phi(x) + \Phi(y) - \varrho_4(xy)$$

holds for all $x, y \in \mathrm{Def}(\Phi, X)$ ([CV3, Lemma 3.9]).

LEMMA 2.3. *For q and $s \in H^4(\mathrm{BSpin}(4);\mathbb{Z})$,*

$$\Sigma(q) = 0, \quad \Sigma(s) = 0, \quad \Phi(q) = \varrho_4(s^2 - qs), \quad \Phi(s) = 0.$$

Proof. Since $H^5(\mathrm{BSpin}(4);\mathbb{Z}_2) = H^7(\mathrm{BSpin}(4);\mathbb{Z}_2) = 0$, we have $\Sigma(q) = \Sigma(s) = 0$ and $\mathrm{Indet}(\Phi, \mathrm{BSpin}(4)) = 0$. Since $\Phi(r) = 0$ for $r \in H^4(\mathrm{BSpin}(3);\mathbb{Z})$, using the formula for $\Phi(x+y)$, we get

$$\begin{aligned}\Phi(s) &= \Phi(\vartheta^*)^{-1}(1 \otimes r) = (\vartheta^*)^{-1}\Phi(1 \otimes r) = 0,\\ \Phi(q) &= \Phi(\vartheta^*)^{-1}(r \otimes 1 + 1 \otimes r) = (\vartheta^*)^{-1}\Phi(r \otimes 1 + 1 \otimes r)\\ &= -(\vartheta^*)^{-1}\varrho_4(r \otimes r) = -\varrho_4(q-s)s = \varrho_4(s^2 - qs).\end{aligned}$$

Using the first Steenrod operation with $\mathbb{Z}_3$ coefficients we obtain the other two relations in $H^8(\mathrm{BSpin}(4);\mathbb{Z}_3)$.

LEMMA 2.4. *For q and $s \in H^4(\mathrm{BSpin}(4);\mathbb{Z})$,*

$$P_3^1 \varrho_3 q + \varrho_3 q^2 = \varrho_3(s^2 - sq), \qquad P_3^1 \varrho_3 s + \varrho_3 s^2 = 0.$$

Proof. According to the proof of Theorem 3.8 in [CV3] we know that

$$P_3^1 \varrho_3 r + \varrho_3 r^2 = 0$$

in $H^8(\mathrm{BSpin}(3);\mathbb{Z}_3)$. This immediately yields the second relation. Further, according to [BS],

$$P_3^1 \varrho_3 p_1 = \varrho_3(2p_2 - p_1^2)$$

where $p_2 = e^2$ in $H^8(\mathrm{BSpin}(4);\mathbb{Z})$. Substitute $p_1 = 2q$ and $e = 2s - q$ to get

$$P_3^1 \varrho_3 2q = \varrho_3(8s^2 - 8sq - 2q^2).$$

This yields the first relation in our lemma.

Finally, we recall the cohomology of BSpin(8) and the spin characteristic classes.

LEMMA 2.5. *The cohomology rings of* BSpin(8) *are*

$$H^*(\mathrm{BSpin}(8);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_4, w_6, w_7, w_8, \varepsilon]$$

and

$$H^*(\mathrm{BSpin}(8);\mathbb{Z}) \cong \mathbb{Z}[q_1, q_2, e, \delta w_6]/\langle 2\delta w_6\rangle$$

where q_1, q_2 *and* ε *are defined by the relations*

$$p_1 = 2q_1, \quad p_2 = q_1^2 + 2e + 4q_2, \quad \varrho_2 q_2 = \varepsilon.$$

Proof. See [Q] and [CV2].

Let ξ be an oriented 8-dimensional vector bundle over a CW-complex X given by the homotopy class of some mapping $\xi : X \to \mathrm{BSO}(8)$. ξ has a spin structure iff $w_2(\xi) = 0$. If some lifting $\overline{\xi} : X \to \mathrm{BSpin}(8)$ is fixed, we talk about a given spin structure. In this case we can define the spin characteristic classes

$$q_1(\xi) = \overline{\xi}^* q_1, \quad q_2(\xi) = \overline{\xi}^* q_2.$$

The first spin characteristic class is always independent of the choice of $\overline{\xi}$. Moreover, if $H^4(X;\mathbb{Z})$ has no element of order 4, then it is uniquely determined by the relations

$$2q_1(\xi) = p_1(\xi), \quad \varrho_2 q_1(\xi) = w_4(\xi).$$

The second spin characteristic class is independent of the spin structure $\overline{\xi}$ if X is simply connected or $H^8(X;\mathbb{Z}) \cong \mathbb{Z}$. In the case of an 8-dimensional manifold $q_2(\xi)$ is uniquely determined by the relation

$$16q_2(\xi) = 4p_2(\xi) - p_1^2(\xi) - 8e(\xi).$$

See [CV2].

3. Four-dimensional spin vector bundles over 8-complexes. The previous section enables us to prove the following result on the existence of 4-dimensional spin vector bundles over CW-complexes of dimension 8.

THEOREM 3.1. *Let X be a connected CW-complex of dimension ≤ 8 and let $P, E \in H^4(X;\mathbb{Z})$. Then there exists an oriented 4-dimensional vector bundle η over X with*

$$w_2(\eta) = 0, \quad p_1(\eta) = P, \quad e(\eta) = E$$

if and only if there are $Q, S \in H^4(X;\mathbb{Z})$ such that

(1) $P = 2Q$, $E = 2S - Q$,
(2) $\mathrm{Sq}^2 \varrho_2 Q = \mathrm{Sq}^2 \varrho_2 S = 0$,
(3) $0 \in \Sigma(Q)$, $0 \in \Sigma(S)$,
(4) $\varrho_4(S^2 - QS) \in \Phi(Q)$, $0 \in \Phi(S)$,
(5) $P_3^1 \varrho_3 Q + \varrho_3 Q^2 = \varrho_3(S^2 - QS)$, $P_3^1 \varrho_3 S + \varrho_3 S^2 = 0$.

Proof. Every oriented 4-dimensional spin vector bundle η over a CW-complex X is determined by a mapping $\eta : X \to \mathrm{BSpin}(4)$. Let η have the prescribed characteristic classes. Then $\eta^*(p_1) = P_1$ and $\eta^*(e) = E$. Put $Q = \eta^*(q)$ and $S = \eta^*(s)$. Now Lemmas 2.1, 2.3 and 2.4 imply that Q and S satisfy conditions (1)–(5).

Conversely, let there be Q and S such that (1)–(5) hold. Consider the fibration

$$F \to \mathrm{BSpin}(4) \xrightarrow{\alpha} K(\mathbb{Z},4) \times K(\mathbb{Z},4)$$

where α is determined by elements $q, s \in H^4(\mathrm{BSpin}(4);\mathbb{Z})$. Next consider the mapping $f : X \to K(\mathbb{Z},4) \times K(\mathbb{Z},4)$ determined by elements $Q, S \in H^4(X;\mathbb{Z})$. Then η with the prescribed properties exists if f can be lifted in the fibration α:

$$\begin{array}{ccc} & & \mathrm{BSpin}(4) \\ & \overset{\eta}{\nearrow} & \big\downarrow \alpha \\ X & \xrightarrow{f} & K(\mathbb{Z},4) \times K(\mathbb{Z},4) \end{array}$$

Therefore we will build the Postnikov tower for the fibration α. The fibre F is 4-connected and the next homotopy groups are

$$\pi_5(F) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \qquad \pi_6(F) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \qquad \pi_7(F) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12}.$$

The first invariants can be easily obtained from the Serre exact sequence for the fibration $F \to \mathrm{BSpin}(4) \to K(\mathbb{Z},4) \times K(\mathbb{Z},4)$. They are $\mathrm{Sq}^2 \varrho_2\iota_4 \otimes 1$ and $1 \otimes \mathrm{Sq}^2 \varrho_2\iota_4$. The universal example for the secondary operation Σ is the fibration

$$K(\mathbb{Z}_2,5) \xrightarrow{j_1} Y_1 \xrightarrow{\pi_1} K(\mathbb{Z},4)$$

induced from the path fibration $K(\mathbb{Z}_2,5) \to PK(\mathbb{Z}_2,6) \to K(\mathbb{Z}_2,6)$ by the mapping $\mathrm{Sq}^2 \varrho_2\iota_4 : K(\mathbb{Z},4) \to K(\mathbb{Z}_2,6)$. That is why the first stage of the Postnikov tower is the product $Y_1 \times Y_1$. We have

$$\begin{array}{ccccc} \overline{F}_1 & \longrightarrow & F & \longrightarrow & K(\mathbb{Z}_2,5) \times K(\mathbb{Z}_2,5) \\ & & \big\downarrow & & \big\downarrow j_1 \times j_1 \\ F_1 & \longrightarrow & \mathrm{BSpin}(4) & \xrightarrow{\alpha_1} & Y_1 \times Y_1 \\ & & \alpha \big\downarrow & & \big\downarrow \pi_1 \times \pi_1 \\ & & K(\mathbb{Z},4) \times K(\mathbb{Z},4) & = & K(\mathbb{Z},4) \times K(\mathbb{Z},4) \\ & & & & 1\otimes\mathrm{Sq}^2 \varrho_2\iota_4 \big\downarrow \mathrm{Sq}^2 \varrho_2\iota_4 \otimes 1 \\ & & & & K(\mathbb{Z}_2,6) \times K(\mathbb{Z}_2,6) \end{array}$$

The next invariants are $\sigma \otimes 1, 1 \otimes \sigma \in H^7(Y_1 \times Y_1; \mathbb{Z}_2)$ where $\sigma \in H^7(Y_1; \mathbb{Z}_2)$ is the element defining the operation Σ. The universal example for the tertiary cohomology operation Φ is the fibration

$$K(\mathbb{Z}_2, 6) \xrightarrow{j_2} Y_2 \xrightarrow{\pi_2} Y_1$$

induced from the path fibration $K(\mathbb{Z}_2, 6) \to PK(\mathbb{Z}_2, 7) \to K(\mathbb{Z}_2, 7)$ by the mapping $\sigma : Y_1 \to K(\mathbb{Z}_2, 7)$. Hence the second stage of the Postnikov tower is the product $Y_2 \times Y_2$. We have

$$\begin{array}{ccccc}
\overline{F}_2 & \longrightarrow & F_1 & \longrightarrow & K(\mathbb{Z}_2,6) \times K(\mathbb{Z}_2,6) \\
 & & \downarrow & & \downarrow {\scriptstyle j_2 \times j_2} \\
F_2 & \longrightarrow & \mathrm{BSpin}(4) & \xrightarrow{\alpha_2} & Y_2 \times Y_2 \\
 & & \downarrow {\scriptstyle \alpha_1} & & \downarrow {\scriptstyle \pi_2 \times \pi_2} \\
 & & Y_1 \times Y_1 & = & Y_1 \times Y_1 \\
 & & & & {\scriptstyle 1\otimes\sigma} \downarrow {\scriptstyle \sigma\otimes 1} \\
 & & & & K(\mathbb{Z}_2,7) \times K(\mathbb{Z}_2,7)
\end{array}$$

In the stage $Y_2 \times Y_2$ there are two $\mathbb{Z}_4$-invariants and two $\mathbb{Z}_3$-invariants in dimension 8. (F_2 is 6-connected and $\pi_7(F_2) \cong \mathbb{Z}_{12} \oplus \mathbb{Z}_{12} \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$.)

First, consider $\mathbb{Z}_4$-coefficients. According to [CV3, Section 3], $H^8(Y_2; \mathbb{Z}_4) \cong \mathbb{Z}_4 \oplus \mathbb{Z}_4$ with the generators $\varrho_4 \pi_2^* \pi_1^* \iota_4$ and φ which is the element defining the tertiary cohomology operation Φ. Using the Künneth formula for $\mathbb{Z}_2$-coefficients, the exact sequence associated with the short exact sequence $0 \to \mathbb{Z}_2 \to \mathbb{Z}_4 \to \mathbb{Z}_2 \to 0$ and the knowledge of $H^*(Y_2; \mathbb{Z}_2)$ and $H^*(Y_2; \mathbb{Z}_4)$ from [CV3] (see Section 3), we find that $H^8(Y_2 \times Y_2; \mathbb{Z}_4) \cong (\mathbb{Z}_4)^5$ with the generators $\varrho_4 \pi_2^* \pi_1^* \iota_4^2 \otimes 1$, $1 \otimes \varrho_4 \pi_2^* \pi_1^* \iota_4^2$, $\varrho_4(\pi_2^* \pi_1^* \iota_4 \otimes \pi_2^* \pi_1^* \iota_4)$, $\varphi \otimes 1$ and $1 \otimes \varphi$. Moreover,

$$\alpha_2^*(\varrho_4 \pi_2^* \pi_1^* \iota_4^2 \otimes 1) = \varrho_4 q^2, \qquad \alpha_2^*(1 \otimes \varrho_4 \pi_2^* \pi_1^* \iota_4^2) = \varrho_4 s^2,$$
$$\alpha_2^*(\varrho_4(\pi_2^* \pi_1^* \iota_4 \otimes \pi_2^* \pi_1^* \iota_4)) = \varrho_4(qs),$$

and using Lemma 2.3,

$$\alpha_2^*(\varphi \otimes 1) = \Phi(q) = \varrho_4(s^2 - qs), \qquad \alpha_2^*(1 \otimes \varphi) = \Phi(s) = 0.$$

Hence the invariants are $\varphi \otimes 1 - 1 \otimes \varrho_4 \pi_2^* \pi_1^* \iota_4^2 + \varrho_4(\pi_2^* \pi_1^* \iota_4 \otimes \pi_2^* \pi_1^* \iota_4)$ and $1 \otimes \varphi$.

Analogously, using Lemma 2.4 we find that the $\mathbb{Z}_3$-invariants are $P_3^1(\varrho_3 \pi_2^* \pi_1^* \iota_4 \otimes 1) + \varrho_3 \pi_2^* \pi_1^* \iota_4^2 \otimes 1 - \varrho_3(1 \otimes \pi_2^* \pi_1^* \iota_4^2 - \pi_2^* \pi_1^* \iota_4 \otimes \pi_2^* \pi_1^* \iota_4)$ and $P_3^1(1 \otimes \varrho_3 \pi_2^* \pi_1^* \iota_4) + 1 \otimes \varrho_3 \pi_2^* \pi_1^* \iota_4^2$.

This shows that $f : X \to K(\mathbb{Z}, 4) \times K(\mathbb{Z}, 4)$ given by the cohomology classes Q and S can be lifted to the third stage of the Postnikov tower if and only if the conditions (2)–(5) are satisfied. But because $\dim X \le 8$, we can see that these conditions are necessary and sufficient for the existence of a lift of f to BSpin(4) in the fibration α.

Now we apply Theorem 3.1 to a closed connected smooth spin manifold of dimension 8.

Corollary 3.2. *Let M be a closed connected smooth spin manifold of dimension 8 and let $P, E \in H^4(M;\mathbb{Z})$. Then there exists an oriented 4-dimensional vector bundle η over M with*

$$w_2(\eta) = 0, \quad p_1(\eta) = P, \quad e(\eta) = E$$

if and only if there are $Q, S \in H^4(M;\mathbb{Z})$ such that

(i) $P = 2Q$, $E = 2S - Q$,
(ii) $\mathrm{Sq}^2 \varrho_2 Q = \mathrm{Sq}^2 \varrho_2 S = 0$,
(iii) $\{4E^2 + P^2 - 2Pp_1(M)\}[M] \equiv 0 \bmod 64$,
$\{2(2E+P)p_1(M) - (2E+P)^2\}[M] \equiv 0 \bmod 128$,
(iv) $P_3^1 \varrho_3 P = \varrho_3(2E^2 - P^2)$, $P_3^1 \varrho_3 E = \varrho_3 EP$.

Proof. Let M be as above. Theorems 5.3 and 5.5 of [CV3] assert that

$$\Phi(z) = \varrho_4 \cdot \frac{1}{2}\{zq_1(M) - z^2\}$$

for all $z \in \mathrm{Def}(\Phi, M)$ and

$$0 \in \Sigma(z)$$

for all $z \in \mathrm{Def}(\Sigma, M)$. Using this and the fact that $H^8(M;\mathbb{Z}) \cong \mathbb{Z}$ we will show that (iii) is equivalent to (4) of Theorem 3.1 on M. We have

$$\begin{aligned} 4E^2 + P^2 - 2Pp_1(M) &= 4(2S-Q)^2 + (2Q)^2 - 8Qq_1(M) \\ &= 16S^2 + 8Q^2 - 16QS - 8Qq_1(M) \\ &= 8\{2(S^2 - QS) - (Qq_1(M) - Q^2)\}. \end{aligned}$$

Next

$$2(2E+P)p_1(M) - (2E+P)^2 = 16(\mathrm{Sq}_1(M) - S^2).$$

Similarly, substituting for P and E in (iv) we get (5) of Theorem 3.1:

$$\begin{aligned} 0 = P_3^1 \varrho_3 P - 2\varrho_3 E^2 + \varrho_3 P^2 &= 2P_3^1 \varrho_3 Q + 2\varrho_3 Q^2 - 8\varrho_3 S^2 + 8\varrho_3 SQ \\ &= 2\{P_3^1 \varrho_3 Q + \varrho_3 Q^2 - \varrho_3(S^2 - SQ)\}. \end{aligned}$$

Further, using the fact that $P_3^1 \varrho_3 P = 2\varrho_3 E^2 - \varrho_3 P^2$, we have

$$\begin{aligned} 0 = P_3^1 \varrho_3 E - \varrho_3 EP &= -2P_3^1 \varrho_3 E - \varrho_3 EP - P_3^1 \varrho_3 P + 2\varrho_3 E^2 - \varrho_3 P^2 \\ &= -4P_3^1 \varrho_3 S - \varrho_3(2S-Q)^2 - 2\varrho_3(2S-Q)Q - 4\varrho_3 Q^2 \\ &= -P_3^1 \varrho_3 S - \varrho_3 S^2. \end{aligned}$$

4. Four linearly independent sections and 4-distributions. In this section we will find necessary and sufficient conditions for an oriented 8-dimensional spin vector bundle over an 8-complex to have 4 linearly independent sections or to be a sum of two 4-dimensional spin vector bundles.

THEOREM 4.1. *Let ξ be an oriented 8-dimensional vector bundle over a connected CW-complex X of dimension ≤ 8 with $w_2(\xi) = 0$. Then ξ has 4 linearly independent sections if and only if for some spin structure on ξ there is $S \in H^4(X;\mathbb{Z})$ such that the following conditions are satisfied*:

(1) $w_6(\xi) = 0$, $\mathrm{Sq}^2\,\varrho_2 S = 0$,
(2) $0 \in \Sigma(q_1(\xi))$, $0 \in \Sigma(S)$,
(3) $e(\xi) = 0$,
(4) $q_2(\xi) = S^2 - q_1(\xi)S$,
(5) $\varrho_4 q_2(\xi) \in \Phi(q_1(\xi))$,
(6) $0 \in \Phi(S)$,
(7) $P_3^1\,\varrho_3 S + \varrho_3 S^2 = 0$.

Proof. The vector bundle ξ over X has 4 linearly independent sections if and only if the mapping $\xi : X \to \mathrm{BSpin}(8)$ which is determined up to homotopy by the spin structure of the vector bundle can be lifted in the standard fibration

$$V_{8,4} \to \mathrm{BSpin}(4) \xrightarrow{\kappa} \mathrm{BSpin}(8).$$

So, we will build the Postnikov tower for this fibration.

The Stiefel manifold $V_{8,4}$ is 3-connected and the next homotopy groups are

$$\pi_4(V_{8,4}) \cong \mathbb{Z}, \quad \pi_5(V_{8,4}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \pi_6(V_{8,4}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$
$$\pi_7(V_{8,4}) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4.$$

Moreover, we have

$$\kappa^*(q_1) = q, \quad \kappa^*(e) = 0, \quad \kappa^*(p_2) = (2s-q)^2,$$

and hence

$$\kappa^*(q_2) = s^2 - sq.$$

The first invariant lies in $H^5(\mathrm{BSpin}(8);\mathbb{Z}) \cong 0$ and that is why it is zero. So the first stage is $\mathrm{BSpin}(8) \times K(\mathbb{Z},4)$ and the mapping $\mu = (\kappa,\beta)$: $\mathrm{BSpin}(4) \to \mathrm{BSpin}(8) \times K(\mathbb{Z},4)$ can be chosen to be a fibration in such a way that

$$\beta^*(\iota_4) = s.$$

The next invariants can be easily obtained from the Serre exact sequence for the fibration μ. They are $w_6 \otimes 1$ and $1 \otimes \mathrm{Sq}^2\,\varrho_2\iota_4$. So the first stage of the Postnikov tower for the fibration μ is $E_1 \times Y_1$, where Y_1 is the universal example for the operation Σ and $E_1 \xrightarrow{\kappa_1} \mathrm{BSpin}(8)$ is the fibration induced from the path fibration $K(\mathbb{Z}_2,5) \to PK(\mathbb{Z}_2,6) \to K(\mathbb{Z}_2,6)$ by the mapping $w_6 = \mathrm{Sq}^2\,\varrho_2 q_1$. We have the following commutative diagram:

$$\begin{array}{ccccc}
 & & K(\mathbb{Z}_2,5)\times K(\mathbb{Z}_2,5) & \overset{\equiv}{\dashrightarrow} & K(\mathbb{Z}_2,5)^2 \\
 & & \downarrow & & \downarrow{\scriptstyle j_1\times j_1} \\
\mathrm{BSpin}(4) & \xrightarrow{\gamma_1} & E_1\times Y_1 & \overset{f_1\times\mathrm{id}}{\dashrightarrow} & Y_1\times Y_1 \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \kappa_1\times\pi_1} & & \downarrow{\scriptstyle \pi_1\times\pi_1} \\
\mathrm{BSpin}(8)\times K(\mathbb{Z},4) & = & \mathrm{BSpin}(8)\times K(\mathbb{Z},4) & \xrightarrow{q_1\times\mathrm{id}} & K(\mathbb{Z},4)^2 \\
 & & & & {\scriptstyle 1\otimes\mathrm{Sq}^2\varrho_2\iota_4}\downarrow{\scriptstyle \mathrm{Sq}^2\varrho_2\iota_4\otimes 1} \\
 & & & & K(\mathbb{Z}_2,6)^2
\end{array}$$

where f_1 exists due to the fact that $\mathrm{Sq}^2\,\varrho_2\kappa_1^* q_1 = 0$. The next invariants are generators of $H^7(E_1 \times Y_1;\mathbb{Z}_2)$ and these are $f_1^*(\sigma)\otimes 1 = \Sigma(\kappa_1^* q_1 \otimes 1)$ and $1\otimes\sigma = 1\otimes\Sigma(\pi_1^*\iota_4)$. Consequently, the second stage of the Postnikov tower has the form $E_2\times Y_2$ where Y_2 is the universal example for the operation Φ and $E_2 \xrightarrow{\kappa_2} E_1$ is the fibration induced from the path fibration $K(\mathbb{Z}_2,6) \to PK(\mathbb{Z}_2,7) \to K(\mathbb{Z}_2,7)$ by the mapping $f_1^*(\sigma): E_1 \to K(\mathbb{Z}_2,7)$, which is the same as the fibration induced from $K(\mathbb{Z}_2,6) \to Y_2 \to Y_1$ by the mapping f_1. We have

$$\begin{array}{ccccc}
 & & K(\mathbb{Z}_2,6)\times K(\mathbb{Z}_2,6) & \overset{\equiv}{\dashrightarrow} & K(\mathbb{Z}_2,6)^2 \\
 & & \downarrow & & \downarrow{\scriptstyle j_2\times j_2} \\
\mathrm{BSpin}(4) & \xrightarrow{\gamma_2} & E_2\times Y_2 & \overset{f_2\times\mathrm{id}}{\dashrightarrow} & Y_2\times Y_2 \\
\downarrow{\scriptstyle \gamma_1} & & \downarrow{\scriptstyle \kappa_2\times\pi_2} & & \downarrow{\scriptstyle \pi_2\times\pi_2} \\
E_1\times Y_1 & = & E_1\times Y_1 & \xrightarrow{f_1\times\mathrm{id}} & Y_1\times Y_1 \\
 & & & & {\scriptstyle 1\otimes\sigma}\downarrow{\scriptstyle \sigma\otimes 1} \\
 & & & & K(\mathbb{Z}_2,7)^2
\end{array}$$

The mapping f_2 exists since $\Sigma(\kappa_2^*\kappa_1^* q_1) = 0$.

Further invariants lie in $H^8(E_2\times Y_2;\mathbb{Z})$, $H^8(E_2\times Y_2;\mathbb{Z}_4)$ and $H^8(E_2\times Y_2;\mathbb{Z}_3)$. The cohomologies of E_2 were computed in the proof of Theorem 4.1 of [CV3]. Hence, we have

$$H^8(E_2\times Y_2;\mathbb{Z}) \cong \mathbb{Z}^5$$

with generators $\kappa_2^*\kappa_1^* q_1^2\otimes 1$, $1\otimes\pi_2^*\pi_1^*\iota_4^2$, $\kappa_2^*\kappa_1^* q_1\otimes\pi_2^*\pi_1^*\iota_4$, $\kappa_2^*\kappa_1^* q_2\otimes 1$, $\kappa_2^*\kappa_1^* e\otimes 1$. Since $H^8(\mathrm{BSpin}(4);\mathbb{Z}) \cong \mathbb{Z}^3$ with generators q^2, s^2 and qs, the integral invariants are the generators of $\ker\gamma_2$:

$$A=\kappa_2^*\kappa_1^*e\otimes 1,\qquad B=\kappa_2^*\kappa_1^*q_2\otimes 1-1\otimes\pi_2^*\pi_1^*\iota_4^2+\kappa_2^*\kappa_1^*q_1\otimes\pi_2^*\pi_1^*\iota_4.$$

Next,

$$H^8(E_2\times Y_2;\mathbb{Z}_4)\cong(\mathbb{Z}_4)^7$$

with generators $\kappa_2^*\kappa_1^*\varrho_4q_1^2\otimes 1$, $\kappa_2^*\kappa_1^*\varrho_4q_2\otimes 1$, $\kappa_2^*\kappa_1^*\varrho_4e\otimes 1$, $f_2^*(\varphi)\otimes 1$, $1\otimes\pi_2^*\pi_1^*\varrho_4\iota_4^2$, $1\otimes\varphi$, $\kappa_2^*\kappa_1^*\varrho_4q_1\otimes\pi_2^*\pi_1^*\varrho_4\iota_4$. Further, $H^8(\mathrm{BSpin}(4);\mathbb{Z}_4)\cong(\mathbb{Z}_4)^3$ with generators ϱ_4q^2, ϱ_4s^2, ϱ_4qs. So using Lemma 2.3, we deduce that $\ker\gamma_2$ is generated by

$$\varrho_4A,\quad \varrho_4B, 1\otimes\varphi,\quad f_2^*\varphi\otimes 1-1\otimes\pi_2^*\pi_1^*\varrho_4\iota_4^2+\kappa_2^*\kappa_1^*\varrho_4q_1\otimes\pi_2^*\pi_1^*\varrho_4\iota_4.$$

It remains to compute the $\mathbb{Z}_3$-invariant. We have

$$H^8(E_2\times Y_2;\mathbb{Z}_3)\cong(\mathbb{Z}_3)^6$$

with generators $\kappa_2^*\kappa_1^*\varrho_3q_2\otimes 1$, $\kappa_2^*\kappa_1^*\varrho_3e\otimes 1$, $\kappa_2^*\kappa_1^*\varrho_3q_1^2\otimes 1$, $\kappa_2^*\kappa_1^*\varrho_3q_1\otimes\pi_2^*\pi_1^*\varrho_3\iota_4$, $1\otimes\pi_2^*\pi_1^*\varrho_3\iota_4^2$, $1\otimes\pi_2^*\pi_1^*P_3^1\varrho_3\iota_4$. So using Lemma 2.4, we find that $\ker\gamma_2$ is generated by

$$\varrho_3A,\qquad \varrho_3B,\qquad 1\otimes\pi_2^*\pi_1^*P_3^1\varrho_3\iota_4+1\otimes\pi_2^*\pi_1^*\varrho_3\iota_4^2.$$

Now, because $\dim X\le 8$, we can immediately see that the vector bundle $\xi: X\to\mathrm{BSpin}(8)$ has 4 linearly independent sections if and only if all the conditions (1)–(7) of the theorem are satisfied.

COROLLARY 4.2. *Let ξ be an oriented 8-dimensional vector bundle over a closed connected smooth spin 8-manifold M with $w_2(\xi)=0$. Then ξ has 4 linearly independent sections if and only if there is $S\in H^4(M;\mathbb{Z})$ and the following conditions are satisfied*:

(1) $w_6(\xi)=0$, $\mathrm{Sq}^2\,\varrho_2S=0$,
(2) $e(\xi)=0$,
(3) $4p_2(\xi)-p_1^2(\xi)=16S^2-8p_1(\xi)S$,
(4) $\{2p_1(M)p_1(\xi)-p_1^2(\xi)-4p_2(\xi)\}[M]\equiv 0 \bmod 64$,
(5) $\{p_1(M)S-2S^2\}[M]\equiv 0 \bmod 16$,
(6) $P_3^1\varrho_3S+\varrho_3S^2=0$.

P r o o f. It is easy to show that in the case $X=M$, conditions (1)–(7) of Theorem 4.1 are equivalent to conditions (1)–(6) of the corollary. It suffices to use the relations $0\in\Sigma(z)$ for $z\in\mathrm{Def}(\Sigma,M)$ (see [CV3], Theorem 5.5), $p_1(\xi)=2q_1(\xi)$, $p_2(\xi)=q_1^2(\xi)+2e(\xi)+4q_2(\xi)$, $\Phi(z)=\varrho_4\cdot\frac{1}{2}\{zq_1(M)-z^2\}$ for $z\in\mathrm{Def}(\Phi,M)$ and the fact that $H^8(M;\mathbb{Z})\cong\mathbb{Z}$.

Now we shall formulate nontrivial sufficient conditions for the existence of 4 linearly independent vector fields in tangent bundles without any reference to an element S.

Corollary 4.3. *Let M be a closed connected smooth spin manifold of dimension* 8. *If*

(i) $w_6(M) = 0$,
(ii) $e(M) = 0$,
(iii) $\{4p_2(M) - p_1^2(M)\}[M] \equiv 0 \bmod 128$,

and there is $k \in \mathbb{Z}$ such that

(iv) $4p_2(M) = (2k-1)^2 p_1^2(M)$,
(v) $k(k+2)p_2(M)[M] \equiv 0 \bmod 3$,

then M has 4 *linearly independent vector fields.*

Proof. We will prove that for $S = kq_1(M)$ conditions (1)–(6) of Corollary 4.2 hold. Obviously, (1) and (2) are satisfied. As for (3),

$$\begin{aligned}4p_2(M) - p_1^2(M) &= (2k-1)^2 p_1^2(M) - p_1^2(M)\\ &= (4k^2 - 4k)p_1^2 = 16S^2 - 8p_1(M)S.\end{aligned}$$

Next, (4) follows from the fact that

$$4p_2(M) - p_1^2(M) = 16k(k-1)q_1^2(M)$$

and

$$\varrho_2 q_1^2(M) = w_4^2(M) = w_8(M) = 0.$$

Because of

$$8(p_1(M)S - 2S^2) = -(2k-1)^2 p_1^2(M) + p_1^2(M) = p_1^2(M) - 4p_2(M),$$

condition (iii) implies (5).

Finally, condition (6) follows from (v). On spin manifolds

$$\widehat{A}[M] = \frac{1}{2^7 \cdot 45}\{7p_1^2(M) - 4p_2(M)\}[M]$$

is an integer (see [H], Theorem 26.3.1) and that is why $p_1^2(M)[M] \equiv p_2(M)[M] \bmod 3$. So we have

$$\begin{aligned}4(P_3^1\varrho_3 S + \varrho_3 S^2) &= 2kP_3^1\varrho_3 p_1(M) + k^2\varrho_3 p_1^2(M)\\ &= 4k\varrho_3 p_2(M) - 2k\varrho_3 p_1^2(M) + k^2\varrho_3 p_1^2(M)\end{aligned}$$

and

$$\begin{aligned}\{4kp_2(M) - 2kp_1^2(M) + k^2p_1^2(M)\}[M] &\equiv (4k - 2k + k^2)p_2(M)[M]\\ &\equiv k(k+2)p_2(M)[M] \bmod 3.\end{aligned}$$

Now, we will state and prove a result on the existence of 4-dimensional subbundles (4-distributions) in an 8-dimensional spin vector bundle. Since we want to avoid technical difficulties with the use of the Postnikov tower, our assumptions are a little more restrictive than in the case of Theorem 4.1 or Corollary 4.2.

THEOREM 4.4. *Let M be a closed connected smooth spin manifold of dimension 8 such that $H^4(M;\mathbb{Z})$ has no element of order 4. Let ξ be an oriented 8-dimensional vector bundle over M with $w_2(\xi)=0$. Then ξ is the sum of two 4-dimensional spin vector bundles if and only if there are S_1, S_2, Q_1, $Q_2 \in H^4(M;\mathbb{Z})$ and the following conditions are satisfied for $n=1,2$:*

(1) $p_1(\xi)=2(Q_1+Q_2)$,
(2) $e(\xi)=(2S_1-Q_1)(2S_2-Q_2)$,
(3) $p_2(\xi)=(2S_1-Q_1)^2+(2S_2-Q_2)^2+4Q_1Q_2$,
(4) $\mathrm{Sq}^2\varrho_2 Q_n=\mathrm{Sq}^2\varrho_2 S_n=0$,
(5) $\{S_n p_1(M)-2S_n^2\}[M]\equiv 0 \bmod 16$,
(6) $\{4S_n^2-4Q_nS_n\}[M]\equiv\{Q_np_1(M)-2Q_n^2\}[M] \bmod 16$,
(7) $P_3^1\varrho_3 Q_n+\varrho_3 Q_n^2=\varrho_3(S_n^2-Q_nS_n)$,
(8) $P_3^1\varrho_3 S_n+\varrho_3 S_n^2=0$.

Proof. First, we show that all the conditions are necessary. Let $\xi=\xi_1\oplus\xi_2$ where ξ_1 and ξ_2 are 4-dimensional spin vector bundles with the Euler and the first Pontryagin classes $2S_1-Q_1$, $2Q_1$ and $2S_2-Q_2$, $2Q_2$, respectively. According to Theorem 3.1 the classes $Q_1,Q_2,S_1,S_2\in H^4(M;\mathbb{Z})$ satisfy conditions (4)–(8). (Notice that $\Phi(z)=\varrho_4\cdot\frac{1}{2}(zq_1(M)-z^2)$ on $\mathrm{Def}(\Phi,M)$.) Moreover,

$$p_1(\xi)=p_1(\xi_1)+p_1(\xi_2),\qquad e(\xi)=e(\xi_1)\cdot e(\xi_2),$$
$$p_2(\xi)=p_2(\xi_1)+p_2(\xi_2)+p_1(\xi_1)p_1(\xi_2).$$

These conditions read as (1), (2) and (3).

Conversely, let conditions (1)–(8) be satisfied. Then according to Theorem 3.1 there are two 4-dimensional vector bundles ξ_1 and ξ_2 over M. Conditions (1)–(3) say that the bundles ξ and $\xi_1\oplus\xi_2$ have the same characteristic classes. Hence, using Theorem 2 from [CV1] (and just here we need the fact that $H^4(M;\mathbb{Z})$ has no element of order 4), we find that the two oriented vector bundles are isomorphic.

EXAMPLE 4.5. The quaternionic projective space $\mathbb{H}P^2$ does not admit any 4-distribution in its tangent bundle. The characteristic classes are

$$p_1(\mathbb{H}P^2)=2u,\quad p_2(\mathbb{H}P^2)=7u^2,\quad e(\mathbb{H}P^2)=3u^2$$

where $u\in H^4(\mathbb{H}P^2;\mathbb{Z})$ and $H^*(\mathbb{H}P^2;\mathbb{Z})=\mathbb{Z}[u]/\langle u^3\rangle$. There are no Q_1,Q_2, $S_1,S_2\in H^4(\mathbb{H}P^2;\mathbb{Z})$ satisfying (2) and (3) of Theorem 4.4. From (2) it follows that $2S_1-Q_1=\pm3$ and $2S_2-Q_2=\pm1$ or vice versa. Then from (3) we get

$$4Q_1Q_2=7u^2-(2S_1-Q_1)^2-(2S_2-Q_2)^2=-3u^2,$$

which is a contradiction.

Example 4.6. The Grassmannian $G^+_{6,2}(\mathbb{R})$ (the space of oriented 2-planes in $\mathbb{R}^6$) does not admit any spin 4-distribution. We have $H^*(G^+_{6,2}(\mathbb{R});\mathbb{Z}) = \mathbb{Z}[u]/\langle u^5\rangle$ where $u \in H^2(G^+_{6,2}(\mathbb{R});\mathbb{Z})$ and

$$p_1(G^+_{6,2}(\mathbb{R})) = 2u^2, \quad p_2(G^+_{6,2}(\mathbb{R})) = 7u^4, \quad e(G^+_{6,2}(\mathbb{R})) = 5u^4.$$

The same considerations as above show the nonexistence of Q_1, Q_2, S_1, S_2 satisfying (2) and (3) of Theorem 4.4. This implies that $G_{6,2}(\mathbb{R})$ (the space of nonoriented 2-planes in $\mathbb{R}^6$) has no spin 4-distribution either.

Example 4.7. Now consider the tangent bundle of the complex Grassmannian $G_{4,2}(\mathbb{C})$. In this case the situation is more complicated since

$$H^*(G_{4,2}(\mathbb{C});\mathbb{Z}) = \mathbb{Z}[u,v]/\langle u^3 - 2uv, v^2 - u^2v\rangle$$

where $u \in H^2(G_{4,2}(\mathbb{C});\mathbb{Z})$ and $v \in H^4(G_{4,2}(\mathbb{C});\mathbb{Z})$ and

$$p_1(G_{4,2}(\mathbb{C})) = 2u^2, \quad p_2(G_{4,2}(\mathbb{C})) = 14u^2v, \quad e(G_{4,2}(\mathbb{C})) = 6u^2v.$$

Nevertheless, neither in this case is there any spin 4-distribution in the tangent bundle. To prove this we have to explore nontrivial conditions (6) and (7) of Theorem 4.4 since for instance

$$Q_1 = -2u^2 - 2v, \quad S_1 = -5u^2, \quad Q_2 = 3u^2 + 2v, \quad S_2 = 4v$$

satisfy (1)–(5), (8) and (9) of Theorem 4.4. The proof that the system of equations (1)–(7) has no solution can be carried out considering equations (1)–(3) "modulo 8" and using a computer to go through all the possible values.

Example 4.8. Using Theorem 3.1 and the characterization of 6-dimensional vector bundles over 6-complexes from [W], it can be shown that the tangent bundle of the complex projective space $\mathbb{C}P^3$ has only spin 4-distributions α with the characteristic classes

$$q(\alpha) = 0, \quad s(\alpha) = \pm y^2, \quad e(\alpha) = \pm 2y^2$$

where $y \in H^2(\mathbb{C}P^3;\mathbb{Z})$ and $H^*(\mathbb{C}P^3;\mathbb{Z}) = \mathbb{Z}[y]/\langle y^4\rangle$. So

$$\tau(\mathbb{C}P^3) = \alpha \oplus \beta,$$

where β is a 2-distribution with $e(\beta) = \pm 2y$. Hence the tangent bundle of $S^2 \times \mathbb{C}P^3$ is a sum of two spin 4-distributions

$$\tau(S^2 \times \mathbb{C}P^3) = \alpha \oplus (\beta \oplus \tau(S^2))$$

with

$$q(\beta \oplus \tau(S^2)) = 2y^2, \quad s(\beta \oplus \tau(S^2)) = \pm 2xy + y^2,$$
$$e(\beta \oplus \tau(S^2)) = \pm 4xy$$

where x is a generator of $H^2(S^2;\mathbb{Z})$.

However, the tangent bundle of $S^2 \times \mathbb{C}P^3$ can also be written as a sum $\gamma \oplus \delta$ of spin 4-distributions with different characteristic classes, for instance

$$q(\gamma) = Q_1 = -4xy - 6y^2, \quad s(\gamma) = S_1 = -4xy + y^2, \quad e(\gamma) = -4xy + 8y^2$$

and

$$q(\delta) = Q_2 = 4xy + 8y^2, \quad s(\delta) = S_2 = -2xy - 5y^2, \quad e(\delta) = -8xy - 18y^2.$$

It is easy to show that these Q_1, S_1, Q_2, S_2 satisfy the assumptions of Theorem 4.4.

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Martin Čadek
Department of Algebra and Geometry
Masaryk University
Janáčkovo nám. 2a
662 95 Brno, Czech Republic
E-mail: cadek@math.muni.cz

Jiří Vanžura
Institute of Mathematics
Academy of Sciences of the Czech Republic
Žižkova 22
616 62 Brno, Czech Republic
E-mail: vanzura@ipm.cz

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