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L^2 ESTIMATES FOR OSCILLATORY INTEGRALS

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0. Introduction. In earlier papers, [1], [2], we obtained the complete L^p-mapping properties for a class of operators that includes

$$Tf(x) = \int_0^\infty \frac{e^{ix^b y^a}}{|x-y|^r} f(y)\, dy, \qquad x \in \mathbb{R},$$

with the phase function $g(x,y) = x^b y^a$, $b, a \geq 1$ and $0 \leq r < 1$. Included among these operators is the Fourier transform.

In [1] and [2] (Theorem 3.1 of [1]) we showed

THEOREM A. *Let $a, b \geq 1$ and $0 \leq r < 1$. Then*

$$\|Tf\|_p^p \leq C\|f\|_p^p$$

if and only if

$$\frac{b+a}{b+ar} \leq p \leq \frac{b+a}{b(1-r)}.$$

The driving force behind proving Theorem A is to solve the (L^2, L^2) mapping problem in the case $r = (b-a)/(2b) + i\alpha$ for $\alpha \in \mathbb{R}$.

In this article we wish to obtain L^2-estimates for similar non-convolution operators with more general phase functions. To be more precise, we consider the operator

$$Tf(x) = \int_0^\infty k(x,y) f(y)\, dy, \qquad x \in \mathbb{R}, \tag{0.1}$$

with

$$k(x,y) = \varphi(x,y) e^{ig(x,y)}$$

where $g(x,y)$ is real-valued. In Theorem 0.1, we study the cases where

$$g(x,y) = x^b \gamma_1(y) + x^m \gamma_2(y), \quad b > a \geq 1. \tag{0.2}$$

The previous case was when $\gamma_1(y) = y^a$ and $m = 0$. In Theorem 2.4 for $1 \leq a < 2$ we obtain a more general result.

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Furthermore, we suppose throughout this paper that for $|x-y|>0$,

$$\text{(0.3)}\qquad \begin{aligned} &\text{(a)}\quad |\varphi(x,y)| \le C|x-y|^{-(b-a)/(2b)},\\ &\text{(b)}\quad |\partial_x\varphi(x,y)| \le C|x-y|^{-(b-a)/(2b)-1}. \end{aligned}$$

The cases where $b \ge a$ are considered in [1]. Note that without any loss we can suppose in (0.3) that $\varphi(x,y)$ and $\partial_x\varphi(x,y)$ are both bounded, since for some cut-off function $\lambda(x)$ we get

$$\begin{aligned} Tf(x) &= \int_0^\infty k(x,y)\lambda(x-y)f(y)\,dy + \int_0^\infty k(x,y)(1-\lambda(x-y))f(y)\,dy\\ &= T_1f(x)+T_2f(x). \end{aligned}$$

But T_1 maps L^2 into L^2 since its integrand is in L^1 and we are left with T_2.

In our first result we show that

Theorem 0.1. *Let $b>a\ge 1$, and suppose $k(x,y)$ satisfies* (0.2) *and* (0.3). *If $b\ge 2$, $b>m>0$, $1\le m_1\le a$, and for $u,v\ge 0$,*

$$\text{(0.4)}\qquad \begin{aligned} &\text{(a)}\quad |\gamma_1(u)-\gamma_1(v)| \ge C|u-v|(u^{a-1}+v^{a-1}),\\ &\text{(b)}\quad |\gamma_2(u)-\gamma_2(v)| \le C|u-v|(u^{m_1-1}+v^{m_1-1}), \end{aligned}$$

then $\|Tf\|_2 \le C\|f\|_2$.

Our second result appears in Theorem 2.4.

We find it convenient to let $\Psi(x,y,y') = g(x,y)-g(x,y')$ throughout this paper. The letter C stands for a positive constant that may change line-by-line.

1. Preliminaries. Here we wish to show that

$$I = \int_{-\infty}^{\infty} |Tf(x)|^2\,dx \le C\int |f|^2\,dy = C\|f\|_2^2$$

for the operators defined in (0.1). For some constant A that still needs to be determined, we consider

$$\text{(1.1)}\qquad I = I_1+I_2 = \int_{|x|\le A} |Tf(x)|^2\,dx + \int_{|x|\ge A} |Tf(x)|^2\,dx$$

and we wish to show that

$$I_1 \le C\|f\|_2^2 \quad\text{and}\quad I_2 \le C\|f\|_2^2.$$

Begin with the term I_1 and note that

$$I_1 \le C \int_{|x|\le A} \Big| \int_0^\infty e^{ig(x,y)} f(y)(\varphi(x,y) - \varphi(0,y))\,dy\Big|^2\,dx$$
$$+ C \int_{|x|\le A} \Big| \int_0^\infty e^{ig(x,y)} \varphi(0,y) f(y)\,dy\Big|^2\,dx = A_1 + A_2.$$

By (0.3)(b) it follows that

$$A_1 \le C\|f\|_2^2,$$

since the integrand is in L^1. We are left with the A_2-piece, namely,

$$A_2 = C\int_0^\infty dy\, f(y) \int_0^\infty dy'\, \overline{f}(y') \int_{|x|\le A} e^{i\Psi(x,y,y')}\varphi(0,y')\varphi(0,y)\,dx.$$

Consider the condition

$$\Big|\int_{|x|\le A} e^{i\Psi(x,y,y')}\,dx\Big| \le \frac{C}{|y^a - y'^a|^{1/b}}. \tag{1.2}$$

Proposition 1.1. *Let $b > a \ge 1$. If $\varphi(x,y)$ satisfies* (0.3) *and* (1.2) *holds, then $I_1 \le C\|f\|_2^2$.*

Proof. Since $I_1 \le A_1 + A_2$ and by (0.3)(b) we get $A_1 \le C\|f\|_2^2$, it suffices to estimate A_2. But by (0.3) and (1.2) we get

$$A_2 \le C\int_0^\infty dy\,|f(y)| \int_0^\infty dy'\,|f(y')| A(y,y')$$

where

$$A(y,y') = \frac{1}{|y^a - y'^a|^{1/b}(y\cdot y')^{(b-a)/(2b)}}.$$

But by Schur's lemma [4], $A(y,y')$ is the kernel of an operator that maps L^2 into L^2. ∎

We point out the following useful but elementary result.

Lemma 1.2. *Let $\Phi(x,y,y')$ be a real-valued function and suppose that*

$$A(y,y') = \int_c^d |\partial_x K(x,y,y')| \Big|\int_c^x e^{i\Phi(v,y,y')}\,dv\Big|\,dx \tag{1.3}$$

is defined for almost all $y, y' \ge 0$. *Then*

$$J = \Big|\int_a^b dy\, f(y) \int_a^b dy'\, f(y') \int_c^d dx\, e^{i\Phi(x,y,y')} K(x,y,y')\Big|$$
$$\le \Big(\int_a^b |f|^2\, dy\Big)^{1/2} \Big(\int_a^b dy'\, |f(y')|^2 \int_a^b dy\, A(y,y')\Big)^{1/2} \Big(\int_a^b A(y,y')\, dy'\Big)^{1/2}.$$

Proof. Set $B = \int_c^d e^{i\Phi(x,y,y')} K(x,y,y')\, dx$. Using integration by parts we see that

$$|B| \le \int_c^d |\partial_x K(x,y,y')| \Big|\int_c^x e^{i\Phi(v,y,y')}\, dv\Big|\, dx = A(y,y').$$

We get our result by repeated application of Schwarz's inequality. ■

We now consider the term I_2 (we bounded I_1 in Proposition 1.1). Let

$$\eta(y) + \sum_{l=0}^{\infty} \psi_l(y) = 1, \qquad \eta, \psi \in C^\infty,\ \psi_l(y) = \psi(y/2^l),$$

and $\psi(y)$ is supported in $1/2 \le |y| \le 2$ and $\eta(y)$ in $|y| \le 1$. We get

$$(1.4) \qquad I_2 \le C \int_{|x|\ge A} \Big|\int_0^\infty k(x,y)\eta(x-y) f(y)\, dy\Big|^2 dx$$
$$+ C \int_{|x|\ge A} \Big|\sum_{l=0}^{\infty} \int_0^\infty k(x,y)\psi_l(x-y) f(y)\, dy\Big|^2 dx$$
$$= I_{21} + I_{22}.$$

The term I_{21} is estimated in a straightforward manner, and we shall do that below; the bounds for I_{22} will be done later in Propositions 1.4 and 1.5. Notice that

$$I_{21} = \int_0^\infty dy\, f(y) \int_0^\infty dy'\, \overline{f}(y') \int_{|x|\ge A} dx\, e^{i\Psi(x,y,y')} \eta(x-y)\eta(x-y')\varphi(x,y)\overline{\varphi}(x,y').$$

Note $A(y,y')$ is defined by (1.3) and is supported in $|y-y'| \le 2$. From Lemma 1.2 it follows that we need only show the L^1 conditions,

$$(1.5) \qquad \text{(a)} \quad \int_0^\infty A(y,y')\, dy \le C, \qquad \text{(b)} \quad \int_0^\infty A(y,y')\, dy' \le C.$$

The next result follows immediately from Lemma 1.2, where $A(y,y')$ is defined by (1.3). From here on we use a parameter λ and the relevant constants A and C do not depend on λ.

PROPOSITION 1.3. *If* (1.5) *holds*, *then* $I_{21} \le C\|f\|_2^2$.

In order to obtain bounds for I_{22}, we utilize the following condition. Let $\lambda > 0$. Then there exists a constant A so that for $x \ge A/\lambda$, where A does not depend upon λ,

$$\Big|\int\limits_{A/\lambda}^{x} e^{i\Psi(\lambda v,\lambda y,\lambda y')}\,dv\Big| \le \frac{C}{\lambda^{(b+a)/b}|y^a - y'^a|^{1/b}} \quad \text{for } y,y' \ge 0. \tag{1.6}$$

Next let

$$\tilde{I}_{2l} = \int\limits_{|x|\ge A}\Big|\int\limits_0^\infty k(x,y)\psi_l(x-y)f(y)\,dy\Big|^2\,dx,$$

then set $\tilde{I}_{2l} = I_{2l,1} + I_{2l,2}$ with

$$\begin{aligned} I_{2l,1} = \lambda^3 &\int\limits_0^{\lambda^{-1}} dy\, f(\lambda y)\int\limits_0^\infty dy'\,\bar{f}(\lambda y') \\ &\times \int\limits_{|x|\ge A/\lambda} dx\,\psi(x-y)\psi(x-y')\varphi(\lambda x,\lambda y)\overline{\varphi}(\lambda x,\lambda y')e^{i\Psi(\lambda x,\lambda y,\lambda y')}, \end{aligned}$$

with $\lambda = 2^{-l}$.

From (1.4) we get

$$I_{22}^{1/2} \le \sum_{l=0}^{\infty}\tilde{I}_{2l}^{1/2}$$

and so in estimating I_{22} our problem is reduced to seeing that the terms $I_{2l,1}$ and $I_{2l,2}$ sum.

PROPOSITION 1.4. *Let* $b > a \ge 1$. *If* (0.3) *and* (1.6) *hold*, *then*

$$I_{2l,1} \le \frac{C}{\lambda^{(1-a/b)/2}}\|f\|_2^2.$$

Proof. By (0.3), (1.6) and Lemma 1.2, it follows that

$$I_{2l,1} \le C\lambda^3\int\limits_0^{\lambda^{-1}} dy\,|f(\lambda y)|\int\limits_0^\infty dy'\,|f(\lambda y')|A(y,y')$$

where

$$A(y,y') = \frac{\chi(|y-y'|\le 4)}{\lambda^2|y-y'|^{1/b}(y^{a-1}+y'^{(a-1)})^{1/b}}.$$

We can easily see that

$$(1.7)\qquad \begin{aligned}&\text{(a)}\quad \int_0^\infty A(y,y')\,dy' \le C\lambda^{-2} \qquad \text{for } 0\le y\le \lambda^{-1},\\ &\text{(b)}\quad \int_0^{\lambda^{-1}} A(y,y')\,dy \le C\lambda^{-(1-a/b)}\lambda^{-2}.\end{aligned}$$

Now by Lemma 1.2 and (1.7) we get

$$\begin{aligned} I_{2l,1} &\le \frac{C\lambda^3}{\lambda^2}\cdot \lambda^{-(1-a/b)/2}\int_0^\infty |f(\lambda y)|^2\,dy\\ &\le C\lambda^{-(1-a/b)/2}\|f\|_2^2\end{aligned}$$

after changing variables. ■

We still need another estimate for the left-hand term that appears in (1.6). It will be used to bound $I_{2l,2}$.

Let $\lambda > 0$. Then there exists a constant A (independent of λ) and an $\alpha > 0$ so that for $x \ge A/\lambda$ and $y, y' \ge 0$,

$$(1.8)\qquad \Big|\int_{A/\lambda}^{x} e^{i\Psi(\lambda v,\lambda y,\lambda y')}\,dv\Big| \le \frac{C}{\lambda^\alpha |y^a - y'^a|} \qquad \text{for } y+y' \ge \lambda^{-1}.$$

Proposition 1.5. *Let $b > a \ge 1$. If* (0.3) *and* (1.8) *hold then*

$$I_{2l,2} \le \frac{C\log(1+\lambda)}{\lambda^{(b-a)/b}\lambda^{\alpha-a-1}}\|f\|_2^2.$$

Proof. Here we have

$$\begin{aligned} I_{2l,2} \le \lambda^3 &\int_{\lambda^{-1}}^\infty dy\,|f(\lambda y)|\int_0^\infty dy'\,|f(\lambda y')|\\ &\times\Big|\int_{|x|\ge A/\lambda} dx\,\psi(x-y)\psi(x-y')\varphi(\lambda x,\lambda y)\overline{\varphi}(\lambda x,\lambda y')e^{i\Psi(\lambda x,\lambda y,\lambda y')}\Big|.\end{aligned}$$

But by Lemma 1.2, (0.3) and (1.8) it follows that

$$A(y,y') = \frac{C\chi(|y-y'|\le 4)}{\lambda^{(b-a)/b}[1+\lambda^\alpha|y^a - y'^a|]}.$$

We easily see that for $a \ge 1$,

$$(1.9)\qquad \begin{aligned}&\text{(a)}\quad \int_0^\infty A(y,y')\,dy' \le C\log(1+\lambda)\lambda^{-(b-a)/b}\lambda^{a-1-\alpha} \qquad \text{if } y\ge \lambda^{-1},\\ &\text{(b)}\quad \int_{\lambda^{-1}}^\infty A(y,y')\,dy \le C\log(1+\lambda)\lambda^{-(b-a)/b}\lambda^{a-1-\alpha}.\end{aligned}$$

Thus by Lemma 1.2, from (1.9) we get

$$I_{2l,2} \le \frac{C\log(1+\lambda)\lambda^3}{\lambda^{(b-a)/b}\lambda^{\alpha-a+1}} \int_0^\infty |f(\lambda y)|^2\,dy,$$

and after changing variables we get our result. ■

Now we put all these results together to obtain

THEOREM 1.6. *Let $b > a \ge 1$. If* (0.3), (1.2), (1.5), (1.6) *all hold and* (1.8) *holds with $\alpha > a + a/b$, then*

$$\|Tf\|_2 \le C\|f\|_2.$$

Proof. We note that by (1.1) and (1.4), $I = I_1 + I_2$ and $I_2 \le I_{21} + I_{22}$, and we need to show that $I \le C\|f\|_2^2$.

By Propositions 1.1 and 1.3 it follows that

$$I_1 + I_{21} \le C\|f\|_2^2. \tag{1.10}$$

Also since $\tilde{I}_{2l} = I_{2l,1} + I_{2l,2}$ we see by Propositions 1.4 and 1.5 ($\lambda = 2^{-l}$) with $\alpha > a + a/b$ that $I_{2l,1}$ and $I_{2l,2}$ sum, and thus

$$I_{22} \le C\|f\|_2^2. \tag{1.11}$$

Now our result follows from (1.10) and (1.11). ■

2. Proof of Theorem 0.1. We prove Theorem 0.1 by showing that the kernel $k(x,y)$ defined there satisfies the conditions of Theorem 1.6. We begin with the following result which is an easy consequence of Lemmas 7– 9 of [3].

LEMMA 2.1. *Assume that $b \ne m$, $\alpha(t) = t^b\xi + t^m\eta$, and $\xi,\eta \in \mathbb{R}$. If $m > 0$ and $b \ge 2$, then*

$$\Big|\int_0^T e^{i\alpha(t)}\,dt\Big| \le C|\xi|^{-1/b} \quad \text{for } T \ge 0,$$

and C does not depend upon ξ,η or T.

Proof. Without any loss, we can suppose that $\xi > 0$ and $T' = T^m\xi^{m/b} \ge 1$. Then

$$\Big|\int_0^T e^{i\alpha(t)}\,dt\Big| = \frac{1}{\xi^{1/b}}\Big|\int_0^{T'} \frac{e^{itb/m}e^{it\lambda}}{t^{1-1/m}}\,dt\Big|$$

and $\lambda = \eta/\xi^{m/b}$. But since $m > 0$, it suffices to bound

$$\Big|\int_1^{T'} \frac{e^{itb/m}e^{it\lambda}}{t^{1-1/m}}\,dt\Big| = \Big|\int_1^{T'} \frac{e^{itb/m}e^{it\lambda}}{t^{1-b/(2m)}t^{(1/m)(b/2-1)}}\,dt\Big| \le C,$$

which follows from Lemmas 7–9 of [3], since $b \ge 2$ and $m > 0$. ■

REMARK. If for the term I_{21}, we suppose (0.4)(a), $b \neq m$, $m > 0$, $b \geq 2$, then by using (1.3) we see from Lemma 2.1 that

$$A(y,y') = \frac{\chi(|y-y'| \leq 2)}{|y-y'|^{1/b}(y^{a-1} + y'^{(a-1)})^{1/b}}.$$

Note that in (1.3), $\Phi(x,y,y') = x^b(\gamma_1(y) - \gamma_1(y')) + x^m(\gamma_2(y) - \gamma_2(y'))$.

We also employ and prove here the following result.

LEMMA 2.2. *Let $y, y' \geq 0$. Suppose that*

$$g(x,y) = x^b\gamma_1(y) + x^m\gamma_2(y), \qquad 1 \leq m_1 \leq a,\ b > m.$$

If (0.4) *holds, then for any $\lambda > 0$ there exists an A large enough so that if $x \geq A/\lambda$ then*

$$\begin{aligned} (2.1)\qquad &|\partial_x \Psi(\lambda x, \lambda y, \lambda y')| \\ &\geq C\lambda^{a+1}|y-y'|(y^{a-1} + y'^{(a-1)}) \quad \text{for } y + y' \geq \lambda^{-1}. \end{aligned}$$

Proof. We have

$$\partial_v \Psi(\lambda v, \lambda y, \lambda y') = b\lambda^b v^{b-1}(\gamma_1(\lambda y) - \gamma_1(\lambda y')) + m\lambda^m v^{m-1}(\gamma_2(\lambda y) - \gamma_2(\lambda y')).$$

Thus

$$\begin{aligned} (2.2)\quad |\partial_v \Psi(\lambda v, \lambda y, \lambda y')| \geq v^{m-1}\lambda^{m+m_1}|y-y'|[&C_1\lambda^{b+a-m-m_1}v^{b-m} \\ &\times (y^{a-1} + y'^{(a-1)}) - mC_2(y^{m_1-1} + y'^{(m_1-1)})] \end{aligned}$$

where we used (0.4). Since $v \geq A/\lambda$ we get

$$C_1 A^{b-m}\lambda^{a-m_1}(y^{a-1} + y'^{(a-1)}) \geq mC_2(y^{m_1-1} + y'^{(m_1-1)}).$$

But since $1 \leq m_1 \leq a$ and $b > m$, we can choose A large enough to obtain the above inequality. ■

In Lemma 2.2 we have determined the value of A from the beginning of the article. Also notice that if $m_1 = a$ in Lemma 2.2, the restriction $y + y' \geq \lambda^{-1}$ could be dropped.

The next result follows from Lemmas 2.1 and 2.2.

PROPOSITION 2.3. *Let $g(x,y) = x^b\gamma_1(y) + x^m\gamma_2(y)$.*

(a) *If $b \geq 2$, $b \neq m$ and $m > 0$, then*

$$\Big|\int_0^x e^{i\Psi(\lambda v, \lambda y, \lambda y')}\,dv\Big| \leq \frac{C}{\lambda|\gamma_1(\lambda y) - \gamma_1(\lambda y')|^{1/b}}.$$

(b) *If $1 \leq m_1 \leq a$, $b > m$, $y + y' \geq \lambda^{-1}$, $y, y' \geq 0$ and* (0.4) *holds, then for $x \geq A/\lambda$,*

$$\Big|\int_{A/\lambda}^x e^{i\Psi(\lambda v, \lambda y, \lambda y')}\,dv\Big| \leq \frac{C}{\lambda^{a+1}|y-y'|(y^{a-1} + y'^{(a-1)})}.$$

Proof. Part (a) follows from Lemma 2.1, while part (b) follows from Lemma 2.2. ■

It follows from Proposition 2.3 that the operator in (0.1) with $g(x,y) = x^b\gamma_1(y)+x^m\gamma_2(y)$ satisfies estimates like (1.2), (1.6) and (1.8). We are now in a position to prove Theorem 0.1.

Proof of Theorem 0.1. According to Theorem 1.6, we need to see that (1.2), (1.5), (1.6) and (1.8) hold, with $\alpha > a+a/b$. We notice that (1.2) and (1.6) follow from Proposition 2.3(a) and (0.4)(a). Next (1.8) follows from Proposition 2.3(b) and (0.4), with $\alpha = a+1 > a+a/b$, since here $b > a$. We are finished once we show (1.5).

To see (1.5), we use the remark following Lemma 2.1 and get

$$A(y,y') = \frac{\chi(|y-y'|\le 2)}{|y-y'|^{1/b}(y^{a-1}+y'^{(a-1)})^{1/b}}.$$

Since $b > a$ we see that (1.5) holds, and this now completes our argument. ■

We say that a function $h(x,y,y')$ is "monotonic" in x for each $y,y' \ge 0$ if there exists a number M independent of x,y,y' so that $h(x,y,y')$ is monotonic in x for $x \in [a_{j-1},a_j]$ with $1 \le j \le N+1$, $a_0 = 0$, $a_{N+1} = \infty$ and $N \le M$. Note that these intervals may depend upon y or y'.

We are able to show that

THEOREM 2.4. *Let $\alpha > a+2a/b$ with $b > a \ge 1$ and $a < 2$. Suppose there exists an A large enough so that*

$$\text{(2.3)}\qquad \begin{aligned} &\text{(a)}\quad \Psi(x,y,y') \text{ is "monotonic" in } x,\\ &\text{(b)}\quad |\partial_x\Psi(\lambda y,\lambda y,\lambda y')| \ge C\lambda^\alpha|y-y'|(y^{a-1}+y'^{(a-1)}), \end{aligned}$$

for each $y,y' \ge 0$, $\lambda > 0$ and $x \ge A/\lambda$. If, furthermore (1.2), (1.5) *both hold with $A(y,y')$ taken from* (1.3), *then*

$$\|Tf\|_2 \le C\|f\|_2.$$

Proof. From (1.3) and (2.3) it follows that

$$A(y,y') = \frac{C\chi(|y-y'|\le 4)}{\lambda^{(b-a)/b}(1+\lambda^\alpha|y-y'|(y^{a-1}+y'^{(a-1)})}$$

just as in the proof of Proposition 1.5. Next,

$$\begin{aligned}\tilde I_{2l} &\le \lambda^3\Big(\int_0^{\lambda^{-1}} + \int_{\lambda^{-1}}^{\infty}\Big)\,dy\,|f(\lambda y)|\int_0^\infty dy'\,|f(\lambda y')|A(y,y')\\ &= I_{2l,1}+I_{2l,2}.\end{aligned}$$

In order to estimate $I_{2l,1}$ we can easily see that with $\lambda = 2^l$ and $a \ge 1$,

$$(2.4)\qquad \begin{aligned} &\text{(a)} \quad \int_0^\infty A(y,y')\,dy' \le Cy(1-a)\frac{l}{\lambda^{(b-a)/b+\alpha}},\\ &\text{(b)} \quad \int_0^{\lambda^{-1}} \frac{A(y,y')}{y^{a-1}}\,dy \le \frac{C}{\lambda^{(b-a)/b+2-a}}. \end{aligned}$$

It follows from (2.4) and Lemma 1.2 that

$$I_{2l,1} \le \frac{Cl^{1/2}\|f\|_2^2}{\lambda^{\alpha/2-a/b-a/2}}.$$

For the term $I_{2l,2}$ we can easily see that for $y \ge \lambda^{-1}$ and $a \ge 1$,

$$(2.5)\qquad \begin{aligned} &\text{(a)} \quad \int_0^\infty A(y,y')\,dy' \le \frac{Cl}{\lambda^{(b-a)/b+\alpha+1-a}},\\ &\text{(b)} \quad \int_{\lambda^{-1}}^\infty A(y,y')\,dy \le \frac{Cl}{\lambda^{(b-a)/b+\alpha+1-a}}. \end{aligned}$$

Thus from Lemma 1.2 and (2.5) we get

$$I_{2l,2} \le \frac{Cl\|f\|_2^2}{\lambda^{(b-a)/b+\alpha-1-a}}.$$

But

$$\tilde{I}_{2l} = I_{2l,1} + I_{2l,2} \le C\|f\|_2^2\left(\frac{l^{1/2}}{\lambda^{\alpha/2-a/b-a/2}} + \frac{l}{\lambda^{\alpha-a/b-a}}\right)$$

and $\alpha > a + 2a/b$ ($\lambda = 2^l$), therefore $I_{22}^{1/2} \le \sum_l \tilde{I}_{2l}^{1/2}$ sums and we get

$$(2.6)\qquad I_{22} \le C\|f\|_2^2.$$

Our proof rests on showing (1.1), that is,

$$(2.7)\qquad I_1 + I_2 \le C\|f\|_2^2.$$

Because of (1.2) we see by Proposition 1.1 that

$$(2.8)\qquad I_1 \le C\|f\|_2^2.$$

By (1.4) and (2.6) it suffices to estimate I_{21}. But by (1.5) and Proposition 1.3 we get

$$(2.9)\qquad I_{21} \le C\|f\|_2^2.$$

Putting the estimates (2.6), (2.8) and (2.9) together, we get our result. ∎

We obtain dual results to both Theorems 0.1 and 2.4. We shall work through the case of Theorem 0.1 here. This time we consider the operator

$$T^*f(x) = \int_0^\infty \varphi(y,x)e^{i(y^b\gamma_1(x)+y^m\gamma_2(x))}f(y)\,dy$$

and show that it maps L^2 into itself. In fact, we get

THEOREM 2.5. *Let $b > a \geq 1$, and assume that $k(x,y)$ satisfies* (0.2)–(0.4). *If $b \geq 2$, $b > m > 0$, $1 \leq m_1 \leq a$, then*

$$\|T^*f\|_2 \leq C\|f\|_2.$$

Proof. Just employ duality with Theorem 0.1, i.e., consider

$$\int_0^\infty g(x)Tf(x)\,dx = \int_0^\infty dy\,f(y)\int_0^\infty \varphi(x,y)e^{i(x^b\gamma_1(y)+x^m\gamma_2(y))}g(x)\,dx.$$

Then

$$\Big|\int_0^\infty gTf\,dx\Big| = \Big|\int_0^\infty fT^*g\,dy\Big| \leq \|g\|_2\|Tf\|_2 \leq C\|g\|_2\|f\|_2,$$

where we used Theorem 0.1 in the last step. ∎

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